NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE FIRST DERIVATIVES ABSOLUTE VALUES ARE s-CONVEX

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Abstract. In this paper, some new inequalities of the left-hand side of Hermite-Hadamard-type are obtained for functions whose first derivatives absolute values are s-convex.

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1. Introduction

If $f : I \subset R_+ \to R_+$ where $R_+ = [0, \infty)$ is said to be s-convex on I if the inequality

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\alpha \in [0, 1]$. It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

One of the most famous inequality for the class of convex functions is so called Hermite-Hadamard inequality, which states that: Let $f : I \subset R \to R$ be a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1], [3], [4], [7], [8], [9] and [10]).

In [6], Dragomir and Fitzpatrick established a variant of Hermite-Hadamard inequality which holds for the s-convex functions.

Theorem 1.1 *Suppose that $f : I \subset [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a,b \in [0,\infty)$, $a < b$. If $f \in L[0,1]$, then the following inequality holds*

$$2^{s-1} f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

Along this paper, we consider a real interval $I \subset R$, and we denote that I^0 is the interior of I.

In [5], Dragomir and Agarwal obtained the following Hermite-Hadamard type integral inequality.

Theorem 1.2 *Let $f : I \subset R \to R$ be differentiable mapping on I^0, where $a,b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds*

$$\Big|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\Big| \leq \frac{b-a}{8}[|f'(a)| + |f'(b)|]. \tag{1.4}$$

In [2], Alomari, Darus and Kirmaci proved the following inequalities of Hermite-Hadamard type for differentiable convex mappings.

Theorem 1.3 *Let $f : I \subset [0,\infty) \to R$ be a differentiable mapping on I^0, such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned}
&\Big|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \\
&\leq \frac{b-a}{4(s+1)(s+2)}\Big[|f'(a)| + 2(s+1)\Big|f'\left(\frac{a+b}{2}\right)\Big| + |f'(b)|\Big] \\
&\leq \frac{(2^{2-s}+1)(b-a)}{4(s+1)(s+2)}\Big[|f'(a)| + |f'(b)|\Big].
\end{aligned}$$

In [12], Pearce and Pečarić proved the following theorem.

Theorem 1.4 *Let $f : I \subset R \to R$ be a differentiable mapping on I^0, such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$, for some $q \geq 1$, then the following inequality holds:*

$$\Big|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \leq \frac{b-a}{4}\Big[\frac{|f'(a)|^q + |f'(b)|^q}{2}\Big]^{1/q}. \tag{1.5}$$

If $|f'|^q$ is concave on $[a,b]$, for some $q \geq 1$, then

$$\Big|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \leq \frac{b-a}{4}\Big|f'\Big(\frac{a+b}{2}\Big)\Big|. \tag{1.6}$$

For recent results and generalizations concerning Hermite-Hadamard's inequality, see [6]-[12] and the references given therein.

In this paper, we establish some new inequalities of Hadamard's type for the class of s-convex functions in the second sense.

2. Lemmas

To prove our main results, we consider the following lemma:

Lemma 2.1 *Let $f : I \subset R \to R$ be differentiable mapping on I^0, where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following inequality holds*

$$\begin{aligned} f(\lambda a+(1-\lambda)b) &- \frac{1}{b-a}\int_a^b f(t)dt \\ &= (b-a)(1-\lambda)^2\int_0^1 tf'\Big(t(\lambda a+(1-\lambda)b)+(1-t)a\Big)dt \\ &+(b-a)\lambda^2\int_0^1 (t-1)f'\Big(tb+(1-t)(\lambda a+(1-\lambda)b)\Big)dt. \end{aligned}$$

for each $\lambda \in [0,1]$.

Proof. We note that

$$\begin{aligned} I_1 &= \int_0^1 tf'\Big(t(\lambda a+(1-\lambda)b)+(1-t)a\Big)dt \\ &= \frac{1}{(b-a)(1-\lambda)}tf\Big(t(\lambda a+(1-\lambda)b)+(1-t)a\Big)\Big|_0^1 \\ &\quad -\frac{1}{(b-a)(1-\lambda)}\int_0^1 f\Big(t(\lambda a+(1-\lambda)b)+(1-t)a\Big)dt \\ &= \frac{1}{(b-a)(1-\lambda)}f(\lambda a+(1-\lambda)b) \\ &\quad -\frac{1}{(b-a)(1-\lambda)}\int_0^1 f\Big(t(\lambda a+(1-\lambda)b)+(1-t)a\Big)dt. \end{aligned}$$

Setting $x = t(\lambda a+(1-\lambda)b)+(1-t)a$, and $dx = (b-a)(1-\lambda)dt$, which gives

$$I_1 = \frac{1}{(b-a)(1-\lambda)}f(\lambda a+(1-\lambda)b) - \frac{1}{(b-a)^2(1-\lambda)^2}\int_a^{\lambda a+(1-\lambda)b} f(x)dx.$$

Similarly, we can show that

$$\begin{aligned} I_2 &= \int_0^1 (t-1)f'\Big(tb+(1-t)(\lambda a+(1-\lambda)b)\Big)dt \\ &= \frac{1}{(b-a)\lambda}f(\lambda a+(1-\lambda)b) - \frac{1}{(b-a)^2\lambda^2}\int_{\lambda a+(1-\lambda)b}^b f(x)dx, \end{aligned}$$

and therefore,

$$\begin{aligned} I &= (b-a)(1-\lambda)^2 I_1 + (b-a)\lambda^2 I_2 \\ &= f(\lambda a + (1-\lambda)b) - \frac{1}{b-a}\int_a^b f(x)dx, \end{aligned}$$

which completes the proof. ∎

Remark 1. Applying Lemma 2.1 for $\lambda = \frac{1}{2}$, we get the Lemma 2.1 in [2].

3. The new Hermite-Hadamard type inequalities

Theorem 3.1 *Let $f : I \subset R \to R$ be a differentiable mapping on I^0, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds*

$$\begin{aligned} &\left| f(\lambda a + (1-\lambda)b) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\leq (b-a)(1-\lambda)^2\Big(\frac{1}{(s+1)(s+2)}|f'(a)| + \frac{1}{s+2}|f'(\lambda a + (1-\lambda)b)|\Big) \\ &\quad +(b-a)\lambda^2\Big(\frac{1}{(s+1)(s+2)}|f'(b)| + \frac{1}{s+2}|f'(\lambda a + (1-\lambda)b)|\Big), \end{aligned}$$

for each $\lambda \in [0,1]$.

Proof. From Lemma 2.1, we have

$$\begin{aligned} &\left| f(\lambda a + (1-\lambda)b) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\leq (b-a)(1-\lambda)^2 \int_0^1 t\Big|f'\Big(t(\lambda a + (1-\lambda)b) + (1-t)a\Big)\Big|dt \\ &\quad +(b-a)\lambda^2 \int_0^1 (1-t)\Big|f'\Big(tb + (1-t)(\lambda a + (1-\lambda)b)\Big)\Big|dt \end{aligned}$$

Because $|f'|$ is s-convex, we have

$$\begin{aligned} &\left| f(\lambda a + (1-\lambda)b) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\leq (b-a)(1-\lambda)^2 \int_0^1 t\Big(t^s|f'(\lambda a + (1-\lambda)b)| + (1-t)^s|f'(a)|\Big)dt \\ &\quad +(b-a)\lambda^2 \int_0^1 (1-t)\Big(t^s|f'(b)| + (1-t)^s|f'(\lambda a + (1-\lambda)b)|\Big)dt \\ &= (b-a)(1-\lambda)^2\Big(\frac{1}{(s+1)(s+2)}|f'(a)| + \frac{1}{s+2}|f'(\lambda a + (1-\lambda)b)|\Big) \\ &\quad +(b-a)\lambda^2\Big(\frac{1}{(s+1)(s+2)}|f'(b)| + \frac{1}{s+2}|f'(\lambda a + (1-\lambda)b)|\Big), \end{aligned}$$

which completes the proof. ∎

Remark 2. Applying Theorem 3.1 for $\lambda = \frac{1}{2}$, we get the result in Theorem 1.3.

Theorem 3.2 *Let $f : I \subset R \to R$ be a differentiable mapping on I^0, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, $p > 1$, then the following inequality holds*

$$\begin{aligned}&\Big|f(\lambda a + (1-\lambda)b) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \\ &\leq (b-a)\Big(\frac{1}{p+1}\Big)^{1/p}\Big(\frac{1}{s+1}\Big)^{1/q}\Big((1-\lambda)^2\Big[(\lambda^s+1)|f'(a)|^q + (1-\lambda)^s|f'(b)|^q\Big]^{1/q} \\ &+\lambda^2\Big[\lambda^s|f'(a)|^q + ((1-\lambda)^s+1)|f'(b)|^q\Big]^{1/q}\Big),\end{aligned}$$

for each $\lambda \in [0,1]$ and p is the conjugate of q, $q = p/(p-1)$.

Proof. From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned}&\Big|f(\lambda a + (1-\lambda)b) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \\ &\leq (b-a)(1-\lambda)^2\int_0^1 t|f'\Big(t(\lambda a + (1-\lambda)b) + (1-t)a\Big)|dt \\ &+(b-a)\lambda^2\textstyle\int_0^1(1-t)|f'\Big(tb + (1-t)(\lambda a + (1-\lambda)b)\Big)|dt \\ &\leq (b-a)(1-\lambda)^2\Big(\int_0^1 t^p dt\Big)^{1/p}\Big(\int_0^1\Big|f'\Big(t(\lambda a + (1-\lambda)b) + (1-t)a\Big)\Big|^q dt\Big)^{1/q} \\ &+(b-a)\lambda^2\Big(\int_0^1 (1-t)^p dt\Big)^{1/p}\Big(\int_0^1\Big|f'\Big(tb + (1-t)(\lambda a + (1-\lambda)b)\Big)\Big|^q dt\Big)^{1/q}.\end{aligned}$$

Because $|f'|^q$ is s-convex, we have

$$\int_0^1\Big|f'\Big(t(\lambda a + (1-\lambda)b) + (1-t)a\Big)\Big|^q dt \leq \frac{|f'(\lambda a + (1-\lambda)b)|^q + |f'(a)|^q}{s+1},$$

and

$$\int_0^1\Big|f'\Big(tb + (1-t)(\lambda a + (1-\lambda)b)\Big)\Big|^q dt \leq \frac{|f'(b)|^q + |f'(\lambda a + (1-\lambda)b)|^q}{s+1}.$$

Therefore, we have

$$\begin{aligned}&\Big|f(\lambda a + (1-\lambda)b) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \\ &\leq (b-a)\Big(\frac{1}{p+1}\Big)^{1/p}\Big(\frac{1}{s+1}\Big)^{1/q}\Big((1-\lambda)^2\Big[|f'(\lambda a + (1-\lambda)b)|^q + |f'(a)|^q\Big]^{1/q} \\ &+\lambda^2\Big[|f'(\lambda a + (1-\lambda)b)|^q + |f'(b)|^q\Big]^{1/q}\Big).\end{aligned}$$

Now, since $|f'|^q$ is s-convex on [a,b], for any $\lambda \in [0,1]$, then by (1) we have

$$|f'(\lambda a+(1-\lambda)b)|^q \le \lambda^s|f'(a)|^q+(1-\lambda)^s|f'(b)|^q. \tag{3.1}$$

Combining all the above inequalities, we obtain

$$\begin{aligned}&\Big|f(\lambda a+(1-\lambda)b)-\frac{1}{b-a}\int_a^b f(t)dt\Big|\\ &\le (b-a)\Big(\frac{1}{p+1}\Big)^{1/p}\Big(\frac{1}{s+1}\Big)^{1/q}\Big((1-\lambda)^2\Big[\lambda^s|f'(a)|^q+(1-\lambda)^s|f'(b)|^q+|f'(a)|^q\Big]^{1/q}\\ &+\lambda^2\Big[\lambda^s|f'(a)|^q+(1-\lambda)^s|f'(b)|^q+|f'(b)|^q\Big]^{1/q}\Big).\end{aligned}$$

This proves the theorem. ■

Theorem 3.3 *Let $f : I \subset R \to R$ be a differentiable mapping on I^0, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, $q \ge 1$, then the following inequality holds*

$$\begin{aligned}&\Big|f(\lambda a+(1-\lambda)b)-\frac{1}{b-a}\int_a^b f(t)dt\Big|\\ &\quad\le (b-a)\Big(\frac{1}{(s+1)(s+2)}\Big)^{1/q}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big((1-\lambda)^2\Big[((s+1)\lambda^s+1)|f'(a)|^q\\ &\quad+(s+1)(1-\lambda)^s|f'(b)|^q\Big]^{1/q}\\ &\quad+\lambda^2\Big[(s+1)\lambda^s|f'(a)|^q+((s+1)(1-\lambda)^s+1)|f'(b)|^q\Big]^{1/q}\Big),\end{aligned}$$

for each $\lambda \in [0,1]$.

Proof. From Lemma 2.1 and using the using the well-known power-mean inequality, we have

$$\begin{aligned}&\Big|f(\lambda a+(1-\lambda)b)-\frac{1}{b-a}\int_a^b f(t)dt\Big|\\ &\le (b-a)(1-\lambda)^2\int_0^1 t|f'(t(\lambda a+(1-\lambda)b)+(1-t)a)|dt\\ &+(b-a)\lambda^2\int_0^1(1-t)|f'(tb+(1-t)(\lambda a+(1-\lambda)b))|dt\\ &\le (b-a)(1-\lambda)^2\Big(\int_0^1 tdt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t\Big|f'(t(\lambda a+(1-\lambda)b)+(1-t)a)\Big|^q dt\Big)^{1/q}\\ &+(b-a)\lambda^2\Big(\int_0^1(1-t)dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1(1-t)|f'(tb+(1-t)(\lambda a+(1-\lambda)b))|^q dt\Big)^{1/q}.\end{aligned}$$

Because $|f'|^q$ is s-convex, by (1) we have

$$\int_0^1 t\Big|f'\Big(t(\lambda a+(1-\lambda)b)+(1-t)a\Big)\Big|^q dt$$
$$\leq \int_0^1 \Big(t^{s+1}|f'(\lambda a+(1-\lambda)b)|^q+t(1-t)^s|f'(a)|^q)dt$$
$$=\frac{1}{(s+1)(s+2)}|f'(a)|^q+\frac{1}{s+2}|f'(\lambda a+(1-\lambda)b)|^q,$$

and

$$\int_0^1 (1-t)\Big|f'\Big(tb+(1-t)(\lambda a+(1-\lambda)b)\Big)\Big|^q dt$$
$$\leq \int_0^1 \Big((1-t)t^s|f'(b)|^q+(1-t)^{s+1}|f'(\lambda a+(1-\lambda)b)|^q\Big)dt$$
$$=\frac{1}{(s+1)(s+2)}|f'(b)|^q+\frac{1}{s+2}|f'(\lambda a+(1-\lambda)b)|^q.$$

Therefore, we have

$$\Big|f(\lambda a+(1-\lambda)b)-\frac{1}{b-a}\int_a^b f(t)dt\Big|$$
$$\leq (b-a)\Big(\frac{1}{(s+1)(s+2)}\Big)^{1/q}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big((1-\lambda)^2[(s+1)|f'(\lambda a+(1-\lambda)b)|^q+|f'(a)|^q]^{1/q}$$
$$+\lambda^2[(s+1)|f'(\lambda a+(1-\lambda)b)|^q+|f'(b)|^q]^{1/q}\Big).$$

Now, since $|f'|^q$ is s-convex on [a,b], for any $\lambda\in[0,1]$, then by (1) we have

$$|f'(\lambda a+(1-\lambda)b)|^q\leq \lambda^s|f'(a)|^q+(1-\lambda)^s|f'(b)|^q. \tag{3.2}$$

Combining all the above inequalities, we have

$$\Big|f(\lambda a+(1-\lambda)b)-\frac{1}{b-a}\int_a^b f(t)dt\Big|$$
$$\leq (b-a)\Big(\frac{1}{(s+1)(s+2)}\Big)^{1/q}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big((1-\lambda)^2\Big[((s+1)\lambda^s+1)|f'(a)|^q$$
$$+(s+1)(1-\lambda)^s|f'(b)|^q\Big]^{1/q}$$
$$+\lambda^2\Big[(s+1)\lambda^s|f'(a)|^q+((s+1)(1-\lambda)^s+1)|f'(b)|^q\Big]^{1/q}\Big),$$

we get the desired result. ∎

Theorem 3.4 *Let $f: I\subset R\to R$ be a differentiable mapping on I^0, such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is s-concave on $[a,b]$, for some fixed $s\in(0,1]$, $q>1$, then the following inequality holds*

$$\Big|f(\lambda a+(1-\lambda)b)-\frac{1}{b-a}\int_a^b f(t)dt\Big|$$
$$\leq (b-a)\Big(\frac{q-1}{2q-1}\Big)^{1-\frac{1}{q}}2^{\frac{s-1}{q}}\Big((1-\lambda)^2\Big|f'\Big(\frac{(1+\lambda)a+(1-\lambda)b}{2}\Big)\Big|+\lambda^2\Big|f'(\frac{\lambda a+(2-\lambda)b}{2})\Big|\Big),$$

for each $\lambda \in [0,1]$.

Proof. From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned}
&\Big|f(\lambda a+(1-\lambda)b)-\frac{1}{b-a}\int_a^b f(t)dt\Big|\\
&\le (b-a)(1-\lambda)^2\int_0^1 t|f'\Big(t(\lambda a+(1-\lambda)b)+(1-t)a\Big)|dt\\
&+(b-a)\lambda^2\textstyle\int_0^1(1-t)|f'\Big(tb+(1-t)(\lambda a+(1-\lambda)b)\Big)|dt\\
&\le (b-a)(1-\lambda)^2\Big(\int_0^1 t^p dt\Big)^{1/p}\Big(\int_0^1\Big|f'\Big(t(\lambda a+(1-\lambda)b)+(1-t)a\Big)\Big|^q dt\Big)^{1/q}\\
&+(b-a)\lambda^2\Big(\int_0^1(1-t)^p dt\Big)^{1/p}\Big(\int_0^1\Big|f'\Big(tb+(1-t)(\lambda a+(1-\lambda)b)\Big)\Big|^q dt\Big)^{1/q}.
\end{aligned}$$

Because $|f'|^q$ is s-concave, by the reversed direction of (3) we have

$$\int_0^1\Big|f'\Big(t(\lambda a+(1-\lambda)b)+(1-t)a\Big)\Big|^q dt\le 2^{s-1}\Big|f'\Big(\frac{(1+\lambda)a+(1-\lambda)b}{2}\Big)\Big|^q,$$

and

$$\int_0^1\Big|f'\Big(tb+(1-t)(\lambda a+(1-\lambda)b)\Big)\Big|^q dt\le 2^{s-1}\Big|f'\Big(\frac{\lambda a+(2-\lambda)b}{2}\Big)\Big|^q,$$

so

$$\begin{aligned}
&\Big|f(\lambda a+(1-\lambda)b)-\frac{1}{b-a}\int_a^b f(t)dt\Big|\\
&\le (b-a)\Big(\frac{1}{p+1}\Big)^{1/p}2^{\frac{s-1}{q}}\Big((1-\lambda)^2\Big|f'\Big(\frac{(1+\lambda)a+(1-\lambda)b}{2}\Big)\Big|+\lambda^2\Big|f'\Big(\frac{\lambda a+(2-\lambda)b}{2}\Big)\Big|\Big)\\
&=(b-a)\Big(\frac{q-1}{2q-1}\Big)^{1-\frac{1}{q}}2^{\frac{s-1}{q}}\Big((1-\lambda)^2\Big|f'\Big(\frac{(1+\lambda)a+(1-\lambda)b}{2}\Big)\Big|+\lambda^2\Big|f'(\frac{\lambda a+(2-\lambda)b}{2})\Big|\Big),
\end{aligned}$$

which yields the desired result. ∎

Remark 3. Applying Theorem 3.4 for $\lambda=\frac{1}{2}$, we get

$$\begin{aligned}
&\Big|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(t)dt\Big|\\
&\le\frac{b-a}{4}\Big(\frac{q-1}{2q-1}\Big)^{1-\frac{1}{q}}2^{\frac{s-1}{q}}\Big(\Big|f'\Big(\frac{3a+b}{4}\Big)\Big|+\Big|f'\Big(\frac{a+3b}{4}\Big)\Big|\Big)\\
&\le\frac{b-a}{4}\Big(\frac{q-1}{2q-1}\Big)^{1-\frac{1}{q}}\Big(\Big|f'\Big(\frac{3a+b}{4}\Big)\Big|+\Big|f'\Big(\frac{a+3b}{4}\Big)\Big|\Big),
\end{aligned}$$

which is an improved result comparing with Theorem 2.5 in [2].

4. Applications to special means

Now, using the results of Section 3, we give some applications to special means of real numbers.

We shall consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$.

(1) Weighted mean

$$W(\alpha,\beta) = \lambda\alpha + (1-\lambda)\beta, \quad \alpha,\beta \in R, \quad \lambda \in [0,1].$$

(2) Generalized log-mean:

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \;\; n \in Z\backslash\{0,1\}, \; \alpha,\beta \in R, \; (\alpha \neq \beta).$$

Therefore, by applying the s-convex mapping $f : [0,1] \to [0,1]$, $f(x) = x^s$, the following inequalities hold:

Proposition 4.1 *Let $a, b \in I^0$, $a < b$ and $0 < s < 1$. Then, we have*

$$\begin{aligned} &|W^s(a,b) - L_s^s(a,b)| \\ &\quad \leq (b-a)(1-\lambda)^2\Big(\frac{s}{(s+1)(s+2)}|a|^{s-1} + \frac{s}{s+2}|\lambda a + (1-\lambda)b|^{s-1}\Big) \\ &\quad +(b-a)\lambda^2\Big(\frac{s}{(s+1)(s+2)}|b|^{s-1} + \frac{s}{s+2}|\lambda a + (1-\lambda)b|^{s-1}\Big), \end{aligned}$$

for each $\lambda \in [0,1]$.

Proposition 4.2 *Let $a, b \in I^0$, $a < b$ and $0 < s < 1$. Then, for all $q > 1$, we have*

$$\begin{aligned} &|W^s(a,b) - L_s^s(a,b)| \\ &\quad \leq s(b-a)\Big(\tfrac{1}{p+1}\Big)^{1/p}\Big(\tfrac{1}{s+1}\Big)^{1/q}\Bigg((1-\lambda)^2\Big[(\lambda^s+1)|a|^{q(s-1)} + (1-\lambda)^s|b|^{q(s-1)}\Big]^{1/q} \\ &\quad +\lambda^2\Big[\lambda^s|a|^{q(s-1)} + ((1-\lambda)^s+1)|b|^{q(s-1)}\Big]^{1/q}\Bigg), \end{aligned}$$

for each $\lambda \in [0,1]$.

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