ON THOMPSON'S CONJECTURE FOR Aut(J_2) AND Aut(McL)

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Abstract. Let G be a finite group and $N(G)$ be the set of conjugacy class sizes of G. In 1980s J. G. Thompson conjectured: If G is a finite group with trivial center and S is a non-abelian finite simple group satisfying that $N(G) = N(S)$, then $G \cong S$. Here, we generalize the conjecture to the automorphism groups of J_2 and McL. As a corollary this result extends the conjecture to all almost sporadic simple groups.

Keywords: finite group, almost sporadic simple groups, conjugacy class sizes, Thompson's conjecture.

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1. Introduction

All groups considered in this paper are finite and simple groups mentioned are non-abelian.

Let G be a group. For an element $x \in G$, we denote by x^G the conjugacy class of x in G and $N(G)$ the set of conjugacy class sizes of G, that is, $N(G) = \{|x^G| \mid x \in G\}$. In 1987, J.G. Thompson posed the following conjecture (ref. to [9], Problem 12.38):

Thompson's conjecture. *Let G be a group with $Z(G) = 1$ and S is a simple group satisfying that $N(G) = N(S)$, then $G \cong S$.*

The *prime graph* of a group G is defined as follows, whose vertices are the prime divisors of $|G|$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq. J.S. Williams in [17] and A.S. Kondratiev in [18] obtained the classification of simple groups with disconnected prime graph. Based on these results, in 1994 G.Y. Chen proved that Thompson's conjecture is true for all finite simple groups with disconnected prime graph in [1], parts of the proof were given in [2], [3], [4]. The method used by G.Y. Chen requires a group with non-connected prime graph and several authors have worked on such groups in the same way as Chen.

For the simple groups with connected prime graph, there is not any progress on Thompson's conjecture for a long time. Until 2009, A.V. Vasil'ev first dealt with the groups with connected prime graph and proved that Thompson's conjecture holds for A_{10} and $L_4(4)$ (see [5]). Jiang Qinhui et al in [7] and Gerald Pientka in [6] also independently gave a positive answer to Thompson's conjecture for A_{10} in the year of 2011 and 2012, respectively. Note that Gerald Pientka's work does not use the classification theorem of simple groups. In 2011, N. Ahanjideh in [8] proved that Thompson's conjecture is true for some projective special linear groups $PSL_n(q)$ which have connected prime graphs. Recently, I.B. Gorshkov in [9] established the validity of Thompson's conjecture for $U_4(4)$, $U_4(5)$, $S_6(4)$, $O_8^+(4)$, and A_{16}.

It is worth to mention that some authors generalize Thompson's conjecture to some almost simple groups, that is, a group M is said to be an almost simple related to S if and only if $S \leq M \leq \mathrm{Aut}(S)$ for some simple group S. For example, in 2002 A. Khosravi and B. Khosravi in [11] generalized Thompson's conjecture to almost sporadic simple groups except $\mathrm{Aut}(J_2)$ and $\mathrm{Aut}(McL)$. In 2005, S.H. Alavi and A. Daneshkhah in [16] generalized Thompson's conjecture to symmetric groups S_n, where $n = p, p+1$, and p is an odd prime number. In 2011, B. Khosravi and M. Khatami in [12], [13] established the validity of Thompson's conjecture for the general projective groups $PGL(2, q)$. It is pity that they still need to assume that the groups discussed have the non-connected prime graphs. Surely, it is an interesting topic to check Thompson's conjecture for some almost simple groups which have the connected prime graphs. In this paper, we give a positive answer to Thompson's conjecture for $\mathrm{Aut}(J_2)$ and $\mathrm{Aut}(McL)$, both of which have the connected prime graphs. Further more, we get that Thompson's conjecture can be generalized to all the almost sporadic simple groups.

Our main contribution is the following theorem and we will give the proofs in Sections 3 and 4.

Main Theorem. *Let G be a group with $Z(G) = 1$ and M one of $Aut(J_2)$ and $Aut(McL)$ satisfying that $N(G) = N(M)$, then $G \cong M$.*

Let M be an almost simple group related to sporadic simple group S. Then, by [14], we have that $|\mathrm{Aut}(S) : S| \leq 2$ such that $M = S$ or $\mathrm{Aut}(S)$. G.Y. Chen proved Thompson's conjecture for all sporadic simple groups in [2]. A. Khosravi and B. Khosravi in [11] proved Thompson's conjecture for other almost sporadic simple groups except Aut(J_2) and Aut(McL). Therefore, by our main conclusion the following corollary holds.

Corollary. *Thompson's conjecture holds for all the almost sporadic simple groups.*

For a group G, we denote by $\pi(G)$ the set of prime divisors of $|G|$ and denote by $Soc(G)$ the socle of G which is a subgroup generated by all minimal normal subgroups of G. In addition, we also denote by $\pi(n)$ the set of all primes dividing n where n is a positive integer, and then n_π to denote π-part of n for $\pi \subseteq \pi(n)$. The other notation and terminologies in this paper are standard and the reader is referred to [14] and [19] if necessary.

2. Preliminaries

First, we cite here some known results which are useful in the sequel.

Lemma 2.1. ([5], Lemma 5, and [9], Lemma 1.4]) *Let K be a normal subgroup of G and $\overline{G} = G/K$. Then*

(1) *If $\overline{x}$ is the image of an element x of G in the group $\overline{G}$, then $|x^K| \big| |x^G|$ and $|\overline{x}^{\overline{G}}| \big| |x^G|$.*

(2) *If $x \in G$ and $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.*

(3) *If $x,\ y \in G$, $(|x|, |y|) = 1$, and $xy = yx$, then $C_G(xy) = C_G(x) \cap C_G(y)$.*

Lemma 2.2. ([5], Lemma 4) *Let G be a group with trivial center, $p \in \pi(G)$ and p^2 not divide n for any $n \in N(G)$. Then a Sylow p-subgroup of G is elementary abelian.*

Lemma 2.3. ([9], Lemma 1.10) *Let a Sylow p-subgroup of G be of order p, x be an element of order p, and $|x^G|$ be a number that is maximal with respect to divisibility in $N(G)$. Then $C_G(x)$ is an abelian group.*

Lemma 2.4. ([9], Lemma 1.9) *Let G be a group, and p and q be numbers in $\pi(G)$. If G satisfies the following conditions:*

(a) *$N(G)$ contains no number divisible by p^2 or q^2;*

(b) *$N(G)$ contains no number except 1 co-prime to pq;*

(c) *$N(G)$ contains a number h_q such that any n in $N(G)$ not divisible by q does not divide h_q and $N(G)$ contains no number divisible by h_q and n;*

(d) *$N(G)$ contains a number h_p such that any l in $N(G)$ not divisible by p does not divide h_p and $N(G)$ contains no number divisible by h_p and l.*

Then, the Sylow p-subgroups and q-subgroups of G are cyclic groups of prime order. In addition, G has no element of order pq.

Lemma 2.5. ([9], Lemma 1.11) *Let G be a group such that $\pi(G)$ has no number greater than 17, and there are numbers p and q in $\pi(G)$ such that:*

(a) *p and q are nonadjacent in the prime graph of G;*

(b) *Sylow p-subgroups and q-subgroups of G have prime orders;*

(c) *$q-1$ is not divisible by p, while $p-1$ is not divisible by q ;*

(d) *the centralizer of any element of order p or q is abelian;*

(e) *p and q are greater than 5.*

Then G is insoluble and possesses a unique non-abelian composition factor S, whose order is divisible by pq. Moreover, If K is the soluble radical of G, then $S \leq G/K \leq \mathrm{Aut}(S)$.

Lemma 2.6. ([9], Lemma 1.12) *Let G be a group, K the soluble radical of G, and $G/K = S$ a simple non-abelian group. Suppose that there exists a prime p such that $p \in \pi(G) \setminus \pi(K)$. Assume that an element g of order p of G satisfies the following conditions:*

(a) *$|g^G| = |\overline{g}^S|$, where $\overline{g}$ is the image of an element g in the group S;*

(b) *the number $|g^G|$ is maximal with respect to divisibility in $N(G)$;*

(c) *the subgroup $C_G(g)$ is abelian.*

Then $K \leq Z(G)$.

By [14], we know that $|\mathrm{Out}(J_2)| = |\mathrm{Out}(McL)| = 2$ such that M is isomorphic to $\mathrm{Aut}(J_2) = J_2 : 2$ or $\mathrm{Aut}(McL) = McL : 2$. Information on the set $N(M)$ and the order of M are given in the next two lemmas is obtained via [14].

Lemma 2.7. *Let $M \cong J_2 : 2$. Then*

(1) $|M| = 2^8 \cdot 3^3 \cdot 5^2 \cdot 7$;

(2) $N(M) = \{n_1 = 1,\ n_2 = 3^2 \cdot 5 \cdot 7,\ n_3 = 2^3 \cdot 3^2 \cdot 5 \cdot 7,\ n_4 = 2^4 \cdot 5 \cdot 7,\ n_5 = 2^5 \cdot 3 \cdot 5^2 \cdot 7,\ n_6 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7,\ n_7 = 2^6 \cdot 3^2 \cdot 7,\ n_8 = 2^7 \cdot 3^3 \cdot 7,\ n_9 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7,\ n_{10} = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7,\ n_{11} = 2^7 \cdot 3^3 \cdot 5^2,\ n_{12} = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7,\ n_{13} = 2^6 \cdot 3^3 \cdot 5 \cdot 7,\ n_{14} = 2^7 \cdot 3^3 \cdot 5 \cdot 7,\ n_{15} = 2^8 \cdot 3^2 \cdot 5 \cdot 7,\ n_{16} = 2^3 \cdot 3^2 \cdot 5^2,\ n_{17} = 2^3 \cdot 3^2 \cdot 5^2 \cdot 7,\ n_{18} = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7,\ n_{19} = 2^3 \cdot 3^3 \cdot 5^2 \cdot 7\}$.

Especially,

(3) *$N(M)$ contains no number other than n_{11} and n_{16} not divisible by 7;*

(4) *$N(M)$ contains no number divided by 7^2;*

(5) *n_4 is only one in $N(M)$ not divisible by 3.*

Lemma 2.8. *Let $M \cong McL : 2$. Then*

(1) $|M| = 2^8 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$;

(2) $N(M) = \{n_1 = 1,\ n_2 = 3^4 \cdot 5^2 \cdot 11,\ n_3 = 2^4 \cdot 5^2 \cdot 7 \cdot 11,\ n_4 = 2^5 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11,\ n_5 = 2^2 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11,\ n_6 = 2^6 \cdot 3^5 \cdot 7 \cdot 11,\ n_7 = 2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 11,\ n_8 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11,\ n_9 = 2^5 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11,\ n_{10} = 2^7 \cdot 3^6 \cdot 5^3 \cdot 11,\ n_{11} = 2^4 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11,\ n_{12} = 2^8 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11,\ n_{13} = 2^6 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11,\ n_{14} = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7,\ n_{15} = 2^5 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11,\ n_{16} = 2^7 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11,\ n_{17} = 2^3 \cdot 3^4 \cdot 5^2 \cdot 7,\ n_{18} = 2^3 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11,\ n_{19} = 2^6 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11,\ n_{20} = 2^3 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11,\ n_{21} = 2^3 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11,\ n_{22} = 2^7 \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11,\ n_{23} = 2^6 \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11\}$.

In particular,

(3) *$N(M)$ contains no number other than n_2 and n_{10} not divisible by 7;*

(4) *$N(M)$ contains no number other than n_{14} and n_{17} not divisible by 11;*

(5) *For any $n \in N(M)$ and $p \in \{7,\ 11\}$, it follows that $p^2 \nmid n$;*

(6) *For any $n \in N(M)$, either 7 or 11 divides n.*

Lemma 2.9. *Let M be one of $J_2 : 2$ and $McL : 2$, and G be a group with trivial center. If $N(G) = N(M)$, then $|M|\,\big|\,|G|$ and $\pi(M) = \pi(G)$.*

Proof. Since the number in $N(G)$ divides $|G|$, under the hypothesis we see that $|M|\,\big|\,|G|$ by Lemmas 2.7 and 2.8. $\pi(M) = \pi(G)$ is the result of Lemma 1.2.1 in [1] or Lemma 3 in [5].

Let S be a simple group with $\pi(S) \subseteq \{2, 3, 5, 7, 11\}$. Up to isomorphism, there are finitely many finite non-abelian simple groups S with $\pi(S) \subseteq \{2, 3, 5, 7, 11\}$. Using the classification of finite simple groups one can list them all. For convenience, we list all the cases of S as well as the orders of S, the orders of the outer automorphism of S in Table 1. Especially, we have that $\pi(\mathrm{Out}(S)) \subseteq \{2,\ 3\}$.

3. Proof of Main Theorem for $J_2 : 2$

In this section, we prove the Main Theorem for $J_2 : 2$ according to the information in Lemma 2.7.

Theorem 3.1. *Let $M \cong J_2 : 2$ and G be a group with trivial center. If $N(G) = N(M)$, then $G \cong M$.*

Proof. We divide the proof of this theorem into eight steps.

Table 1. Non-abelian simple groups S with $\pi(S) \subseteq \{2, 3, 5, 7, 11\}$

S	Order of S	$\lvert\mathrm{Out}(S)\rvert$	S	Order of S	$\lvert\mathrm{Out}(S)\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	2	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	$\lvert S_3\rvert$
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	$\lvert D_8\rvert$
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\lvert D_{12}\rvert$	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	$\lvert S_3\rvert$
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	HS	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11$	2
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2	$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	$\lvert S_3\rvert$

Step 1. The Sylow 7-subgroup P of G is order of 7.

Using Lemma 2.2 and (2) of Lemma 2.7, we derive that P is elementary abelian. Assume that 7^2 divides G. Since $N(G) = N(M)$, the centralizer of every element of G contains an element of order 7 by (4) of Lemma 2.7. Consider an element y of G such that $|y^G| = n_8 = 2^7 \cdot 3^3 \cdot 7$.

Suppose that 7 does not divide $|y|$. Let x be an element of order 7 in $C_G(y)$. Then by (3) of Lemma 2.1, $C_G(xy) = C_G(x) \cap C_G(y)$, and so $lcm(|x^G|, |y^G|)$ divides $|(xy)^G|$. Since P is abelian, $C_G(x)$ includes P up to conjugacy. Hence 7 does not divide $|x^G|$. It follows that $|x^G|$ is equal to $n_{11} = 2^7 \cdot 3^3 \cdot 5^2$ or $n_{16} = 2^3 \cdot 3^2 \cdot 5^2$. In both cases, $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ divides $|(xy)^G|$, which is impossible by (2) of Lemma 2.7.

Suppose that 7 divides $|y|$. Let $|y| = 7t$. Since P is elementary abelian, one has that $gcd(7,\ t) = 1$. Put $u = y^7$ and $v = y^t$. Then $y = uv$, and so $C_G(uv) = C_G(u) \cap C_G(v)$ by (3) of Lemma 2.1. Therefore $|v^G|$ divides $|y^G| = n_8 = 2^7 \cdot 3^3 \cdot 7$. On the other hand, the element v is order of 7, and thus $|v^G|$ is equal to $n_{11} = 2^7 \cdot 3^3 \cdot 5^2$ or $n_{16} = 2^3 \cdot 3^2 \cdot 5^2$, a contradiction.

Hence P has order of 7.

Step 2. For an element $x \in G$ of order 7, it follows that $|x^G| = n_{11} = 2^7 \cdot 3^3 \cdot 5^2$ and $C_G(x)$ is abelian.

For any $1 \neq y \in C_G(x)$, since $7 \parallel |G|$ by Step 1, one has that $7 \nmid |y^G|$, and hence $|y^G| = n_{11}$ or n_{16} by (2) of Lemma 2.7. Assume that $|x^G| = n_{16} = 2^3 \cdot 3^2 \cdot 5^2$ and let H be a Sylow 3-subgroup of $C_G(x)$. Then H is a nontrivial group of order $|G|_3/3^2$ by Lemma 2.9. Hence there exists a 3-subgroup K of G such that H is a normal subgroup of K and $|K/H| = 3$. Then $1 \neq H \cap Z(K) \leq C_G(x)$. Take $1 \neq h \in H \cap Z(K)$, we have that $K \leq C_G(h)$, and so $|h^G|_3 \leq 3$. But $|h^G| = n_{11}$ or n_{16}, a contradiction. It follows that $|x^G| = n_{11} = 2^7 \cdot 3^3 \cdot 5^2$.

Since n_{11} is maximal with respect to divisibility in $N(G)$, Lemma 2.3 implies that the group $C_G(x)$ is abelian.

Step 3. Suppose that $q \in \{2,3,5\}$, Q is a Sylow q-subgroup of G, and $Z(Q)$ is its center. Let $x \in Z(Q)$, then $7 \nmid |C_G(x)|$.

Let $x \in Z(Q)$. Then q does not divide $|x^G|$, and so by Lemma 2.7 we have that $|x^G| = n_2 = 3^2 \cdot 5 \cdot 7$ while $q = 2$, $|x^G| = n_4 = 2^4 \cdot 5 \cdot 7$ while $q = 3$, and $|x^G|$ is equal to $n_7 = 2^6 \cdot 3^2 \cdot 7$ or $n_8 = 2^7 \cdot 3^3 \cdot 7$ while $q = 5$. The Step 3 follows.

Step 4. G is non-soluble and $O_{2,\ 2'}(G) = O_2(G)$.

Let $K = O_2(G)$, $\overline{G} = G/K$, and denote by $\overline{x}$ the images of an element x of G in $\overline{G}$. If the statement is false, then there exists $r \in \{3,5,7\}$ such that $O_r(\overline{G}) \neq 1$.

If $O_7(\overline{G}) \neq 1$, then $|O_7(\overline{G})| = 7$ by Step 1. Let y be an element of the center $Z(Q)$ of a Sylow 5-subgroup Q. Obviously, the subgroup $O_7(\overline{G})\langle\overline{y}\rangle$ is cyclic. Hence 7 divides $|C_{\overline{G}}(\overline{y})|$. Since $(5, |K|) = 1$, Lemma 2.1(2) implies that 7 divides $|C_G(y)|$, which is impossible by Step 3. Thus, $O_7(\overline{G}) = 1$.

Let $q \in \{3,\ 5\}$, and Q be a Sylow q-subgroup of G. If $O_q(\overline{G}) \neq 1$, then $V = Z(O_q(\overline{G})$ is a nontrivial normal subgroup of G. If x is an element of order 7 in G, then $V = C_V(\overline{x}) \times [V,\ \overline{x}]$ such that $\overline{x}$ acts fixed-point freely on $[\overline{x}, V]$, and then $|[\overline{x}, V]| - 1$ is divisible by 7. Lemma 2.1 (1) implies that $|\overline{x}^{\overline{G}}|$ is a divisor of $2^7 \cdot 3^3 \cdot 5^2$, and hence $|V : C_V(\overline{x})|$ divides 5^2 or 3^3 so that $|[V,\ \overline{x}]| \leq q^3$. But $n = 6$ is the least number such that 7 divides $q^n - 1$, we have that $[V,\ x] = 1$ and $V = C_V(\overline{x})$. On the other hand, the center $\overline{Z(Q)}$ of a Sylow subgroup Q has a nontrivial intersection with V. Let $\overline{z}$ be of order q from this intersection. Since $(|K|,\ q) = 1$, there exists a pre-image z of $\overline{z}$ in G such that z lies in the center of a Sylow q-subgroup of G by Lemma 2.1 (2). Further, the centralizer of z also contains an element of order 7, which contradicts Step 3. Therefore, $O_q(\overline{G}) = 1$. The Step 4 follows.

Step 5. Let $K = O_2(G)$, $\overline{G} = G/K$. Then, every minimal normal subgroup of $\overline{G}$ is non-soluble. Especially, $Soc(\overline{G}) \trianglelefteq \overline{G} \lesssim \mathrm{Aut}(Soc(\overline{G}))$.

Let N be any minimal normal subgroup of $\overline{G}$ and assume that N is solvable. Then N is an elementary abelian p-group for some $p \in \{3,5,7\}$, and so $N \leq O_p(\overline{G})$. It follows that $O_p(\overline{G})$ is nontrivial, contradicts Step 4. Hence a very minimal normal subgroup of $\overline{G}$ is non-soluble. Let $N_1, N_2, ..., N_s$ be all minimal normal subgroups of $\overline{G}$, where s is a positive integer. Then $Soc(\overline{G}) = N_1 \times N_2 \times \cdots \times N_s$. We assert that $C_{\overline{G}}(Soc(\overline{G})) = 1$. Otherwise, there exists a minimal normal subgroup N of $\overline{G}$ so that $N \leq C_{\overline{G}}(Soc(\overline{G})) \cap Soc(\overline{G})$. Thus N is an abelian group, a contradiction. By N/C theorem, we have $Soc(\overline{G}) \trianglelefteq \overline{G} = \overline{G}/C_{\overline{G}}(Soc(\overline{G})) \lesssim \mathrm{Aut}(Soc(\overline{G}))$, as desired.

Step 6. Let $L = Soc(\overline{G})$. Then L is a non-abelian simple group and $7 \big| |L|$.

By Step 5, we can have that $L = S_1 \times S_2 \times \cdots \times S_k$ is a direct product of non-abelian simple groups of $S_1, S_2, \ldots,$ and S_k. Let g be an element of order 7 of G and suppose that $7 \notin \pi(L)$. Then $\overline{g}$ is of order 7 in $\overline{G}$ and induces a nontrivial outer automorphism of the group L. Suppose that there exists i such that $S_i^g \neq S_i$. Without loss of generality, we assume that $i = 1$. Let $H = \langle s \mid s = s_1 s_1^{\overline{g}} s_1^{\overline{g}^2} \cdots s_1^{\overline{g}^6},\ s_1 \in S_1\rangle$. Then H lies in the centralizer of the element $\overline{g}$ and is isomorphic to S_1, but the centralizer of g is abelian by Step 2, a contradiction. Hence $\overline{g}$ induces a nontrivial outer automorphism of the group S_i such that $7\big||\mathrm{Out}(S_i)|$. In view of $\pi(S_i) \subseteq \pi(G) = \{2, 3, 5, 7\}$ and by Table 1, the prime divisors of $|\mathrm{Out}(S_i)|$ are not greater than 5, a contradiction. Therefore, $7\big||L|$.

If $n > 1$ and $\overline{g} \in S_j$, then $S_i \le C_{\overline{G}}(\overline{g})$ for any $1 \le j \le n,\ j \ne i$. On the other hand, $C_{\overline{G}}(\overline{g})$ is abelian by Step 2, a contradiction. Therefore, $n = 1$, and so L is a non-ableian simple group and $7\big||L|$.

Step 7. $L \cong J_2$.

By Step 6 and Step 1, we have that L is a non-ableian simple group such that $7 \parallel |M|$. Then, by Table 1, L may be isomorphic to one of the following groups:

$$L_2(7),\ L_2(8),\ U_3(3),\ A_7,\ A_8,\ L_3(4),\ U_3(5),\ A_9,\ A_{10},\ U_4(3),\ J_2,\ S_6(2),\ O_8^+(2).$$

Hence, by Table 1 and Step 5, we see that $\pi(\mathrm{Out}(L)) \subseteq \{2,\ 3\}$ and $L \trianglelefteq \overline{G} \lesssim \mathrm{Aut}(L)$. Since $K = O_2(G)$ and $|M|\big||G|$, we have that 5^2 divides $|L|$, which implies that L can be only isomorphic to one of the following groups:

$$U_3(5),\ A_{10},\ J_2,\ O_8^+(2).$$

If $L \cong U_3(5)$ or $O_8^+(2)$, then let x be an element of order 7 in G and $\overline{x}$ be its image in $\overline{G}$, of course $\overline{x} \in L$, $|\overline{x}^{\overline{G}}|$ is a multiple of 5^3 or 3^5, so is $|x^G|$ by (1) of Lemma 2.1. This contradicts (2) of Lemma 2.7.

Assume that $L \cong A_{10}$. Then, there exists an element w of order 5 in L such that $|w^L|$ is a multiple of 3^4, so is $|w^{\overline{G}}|$ by (2) of Lemma 2.1, which implies that $N(G)$ has one number divided by 3^4. This is impossible by (2) of Lemma 2.7. Hence, L must be isomorphic to J_2.

Step 8. $G \cong M$.

Let x be an element of order 7 in G and $\overline{x}$ be its image in $\overline{G}$. It is clear that $\overline{x} \in L$. By Lemma 2.1 and [14], we have that $|\overline{x}^L| = |\overline{x}^{\overline{G}}| = |x^G| = 2^7 \cdot 3^3 \cdot 5^2$ such that $K \le C_G(x)$. If $K \ne 1$, then x centralizes an element from the center of a Sylow 2-subgroup of G, which is impossible by Step 3. Hence $G = \overline{G}$. Recall that $L \trianglelefteq \overline{G} \lesssim \mathrm{Aut}(L)$. Then $G \cong J_2$ or $J_2 : 2$ by Table 1. If $G \cong J_2$, then $|N(G)| = |N(J_2)|$, a contradiction by [14]. Therefore $G \cong J_2 : 2 \cong M$, as claimed.

4. Proof of Main Theorem for $McL : 2$

In this section, we prove the following theorem according to Lemma 2.8.

Theorem 4.1. *Let $M \cong McL : 2$ and G be a group with trivial center. If $N(G) = N(M)$, then $G \cong M$.*

Proof. We divide the proof of this theorem into seven steps.

Step 1. Sylow 7-subgroups and Sylow 11-subgroups of G are cyclic groups of prime order and there are no elements of order 77 in G.

In view of $N(G) = N(M)$ and Lemma 2.9, we can choose $p = 7$ and $q = 11$, and take $h_7 = n_{14}$ and $h_{11} = n_{10}$ such that G meets the hypotheses of Lemma 2.4. Hence Sylow 7-subgroups and Sylow 11-subgroups of G are cyclic groups of prime order and there are no elements of order $7 \cdot 11$ in G.

Step 2. Let $g, h \in G$ be elements of orders 7 and 11, respectively. Then $|g^G| = n_{10} = 2^7 \cdot 3^6 \cdot 5^3 \cdot 11$ and $|h^G| = n_{14} = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7$, and $C_G(g)$ and $C_G(h)$ are abelian.

We only state the case of $p = 7$, and the case of $q = 11$ is similar.

Since the Sylow 7-subgroup of G is order of 7 by Step 1, one has that $7 \nmid |x^G|$ for any $1 \neq x \in C_G(g)$. Hence $|x^G| = n_2$ or n_{10}. Assume that $|g^G| = n_2 = 3^4 \cdot 5^2 \cdot 11$. Let H be a Sylow 5-subgroup of $C_G(g)$. Then H is a nontrivial group of order $|G|_5/5^2$ by Lemma 2.9. Hence there exists a 5-subgroup K of G such that H is a normal subgroup of K and $|K/H| = 5$. Further, $1 \neq H \cap Z(K) \leq C_G(x)$. Taking $1 \neq h \in H \cap Z(K)$, one has that $5^2 \nmid |h^G|_5$ and $7 \nmid |h^G|$, which implies $|h^G| = 1$ by Lemma 2.8, and so $y \in Z(G)$, a contradiction. It follows that $|g^G| = n_{10} = 2^7 \cdot 3^6 \cdot 5^3 \cdot 11$.

Since n_{10} is maximal with respect to divisibility in $N(G)$, Lemma 2.3 implies that the group $C_G(g)$ is abelian.

In the following discussion, we assume that K is the soluble radical of a group G, and $\overline{G} = G/K$.

Step 3. G is non-soluble and has a unique non-abelian composition factor S such that $7 \cdot 11 \big| |S|$ and $S \leq \overline{G} \leq \mathrm{Aut}(S)$. Moreover, S may be isomorphic to one of the following groups:

$$M_{22}, A_{11}, HS, A_{12}, McL, U_6(2).$$

In view of Step 1 and Step 2, we can choose $p = 7$ and $q = 11$ such that G satisfies the hypotheses of Lemma 2.5. Hence G is non-soluble and has a unique non-abelian composition factor S such that $7 \cdot 11 \big| |S|$ and $S \leq \overline{G} \leq \mathrm{Aut}(S)$. Since $\{7, 11\} \subseteq \pi(S) \subseteq \{2, 3, 5, 7, 11\}$, we can easily get that S can be isomorphic to one of the groups: $M_{22}, A_{11}, HS, A_{12}, McL, U_6(2)$ by Table 1.

Step 4. $5^3 \big| |S|$.

Since $N(G)$ contains an integer divisible by 5^3, $|G|$ is divisible by 5^3. Assume $|S|$ is not divisible by 5^3. By Table 1, $|\mathrm{Out}(S)|$ is not divisible by 5. Consequently, $|K|$ is divisible by 5. For $|K|$ is not divisible by 7, there is an element g in G such that $|g| = 7$ and the image $\overline{g}$ of g in the group $\overline{G}$ is nontrivial. By virtue of Step 2, $|g^G|_5 = |n_{10}|_5 = 5^3$, the subgroup K contains an element h of order 5 such that g does not centralize h. Let $R_1 \lhd R_2 \lhd \cdots \lhd R_n \lhd K$ be an un-refinable series of characteristic subgroups in K and l a number such that $h \in R_{l+1} \setminus R_l$. The image $\widetilde{h}$ of h in the group K/R_l is nontrivial and lies in H, which is a normal abelian 5-subgroup of G/R_l. Let $\widetilde{G} = G/R_l$ and $\widetilde{g}$ be the image of g in the group $\widetilde{G}$. Then

$H = C_H(\widetilde{g}) \times [\widetilde{g}, H]$ and g acts fixed-point freely on $[\widetilde{g}, H]$ such that $|[\widetilde{g}, H]| - 1$ is divisible by 7. Thus $|[\widetilde{g}, H]| \geq 5^6$, which implies that

$$5^6 \,\Big|\, \frac{|H|}{|C_H(\widetilde{g})|} \,\Big|\, \frac{|\widetilde{K}|_5}{|C_{\widetilde{K}}(\widetilde{g})|_5}.$$

By Lemma 2.1,

$$C_{\overline{G}}(\overline{g}) = C_{\widetilde{G}}(\widetilde{g})\widetilde{K}/\widetilde{K} = C_{\widetilde{G}}(\widetilde{g})/(C_{\widetilde{G}}(\widetilde{g}) \cap \widetilde{K}) = C_{\widetilde{G}}(\widetilde{g})/C_{\widetilde{K}}(\widetilde{g}),$$

and then

$$|\widetilde{g}^{\widetilde{G}}|_5 = |\overline{G}|_5 |\widetilde{K}|_5/(|C_{\overline{G}}(\overline{g})|_5 |C_{\widetilde{K}}(\widetilde{g})|_5) \geq 5^6,$$

so is $|g^G|_5$. It follows a contradiction by (2) of Lemma 2.8, which implies that $N(G)$ has no number divisible by 5^6. Hence $5^3 \big| |S|$.

Step 5. $S \cong McL$.

By Step 3 and Step 4, we can get that S may be isomorphic to one of groups HS and McL by Table 1. If $M \cong HS$, then let x be an element of order 7 in G and $\overline{x}$ be its image in $\overline{G}$, of course $\overline{x} \in S$, $|\overline{x}^S|$ is a multiple of 2^9, so are $|\overline{x}^{\overline{G}}|$ and $|x^G|$ by Lemma 2.1. This contradicts (2) of Lemma 2.8. Hence $S \cong McL$.

Step 6. $G/K \cong McL : 2$.

By virtue of $S \leq G/K \leq \mathrm{Aut}(S)$ and [14], we have $G/K \cong McL$ or $McL : 2$ and $K \neq 1$. Assume that $G/K \cong McL$. Then, let $g \in G$, $|g| = 7$, and $\overline{g} \in G/K = \overline{G}$ be the image of the element g. In view of Lemma 2.8 (2), Step 2, and [14], the number n_{10} is maximal in $N(G)$ and the number $|g^G| = |\overline{g}^S| = n_{10}$. Thus G satisfies the hypothesis of Lemma 2.6, and so $K \leq Z(G)$, a contradiction to $Z(G) = 1$. Therefore it is impossible that $G/K \cong S \cong McL$, and so G/K must be isomorphic to $McL : 2$.

Step 7. K is a trivial group such that $G \cong M \cong McL : 2$.

Also let $g \in G$, $|g| = 7$, and $\overline{g} \in G/K = \overline{G} \cong M$ be the image of the element g. By Lemma 2.8 (2), Step 2, and [14], the number n_{10} is maximal in $N(G)$ and the number $|g^G| = |\overline{g}^{\overline{G}}| = |\overline{g}^S| = n_{10}$. Thus $K \leq C_G(g)$. If $K \neq 1$ and let $p \big| |K|$, then g centralizes an element from the center of a Sylow p-subgroup of G, which is impossible by Lemma 2.8. Hence K is a trivial group such that $G \cong M \cong McL : 2$.

Proof of Main Theorem. It follows directly from Theorems 3.1 and 4.1.

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