ON SOME GROWTH PROPERTIES OF DIFFERENTIAL POLYNOMIALS IN THE LIGHT OF RELATIVE ORDER

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Abstract. In the paper we establish some newly developed results based on the growth properties of relative order (relative lower order), relative type (relative lower type) and relative weak type of differential polynomials generated by entire and meromorphic functions.

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1. Introduction, definitions and notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Also, let $n_{0j}, n_{1j}, ..., n_{kj}$ $(k \geq 1)$ be non-negative integers such that for each j, $\sum\limits_{i=0}^{k} n_{ij} \geq 1$.

We call $M_j[f] = A_j(f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} \cdots \left(f^{(k)}\right)^{n_{kj}}$, where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The numbers $\gamma_{M_j} = \sum\limits_{i=0}^{k} n_{ij}$ and $\Gamma_{M_j} = \sum\limits_{i=0}^{k} (i+1)\, n_{ij}$ are called, respectively, the degree and weight of $M_j[f]$ ([2], [8]). The expression $P[f] = \sum\limits_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max\limits_{1<j<s} \gamma_{M_j}$ and $\Gamma_P = \max\limits_{1<j<s} \Gamma_{M_j}$ are called, respectively, the degree and weight of $P[f]$ ([2], [8]). Also, we call the numbers $\underline{\gamma}_P = \min\limits_{1<j<s} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$, respectively. If $\underline{\gamma}_P = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f, i.e., for which $n_{0j} = 0$ for $j = 1, 2, ..., s$. We consider only those $P[f], P_0[f]$ singularities of whose individual terms do not cancel each other.

The following definitions are well known.

Definition 1. The quantity $\Theta(a; f)$ of a meromorphic function f is defined as follows

$$\Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, a; f)}{T(r, f)}.$$

Definition 2. [7] For $a \in \mathbb{C} \cup \{\infty\}$,let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \le r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geqslant p$ is counted exactly p times; and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define

$$\delta_p(a; f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

Definition 3. [1] $P[f]$ is said to be admissible if

(i) $P[f]$ is homogeneous, or

(ii) $P[f]$ is non homogeneous and $m(r, f) = S(r, f)$.

The following definitions are also well known.

Definition 4. The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

Definition 5. The type σ_f and lower type $\bar{\sigma}_f$ of a meromorphic function f are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}} \quad \text{and} \quad \bar{\sigma}_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

In this connection, Datta and Jha [3] gave the definition of weak type of a meromorphic function of finite positive lower order in the following way:

Definition 6. [3] The weak type τ_f of a meromorphic function f of finite positive lower order λ_f is defined by

$$\tau_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}}.$$

Similarly, one can define the growth indicator $\bar{\tau}_f$ of a meromorphic function f of finite positive lower order λ_f as

$$\bar{\tau}_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}}.$$

For an entire function g, the Nevanlinna's characteristic function $T_g(r)$ is defined as

$$T_g(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|g(re^{i\theta})\right| d\theta,$$

where $\log^+ x = \max(0, \log x)$ for $x > 0$.

If g is non-constant then $T_g(r)$ is strictly increasing and continuous and its inverse $T_g^{-1} : (T_g(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [6] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 7. [6] Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as

$$\rho_g(f) = \inf \left\{ \begin{array}{c} \mu > 0 : T_f(r) < T_g(r^\mu) \\ \text{for all sufficiently large } r \end{array} \right\} = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

The definition coincides with the classical one [6] if $g(z) = \exp z$.

Similarly, one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

Datta and Biswas [4] gave the definition of relative type and relative weak type of a meromorphic function with respect to an entire function g which are as follows:

Definition 8. [4] The relative type $\sigma_g(f)$ of a meromorphic function f with respect to an entire function g are defined as

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Similarly, one can define the lower relative type $\bar{\sigma}_g(f)$ in the following way

$$\bar{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Definition 9. [4] The relative weak type $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}}.$$

Analogously, one can define the growth indicator $\bar{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\bar{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}}.$$

In this paper we establish some newly developed results based on the growth properties of relative order (relative lower order) relative type(relative lower type) and relative weak type of polynomials generated by entire and meromorphic functions. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [5] and [9].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [1] *Let f be either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$. Then, for homogeneous $P_0[f]$,*

$$\lim_{r\to\infty} \frac{T_{P_0[f]}(r)}{T_f(r)} = \gamma_{P_0[f]}.$$

Lemma 2. [1] *Let $P_0[f]$ be admissible. If f is of finite order or of non zero lower order and $\sum_{a\neq\infty} \Theta(a; f) = 2$, then*

$$\lim_{r\to\infty} \frac{T_{P_0[f]}(r)}{T_f(r)} = \Gamma_{P_0[f]}.$$

Lemma 3. [3] *If f be a meromorphic function of regular growth, i.e., $\rho_f = \lambda_f$ then*

$$\sigma_f = \bar{\sigma}_f = \tau_f = \bar{\tau}_f.$$

3. Theorems

In this section, we present the main results of the paper. It is needless to mention that in the paper, the admissibility and homogeneity of $P_0[f]$ will be needed as per the requirements of the theorems.

Theorem 1. *Let f be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ *or* $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ *and g be an entire function with* $0<\tau_g\leq\overline{\tau}_g<\infty$ *and* $0<\overline{\sigma}_g\leq\sigma_g<\infty$. *Also let* $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ *or* $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$. *Then*

$$\max\left\{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\tau_g}{\overline{\tau}_g}\right)^{\frac{1}{\lambda_g}},\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\overline{\sigma}_g}{\sigma_g}\right)^{\frac{1}{\rho_g}}\right\}$$

$$\leq\ \liminf_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_gT_f(r)}\leq\limsup_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_gT_f(r)}$$

$$\leq\ \min\left\{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\overline{\tau}_g}{\tau_g}\right)^{\frac{1}{\lambda_g}},\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\sigma_g}{\overline{\sigma}_g}\right)^{\frac{1}{\rho_g}}\right\},$$

where $P_0[f]$ and $P_0[g]$ are homogeneous.

Proof. For any $\varepsilon(>0)$, we get from Lemma 1, for all sufficiently large values of r,

(1) $$T_{P_0[f]}(r)\leq\left\{\gamma_{P_0[f]}+\varepsilon\right\}T_f(r)$$

and

(2) $$T_{P_0[f]}(r)\geq\left\{\gamma_{P_0[f]}-\varepsilon\right\}T_f(r).$$

Also, from Lemma 1, we get for all sufficiently large values of r that

$$T_{P_0[g]}(r)\geq\left\{\gamma_{P_0[g]}-\varepsilon\right\}T_g(r)$$

$$i.e.,\ r\geq T^{-1}_{P_0[g]}\left[\left\{\gamma_{P_0[g]}-\varepsilon\right\}T_g(r)\right]$$

(3) $$i.e.,\ T^{-1}_g\left(\frac{r}{\gamma_{P_0[g]}-\varepsilon}\right)\geq T^{-1}_{P_0[g]}(r).$$

and

$$T_{P_0[g]}(r)\leq\left\{\gamma_{P_0[g]}+\varepsilon\right\}T_g(r)$$

$$i.e.,\ r\leq T^{-1}_{P_0[g]}\left[\left\{\gamma_{P_0[g]}+\varepsilon\right\}T_g(r)\right]$$

(4) $$i.e.,\ T^{-1}_g\left(\frac{r}{\gamma_{P_0[g]}+\varepsilon}\right)\leq T^{-1}_{P_0[g]}(r).$$

Now, from (1) and (3), it follows for all sufficiently large values of r,

$$T^{-1}_{P_0[g]}T_{P_0[f]}(r)\ \leq\ T^{-1}_{P_0[g]}\left[\left\{\gamma_{P_0[f]}+\varepsilon\right\}T_f(r)\right]$$

(5) $$i.e.,\ T^{-1}_{P_0[g]}T_{P_0[f]}(r)\ \leq\ T^{-1}_g\left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\gamma_{P_0[g]}-\varepsilon}\right)T_f(r)\right].$$

Again from (2) and (4) it follows for all sufficiently large values of r,

$$T^{-1}_{P_0[g]}T_{P_0[f]}(r) \geq T^{-1}_{P_0[g]}\left[\left\{\gamma_{P_0[f]}-\varepsilon\right\}T_f(r)\right]$$

$$\text{(6)} \qquad i.e.,\ T^{-1}_{P_0[g]}T_{P_0[f]}(r) \geq T^{-1}_g\left[\left(\frac{\gamma_{P_0[f]}-\varepsilon}{\gamma_{P_0[g]}+\varepsilon}\right)T_f(r)\right].$$

Now, for the definition of type and lower type we get for all sufficiently large values of r that

$$T_g\left(\left\{\frac{T_f(r)}{(\sigma_g+\varepsilon)}\right\}^{\frac{1}{\rho_g}}\right) \leq T_f(r)$$

$$\text{(7)} \qquad i.e., \quad T^{-1}_g T_f(r) \geq \left\{\frac{T_f(r)}{(\sigma_g+\varepsilon)}\right\}^{\frac{1}{\rho_g}}$$

and

$$T_g\left(\left\{\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\left(\gamma_{P_0[g]}-\varepsilon\right)\left(\bar{\sigma}_g-\varepsilon\right)}\right)T_f(r)\right\}^{\frac{1}{\rho_g}}\right) \geq \left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\gamma_{P_0[g]}-\varepsilon}\right)T_f(r)\right]$$

$$\text{(8)} \quad i.e., \ \left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\left(\gamma_{P_0[g]}-\varepsilon\right)\left(\bar{\sigma}_g-\varepsilon\right)}\right)T_f(r)\right]^{\frac{1}{\rho_g}} \geq T^{-1}_g\left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\gamma_{P_0[g]}-\varepsilon}\right)T_f(r)\right].$$

Therefore, from (5) and (8), it follows for all sufficiently large values of r that

$$\text{(9)} \qquad T^{-1}_{P_0[g]}T_{P_0[f]}(r) \leq \left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\left(\gamma_{P_0[g]}-\varepsilon\right)\left(\bar{\sigma}_g-\varepsilon\right)}\right)T_f(r)\right]^{\frac{1}{\rho_g}}.$$

Therefore, from (7) and (9), it follows for all sufficiently large values of r that

$$\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_g T_f(r)} \leq \frac{\left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\left(\gamma_{P_0[g]}-\varepsilon\right)\left(\bar{\sigma}_g-\varepsilon\right)}\right)T_f(r)\right]^{\frac{1}{\rho_g}}}{\left\{\frac{T_f(r)}{(\sigma_g+\varepsilon)}\right\}^{\frac{1}{\rho_g}}}$$

$$i.e.,\ \frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_g T_f(r)} \leq \left(\frac{\left(\gamma_{P_0[f]}+\varepsilon\right)(\sigma_g+\varepsilon)}{\left(\gamma_{P_0[g]}-\varepsilon\right)\left(\bar{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\rho_g}}$$

$$\text{(10)} \qquad i.e.,\ \limsup_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_g T_f(r)} \leq \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}.\left(\frac{\sigma_g}{\bar{\sigma}_g}\right)^{\frac{1}{\rho_g}}.$$

Similarly, from (6), it can be shown for all sufficiently large values of r,

$$\liminf_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_g T_f(r)} \geq \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\overline{\sigma}_g}{\sigma_g}\right)^{\frac{1}{\rho_g}}. \tag{11}$$

Therefore, from (10) and (11), we obtain that

$$\begin{aligned}\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\overline{\sigma}_g}{\sigma_g}\right)^{\frac{1}{\rho_g}} &\leq \liminf_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_g T_f(r)}\\ &\leq \limsup_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_g T_f(r)}\\ &\leq \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\sigma_g}{\overline{\sigma}_g}\right)^{\frac{1}{\rho_g}}.\end{aligned} \tag{12}$$

Similarly, using the weak type, one can easily verify that

$$\begin{aligned}\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\tau_g}{\overline{\tau}_g}\right)^{\frac{1}{\lambda_g}} &\leq \liminf_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_g T_f(r)}\\ &\leq \limsup_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_g T_f(r)}\\ &\leq \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\overline{\tau}_g}{\tau_g}\right)^{\frac{1}{\lambda_g}}.\end{aligned} \tag{13}$$

Thus the theorem follows from (12) and (13). ∎

Theorem 2. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\sum_{a\neq\infty}\Theta(a;f)=2$ and g be an entire function with $0<\tau_g\leq\overline{\tau}_g<\infty$ and $0<\overline{\sigma}_g\leq\sigma_g<\infty$. Also let $\sum_{a\neq\infty}\Theta(a;g)=2$. Then*

$$\begin{aligned}&\max\left\{\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\tau_g}{\overline{\tau}_g}\right)^{\frac{1}{\lambda_g}},\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\overline{\sigma}_g}{\sigma_g}\right)^{\frac{1}{\rho_g}}\right\}\\ &\leq \liminf_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_g T_f(r)} \leq \limsup_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T^{-1}_g T_f(r)}\\ &\leq \min\left\{\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\overline{\tau}_g}{\tau_g}\right)^{\frac{1}{\lambda_g}},\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\sigma_g}{\overline{\sigma}_g}\right)^{\frac{1}{\rho_g}}\right\},\end{aligned}$$

where $P_0[f]$ and $P_0[g]$ are admissible.

With the help of Lemma 2, Theorem 2 can be carried out in the line of Theorem 1. So, the proof is omitted.

Corollary 1. *Under the same conditions of Theorem* 1*, if* g *is of regular growth, then by Lemma* 3 *one can easily verify that*

$$\lim_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T_g^{-1}T_f(r)}=\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}.$$

Corollary 2. *Under the same conditions of Theorem* 2*, if* g *is of regular growth, then by Lemma* 3 *one can also verify that*

$$\lim_{r\to\infty}\frac{T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{T_g^{-1}T_f(r)}=\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}.$$

Theorem 3. *Let* f *be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ *or* $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$ *and* g *be an entire function with* $0<\lambda_g\leq\rho_g<\infty$*. Also let* $\Theta(\infty;g)=\sum\limits_{a\neq\infty}\delta_p(a;g)=1$ *or* $\delta(\infty;g)=\sum\limits_{a\neq\infty}\delta(a;g)=1$*. Then, for homogeneous* $P_0[f]$ *and* $P_0[g]$*,*

$$\frac{\lambda_g}{\rho_g}\leq\liminf_{r\to\infty}\frac{\log T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{\log T_g^{-1}T_f(r)}\leq\limsup_{r\to\infty}\frac{\log T^{-1}_{P_0[g]}T_{P_0[f]}(r)}{\log T_g^{-1}T_f(r)}\leq\frac{\rho_g}{\lambda_g}.$$

Proof. From (5) and (6), we get for all sufficiently large values of r that

$$\log T^{-1}_{P_0[g]}T_{P_0[f]}(r)\leq\log T_g^{-1}\left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\gamma_{P_0[g]}-\varepsilon}\right)T_f(r)\right]\tag{14}$$

and

$$\log T^{-1}_{P_0[g]}T_{P_0[f]}(r)\geq\log T_g^{-1}\left[\left(\frac{\gamma_{P_0[f]}-\varepsilon}{\gamma_{P_0[g]}+\varepsilon}\right)T_f(r)\right].\tag{15}$$

Now for the definition of order and lower order we get for all sufficiently large values of r that

$$T_g\left(\{T_f(r)\}^{\frac{1}{\rho_g+\varepsilon}}\right)\leq T_f(r)$$

$$i.e.,\quad \log T_g^{-1}T_f(r)\geq\frac{1}{(\rho_g+\varepsilon)}\log T_f(r).\tag{16}$$

and

$$T_g\left(\left\{\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\left(\gamma_{P_0[g]}-\varepsilon\right)\left(\bar{\sigma}_g-\varepsilon\right)}\right)T_f(r)\right\}^{\frac{1}{\lambda_g-\varepsilon}}\right)\geq\left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\gamma_{P_0[g]}-\varepsilon}\right)T_f(r)\right]$$

$$i.e., \left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\left(\gamma_{P_0[g]}-\varepsilon\right)\left(\bar{\sigma}_g-\varepsilon\right)}\right)T_f\left(r\right)\right]^{\frac{1}{\lambda_g-\varepsilon}} \geq T_g^{-1}\left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\gamma_{P_0[g]}-\varepsilon}\right)T_f\left(r\right)\right]$$

$$(17) \qquad i.e., \ \frac{1}{(\lambda_g-\varepsilon)}\log T_f\left(r\right)+O(1) \geq \log T_g^{-1}\left[\left(\frac{\gamma_{P_0[f]}+\varepsilon}{\gamma_{P_0[g]}-\varepsilon}\right)T_f\left(r\right)\right].$$

Therefore from (14) and (17) it follows for all sufficiently large values of r that

$$(18) \qquad \log T_{P_0[g]}^{-1}T_{P_0[f]}\left(r\right) \leq \frac{1}{(\lambda_g-\varepsilon)}\log T_f\left(r\right)+O(1).$$

Therefore from (16) and (18) it follows for all sufficiently large values of r that

$$\frac{\log T_{P_0[g]}^{-1}T_{P_0[f]}\left(r\right)}{\log T_g^{-1}T_f\left(r\right)} \leq \left(\frac{\rho_g+\varepsilon}{\lambda_g-\varepsilon}\right).\frac{\log T_f\left(r\right)+O(1)}{\log T_f\left(r\right)}$$

$$(19) \qquad i.e., \ \limsup_{r\to\infty}\frac{\log T_{P_0[g]}^{-1}T_{P_0[f]}\left(r\right)}{\log T_g^{-1}T_f\left(r\right)} \leq \frac{\rho_g}{\lambda_g}.$$

Similarly from (15) it can be shown for all sufficiently large values of r that

$$(20) \qquad \liminf_{r\to\infty}\frac{\log T_{P_0[g]}^{-1}T_{P_0[f]}\left(r\right)}{\log T_g^{-1}T_f\left(r\right)} \geq \frac{\lambda_g}{\rho_g}.$$

Therefore from (19) and (20) we obtain that

$$\frac{\lambda_g}{\rho_g} \leq \liminf_{r\to\infty}\frac{\log T_{P_0[g]}^{-1}T_{P_0[f]}\left(r\right)}{\log T_g^{-1}T_f\left(r\right)} \leq \limsup_{r\to\infty}\frac{\log T_{P_0[g]}^{-1}T_{P_0[f]}\left(r\right)}{\log T_g^{-1}T_f\left(r\right)} \leq \frac{\rho_g}{\lambda_g}.$$

Thus the theorem follows from above. ■

Remark 1. The conclusion of Theorem 3 can also drawn under the hypothesis $\sum_{a\neq\infty}\Theta\left(a;f\right)=2$ and $\sum_{a\neq\infty}\Theta\left(a;g\right)=2$ instead of "$\Theta\left(\infty;f\right)=\sum_{a\neq\infty}\delta_p\left(a;f\right)=1$ or $\delta\left(\infty;f\right)=\sum_{a\neq\infty}\delta\left(a;f\right)=1$" and "$\Theta\left(\infty;g\right)=\sum_{a\neq\infty}\delta_p\left(a;g\right)=1$ or $\delta\left(\infty;g\right)=\sum_{a\neq\infty}\delta\left(a;g\right)=1$" where $P_0\left[f\right]$ and $P_0\left[g\right]$ are admissible.

Corollary 3. *Under the same conditions of Theorem* 3 *and Remark* 1 *if* g *is of regular growth then one may get that*

$$\lim_{r\to\infty}\frac{\log T_{P_0[g]}^{-1}T_{P_0[f]}\left(r\right)}{\log T_g^{-1}T_f\left(r\right)}=1.$$

Theorem 4. *If f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and g be an entire function of regular growth having non zero finite order and $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$. Then the relative order and relative lower order of $P_0[f]$ with respect to $P_0[g]$ are same as those of f with respect to g where $P_0[f]$ and $P_0[g]$ are homogeneous.*

Proof. By Corollary 3, we obtain that

$$\begin{aligned}
\rho_{P_0[g]}(P_0[f]) &= \limsup_{r\to\infty} \frac{\log T^{-1}_{P_0[g]} T_{P_0[f]}(r)}{\log r} \\
&= \limsup_{r\to\infty} \frac{\log T^{-1}_{g} T_{f}(r)}{\log r} \cdot \lim_{r\to\infty} \frac{\log T^{-1}_{P_0[g]} T_{P_0[f]}(r)}{\log T^{-1}_{g} T_{f}(r)} \\
&= \rho_g(f).1 = \rho_g(f).
\end{aligned}$$

In a similar manner,

$$\lambda_{P_0[g]}(P_0[f]) = \lambda_g(f).$$

Thus, the theorem follows. ∎

Theorem 5. *If f be a meromorphic function either of finite order or of non-zero lower order such that $\sum_{a\neq\infty} \Theta(a; f) = 2$ and g be an entire function of regular growth having non zero finite order and $\sum_{a\neq\infty} \Theta(a; g) = 2$. Then the relative order and relative lower order of $P_0[f]$ with respect to $P_0[g]$ are same as those of f with respect to g where $P_0[f]$ and $P_0[g]$ are admissible.*

We omit the proof of Theorem 5 because it can be carried out in the line of Theorem 4 and with the help of Corollary 3.

Theorem 6. *If f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and g be an entire function of regular growth having non zero finite type and $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$. Then the relative type and relative lower type of $P_0[f]$ with respect to $P_0[g]$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_g(f)$ is positive finite and $P_0[f]$ and $P_0[g]$ are homogeneous.*

Proof. From Corollary 1 and Theorem 4, we get that

$$\begin{aligned}
\sigma_{P_0[g]}(P_0[f]) &= \limsup_{r\to\infty} \frac{T^{-1}_{P_0[g]} T_{P_0[f]}(r)}{r^{\rho_{P_0[g]}(P_0[f])}} \\
&= \lim_{r\to\infty} \frac{T^{-1}_{P_0[g]} T_{P_0[f]}(r)}{T^{-1}_{g} T_{f}(r)} . \limsup_{r\to\infty} \frac{T^{-1}_{g} T_{f}(r)}{r^{\rho_g(f)}} = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}} \sigma_g(f).
\end{aligned}$$

Similarly,

$$\overline{\sigma}_{P_0[g]}\left(P_0\left[f\right]\right)=\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}.\overline{\sigma}_g\left(f\right).$$

This proves the theorem. ∎

Theorem 7. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta\left(\infty;f\right)=\sum\limits_{a\neq\infty}\delta_p\left(a;f\right)=1$ or $\delta\left(\infty;f\right)=\sum\limits_{a\neq\infty}\delta\left(a;f\right)=1$ and g be an entire function of regular growth having non zero finite type and $\Theta\left(\infty;g\right)=\sum\limits_{a\neq\infty}\delta_p\left(a;g\right)=1$ or $\delta\left(\infty;g\right)=\sum\limits_{a\neq\infty}\delta\left(a;g\right)=1$. Then $\tau_{P_0[g]}\left(P_0\left[f\right]\right)$ and $\overline{\tau}_{P_0[g]}\left(P_0\left[f\right]\right)$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g, i.e., $\tau_{P_0[g]}\left(P_0\left[f\right]\right)=\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}.\tau_g\left(f\right)$ and $\overline{\tau}_{P_0[g]}\left(P_0\left[f\right]\right)=\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}.\overline{\tau}_g\left(f\right)$ when $\lambda_g\left(f\right)$ is positive finite and $P_0\left[f\right]$ and $P_0\left[g\right]$ are homogeneous.*

We omit the proof of Theorem 7 because it can be carried out in the line of Theorem 6 and with the help of Theorem 5 and Corollary 2.

In a similar manner, we can state the following two theorem without proof:

Theorem 8. *If f be a meromorphic function either of finite order or of non-zero lower order such that $\sum\limits_{a\neq\infty}\Theta\left(a;f\right)=2$ and g be an entire function of regular growth having non zero finite type and $\sum\limits_{a\neq\infty}\Theta\left(a;g\right)=2$, then the relative type and relative lower type of $P_0\left[f\right]$ with respect to $P_0\left[g\right]$ are $\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_g\left(f\right)$ is positive finite and $P_0\left[f\right]$ and $P_0\left[g\right]$ are admissible.*

Theorem 9. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\sum\limits_{a\neq\infty}\Theta\left(a;f\right)=2$ and g be an entire function of regular growth having non zero finite type and $\sum\limits_{a\neq\infty}\Theta\left(a;g\right)=2$. Then $\tau_{P_0[g]}\left(P_0\left[f\right]\right)$ and $\overline{\tau}_{P_0[g]}\left(P_0\left[f\right]\right)$ are $\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g i.e., $\tau_{P_0[g]}\left(P_0\left[f\right]\right)=\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}.\tau_g\left(f\right)$ and $\overline{\tau}_{P_0[g]}\left(P_0\left[f\right]\right)=\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}.\overline{\tau}_g\left(f\right)$ when $\lambda_g\left(f\right)$ is positive finite and $P_0\left[f\right]$ and $P_0\left[g\right]$ are homogeneous.*

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