A CHARACTERIZATION OF PROJECTIVE SPECIAL LINEAR GROUP $L_3(5)$ BY nse

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Abstract. Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\text{nse}(G) = \{s_k | k \in \omega(G)\}$. In Khatami et al. and Liu's works, the authors proved that the groups $L_3(2)$ and $L_3(4)$ are unique determined by nse. In this paper, we prove that if G is a group such that $\text{nse}(G) = \text{nse}(L_3(5))$, then $G \cong L_3(5)$.

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1. Introduction

In 1987, Thompson posed a very interesting problem related to algebraic number fields as follows (see [17]).

Thompson's Problem. *Let $T(G) = \{(n, s_n) | n \in \omega(G)$ and $s_n \in nse(G)\}$, where s_n is the number of elements with order n. Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?*

It was proved that: *Let G be a group and M some simple K_i-group, $i = 3, 4$, then $G \cong M$ if and only if $|G| = |M|$ and $nse(G)=nse(M)$* (see [13], [14]). The group A_{12} is characterized by nse and order (see [8]). Recently, all sporadic simple groups are characterizable by nse and these orders (see [1]).

Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the Thompson's Problem, in other words, it remains only nse(G), whether can it characterize finite simple groups? Up to now, some groups especial for $L_2(q)$, where $q = 7, 8, 9, 11, 13$, can be characterized by only the set nse(G) (see [7], [16]). The author has proved that the groups $L_3(4)$, $L_5(2)$

and $U_3(5)$ are characterizable by nse (see [9], [10] and [11], respectively). In this paper, it is shown that the group $L_3(5)$ also can be characterized by nse($L_3(5)$).

We introduce some unfamiliar notations which will be used in this note. Let $a \cdot b$ denote the products of an integer a by an integer b. If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a group and r a prime. Then we denote the number of Sylow r-subgroups P_r of G by n_r or $n_r(G)$. The set of element orders of G is denoted by $\omega(G)$. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\mathrm{nse}(G) = \{s_k \big| k \in \omega(G)\}$. Let $\pi(G)$ denote the set of prime p such that G contains an element of order p. The other notations are standard (see [2]).

2. Some lemmas

Lemma 2.1 [4] *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \big| g^m = 1\}$, then $m \big| |L_m(G)|$.*

Lemma 2.2 [12] *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Lemma 2.3 [16] *Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Lemma 2.4 [5, Theorem 9.3.1] *Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \cdots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$:*

(1) *$q_i^{\beta_i} \equiv 1$ (mod p_j) for some p_j.*

(2) *The order of some chief factor of G is divided by $q_i^{\beta_i}$.*

Definition 2.5 A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$.

Remark 2.6 If G is a simple K_1-group, then G is a cyclic group of order p.

Remark 2.7 If G is a simple K_2-group, then by Thompson' $p^a q^b$ Theorem, G is soluble. Therefore there is no simple K_2-group.

Lemma 2.8 [18] *Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(1) *A_7, A_8, A_9 or A_{10}.*

(2) *M_{11}, M_{12} or J_2.*

(3) *One of the following:*

(a) $L_2(r)$, *where* r *is a prime and* $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ *with* $a \geq 1$, $b \geq 1$, $c \geq 1$, *and* v *is a prime greater than 3.*

(b) $L_2(2^m)$, *where* $2^m - 1 = u$, $2^m + 1 = 3t^b$ *with* $m \geq 2$, u, t *are primes,* $t > 3$, $b \geq 1$.

(c) $L_2(3^m)$, *where* $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ *or* $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, *with* $m \geq 2$, u, t *are odd primes,* $b \geq 1$, $c \geq 1$.

(4) *One of the following* 28 *simple groups:* $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^2D_4(2)$ *or* ${}^2F_4(2)'$.

Lemma 2.9 *Let* G *be a simple* K_4*-group and* $31 \mid |G| \mid 2^5.3.5^3.31$. *Then* $G \cong L_3(5)$ *or* $L_2(31)$.

Proof. Since G is a K_4-group, then by Lemma 2.8(1)(2), order consideration rules out these cases.

So, by Lemma 2.8(3), the following cases are considered.

Case 1. $G \cong L_2(r)$, where $r \in \{3, 5, 31\}$.

- Let $r = 3$, then $|\pi(r^2 - 1)| = 1$, which contradicts $|\pi(r^2 - 1)| = 3$.
- Let $r = 5$, then $|\pi(r^2 - 1)| = 2$, which contradicts $|\pi(r^2 - 1)| = 3$.
- Let $r = 31$, then $|\pi(31^2 - 1)| = 3$. So $G \cong L_2(31)$ since $|G| = 2^5.3.5.31$.

Case 2. $G \cong L_2(2^m)$, where $u \in \{3, 5, 31\}$.

- Let $u = 3$, then $m = 2$ and so $5 = 3t^b$. But the equation has no solution in $\mathbb{N}$, a contradiction.
- Let $u = 5$, then the equation $2^m - 1 = 5$ has no solution in $\mathbb{N}$, a contradiction..
- Let $u = 31$, then $32 = 2^m$ and so $m = 5$. So $33 = 3t^b$, and so $t = 11$, which is a contradiction since $11 \mid |G|$.

Case 3. $G \cong L_2(3^m)$.

The following two cases need to be considered.

- Subcase 3.1. $3^m + 1 = 4t$ and $3^m - 1 = 2u^c$.
  Let $t \in \{3, 5, 31\}$.
  If $t = 3, 5, 31$, the equation $3^m + 1 = 4t$ has no solution. So the case can be ruled out.

- Subcase 3.2. $3^m + 1 = 4t^b$ and $3^m - 1 = 2u$.
  Let $u \in \{3, 5, 31\}$.
  If $u = 3, 5, 31$, then the equation $3^m - 1 = 2u$ has no solution in $\mathbb{N}$, a contradiction.

By Lemma 2.8(4), order consideration rules out the groups except for $L_3(5)$. Therefore, G is isomorphic to $L_2(31)$ or $L_3(5)$.
This completes the proof. ∎

3. Main theorem and its proof

Let G be a group such that nse(G) =nse($L_3(5)$), and s_n be the number of elements of order n. By Lemma 2.3 we have that G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also, we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 2.1 and the above discussion, we have

$$(3.1) \qquad \begin{cases} \phi(m) \mid s_m \\ \\ m \mid \sum_{d \mid m} s_d \end{cases}$$

Theorem 3.1 *Let G be a group with nse(G)=nse($L_3(5)$) $= \{1, 775, 15500, 15624, 18600, 24800, 31000, 37200, 62000, 120000\}$. Then $G \cong L_3(5)$.*

Proof. We first prove that $\pi(G) \subseteq \{2,\ 3,\ 5,\ 31\}$, then show that $|G| = |L_3(5)|$ and so by [14], $G \cong L_3(5)$.

By (3.1), $\pi(G) \subseteq \{2,\ 3,\ 5,\ 7,\ 11,\ 31,\ 37201\}$.

If $k > 2$, then $\phi(k)$ is even, s_2=775, and so $2 \in \pi(G)$.

If $5, 31, 3 \in \pi(G)$, then s_5=15624, s_{31}=120000, s_3=15500, 24800, 62000.

In the following, we prove $37201 \notin \pi(G)$.

If $37201 \in \pi(G)$, then by (3.1), s_{37201}=37200.

If $2.37201 \in \omega(G)$, then $s_{74402} \notin$nse(G).

Therefore $2.37201 \notin \omega(G)$.

It follows that the Sylow 37201-subgroup of G acts fixed point freely on the set of elements of order 2 and so $|P_{37201}| \mid s_2$, a contradiction. Thus, $37201 \notin \pi(G)$.

If $2.7 \in \omega(G)$, then by Lemma 2.3 of [15], $s_{2.7} = s_7.t$ for some integer t. Then $s_{2.7} = s_7$. But by Lemma 2.1, $2.7 \mid 1 + s_2 + s_7 + s_{2.7}(240776)$, a contradiction. It follows that the Sylow 7-subgroup of G acts fixed point freely on the set of elements of order 2 and $7 \mid s_2$, a contradiction. Hence $7 \notin \pi(G)$.

If $2^a \in \omega(G)$, then $\phi(2^a) = 2^{a-1}|s_{2^a}$ and so $0 \leq a \leq 7$.

If $3^a \in \omega(G)$, then $1 \leq a \leq 3$.

If $5^a \in \omega(G)$, then $1 \leq a \leq 5$. If $5^5 \in \omega(G)$, then $s_{3125} \notin$nse(G) and so $1 \leq a \leq 4$.

If $11^a \in \omega(G)$, then $a = 1$.

If $31^a \in \omega(G)$, then $a = 1$ or 2. If $a = 2$, then $s_{961} \notin$nse(G) and so $961 \notin \omega(G)$. Hence $a = 1$.

If $2^a.3^b \in \omega(G)$, then $1 \leq a \leq 6$ and $1 \leq b \leq 3$.

If $3^a.5^b \in \omega(G)$, then $1 \leq a \leq 3$ and $1 \leq b \leq 4$.

If $2^a.3^b.5^c \in \omega(G)$, then $1 \leq a \leq 4$, $1 \leq b \leq 3$ and $1 \leq c \leq 4$.

To remove the prime 11, the fact that the prime 31 which belong to $\pi(G)$ will be proved.

Suppose that $31 \notin \pi(G)$.

- If 3, 5, $11 \notin \pi(G)$, then G is a 2-group and so $372000 + 15500k_1 + 15624k_2 + 18600k_3 + 24800k_4 + 31000k_5 + 37200k_6 + 62000k_7 + 120000k_8 = 2^m$ where $k_i, i = 1, 2, ..., 8$ and m are non-negative integers. Since $|\omega(G)| = 8$, the equation has no solution in $\mathbb{N}$ since the number of elements of nse(G) is ten. So, the following cases are considered: $\{3\}$, $\{3, 5\}$, $\{3, 11\}$, $\{3, 5, 11\}$, $\{5, 11\}$, $\{5, 7, 11\}$ and $\{11\}$,

- Let $11 \in \pi$(G), then as exp(P_{11})=11, $|P_{11}||1 + s_{11}$ and so $|P_{11}|$=11. Since $n_{11} = s_{11}/\phi(11)$=1860, then $31 \in \pi(G)$, a contradiction.

  Similarly, the set which contains the prime 11 can be excluded as the the set $\{11\}$.

- Let $5 \in \pi(G)$. The exponent of P_5 is equal to 5, 25, 125, and 625.

  * If exp(P_5)=5, then $|P_5| \mid 1 + s_5$ and so $|P_5| \mid 5^6$.

    If $|P_5| = 5$, then $n_5 = s_5/\phi(5)$=3906 and so $31 \in \pi$(G), a contradiction. If $|P_5| = 5^2$, then since $31, 11 \notin \pi(G)$, $372000 + 15500k_1 + 15624k_2 + 18600k_3 + 24800k_4 + 31000k_5 + 37200k_6 + 62000k_7 + 120000k_8 = 2^m.3^n.5^2$ where k_i, $i = 1, 2, ..., 8$ and m, n are non-negative integers, and $0 \leq \sum_{i=1}^{8} k_i \leq 77$. Since $372000 \leq |G| = 2^m.3^n.5^2 \leq 37200 + 77.120000$, then if $n = 0$, $m = 12, ..., 18$; if $n = 1$, then $m = 13, ..., 16$; if $n = 2$, then $m = 11, ..., 15$; if $n = 3$, then $m = 10, ..., 13$. But m is at most seven, and so these cases can be excluded. If $|P_5| = 5^3$, then similarly if $n = 0$, then $m = 12, ..., 16$; if $n = 1$, then $m = 10, ..., 14$; if $n = 2$, then $m = 9, ..., 13$; if $n = 3$, then $m = 7, ..., 11$. Therefore only $m = 7$ is considered. In this case, $372000 + 15500k_1 + 15624k_2 + 18600k_3 + 24800k_4 + 31000k_5 + 37200k_6 + +6200k_7 + 120000k_8 = |G| = 2^7.3^3.5^3$ where k_1, ..., k_8 are non-negative integers. By computer computation, the equation has no solution in $\mathbb{N}$.

    Similarly, the other cases $|P_5| = 5^4, 5^5, 5^6$ can be ruled out as the above methods.

  * If exp$(P_5) = 25$, then $|P_5| \mid 1 + s_5 + s_{25}$ and so $|P_5| \mid 125$. If $|P_5| = 25$, then by (1), $n_5 = s_{25}/\pi(25)$ if $s_{25} \in \{15500, 18600, 24800, 31000, 37200\}$ and so $31 \in \pi(G)$, a contradiction.

If $s_{25} = 120000$, then $n_5 = 6000$. By Sylow's theorem, $n_5 = 5k + 1$ for some integer k, but the equation has no solution in $\mathbb{N}$.

If $|P_5| = 125$, then similarly as the case "exp(P_5)=5 and $|P_5| = 25$", we can rule out this case.

* If $\exp(P_5) = 125$, then by Lemma 2.1, $|P_5| \mid 1 + s_5 + s_{25} + s_{125}$ and so $|P_5| \mid 5^4$.

  If $|P_5| = 5^3$, then by (3.1), s_{125}=15500, 24800, 31000, 37200, 120000. If s_{125}=15500, 18600, 24800, 31000, 37200, then $n_5 = s_{125}/\phi(125)$=155, 186, 248, 310, 372 and so 31 belongs to $\pi(G)$, a contradiction. Hence s_{125}=120000 and n_5=1200. But by Sylow's theorem, $n_5 = 5k + 1$ for some integer k, it is easy to see that the equation has no solution in $\mathbb{N}$.

  If $|P_5| = 5^4$, then similarly as the case "exp(P_5)=5 and $|P_5| = 25$", we can rule out this case.

* If $\exp(P_5) = 625$, then by Lemma 2.1, $|P_5| \mid 1 + s_5 + s_{25} + s_{125} + s_{625}$ ($s_{625} = 15500, 31000, 120000$) and so $|P_5| = 625$. If s_{625}=15500, 31000, $n_5 = s_{625}/\phi(625)$=31, 62 and so 31 belongs to $\pi(G)$, a contradiction. Therefore $s_{625} = 120000$ and t=240. But by Sylow's theorem $t = 5k+1$ for some integer k, so the equation has no solution in $\mathbb{N}$.

Similarly, the set which contains the prime 5 can be excluded as the the set $\{5\}$.

- Let $3 \in \pi(G)$. The exponent of P_3 equal to 3, 9, 27.

  * Let exp(P_3)=3. Then by Lemma 2.1, $|P_3| \mid 1 + s_3$ and so $|P_3| \mid 9$.

    If $|P_3| = 3$, then $n_3 = s_3/\phi(3)$ and so 31 belongs to $\pi(G)$, a contradiction.

    If $|P_3| = 9$, then $s_3 = 3t$ for some non-negative integer t. So s_3=15624, 18600, 37200, 120000 and t=5208, 6200, 12400, 40000, respectively. If s_3=15624, 18600, 37200, then $31 \in \pi(G)$. So $s_3 = 120000$. In this case, since 2$\in \pi(G)$ and if 31$\notin \pi(G)$, the primes $11, 5 \notin \pi(G)$, then $\pi(G) = \{2, 3\}$ and so $372000+15500k_1+15624k_2+18600k_3+24800k_4+31000k_5+37200k_6+62000k_7+120000k_8 = 2^m.9$ where $k_i, i = 1, 2, ..., 8$ and m are non-negative integers, and $0 \le \sum_{i=1}^{8} k_i \le 7$. Since $372000 \le |G| = 2^m.3^2 \le 37200 + 7.120000$, then $m = 16$, a contradiction since m is at most 7.

  * Let exp(P_3)=9. Then by Lemma 2.1, $|P_3| \mid 1 + s_3 + s_9$ and so $|P_3| \mid 81$.

    If $|P_3| = 9$, then 5 or 31 belongs to $\pi(G)$, a contradiction.

    If $|P_3| = 27$, then similarly $m = 16$, a contradiction. If $|P_3| = 81$, then $m > 7$, a contradiction.

  * Let exp(P_3)=27. Then by Lemma 2.1, $|P_3| \mid 1 + s_3 + s_9 + s_{27}$ and so $|P_3| = 27$. So $n_3 = s_{27}/\phi(27)$ and 31 belongs to $\pi(G)$, a contradiction.

Similarly, the set which contains the prime 3 can be excluded as the the set $\{3\}$.

Therefore, $31 \in \pi(G)$.

By Lemma 2.1, $|P_{31}| \mid 1 + s_{31}$ and so $|P_{31}| = 31$.

In the following, that the prime 11 do not belong to $\pi(G)$ are proved.

Let $11 \in \pi(G)$. If $11.31 \in \omega(G)$, then by Lemma 2.3 of [15], $s_{11.31} = 10.s_{31}.t$ for some integer t. But the equation has no solution since $s_{11.31} \in$nse(G). Hence $11.31 \notin \omega(G)$. It follows that the Sylow 11-subgroup of G acts fixed freely on the set of elements of order 31 and $11 \mid s_{31}$, a contradiction. So$11 \notin \pi(G)$.

Therefore, $\pi(G) \subseteq \{2, 3, 5, 31\}$.

From the above arguments, $2, 31 \in \pi(G)$, so the following cases will be considered: $\{2, 31\}$, $\{2, 3, 31\}$, $\{2, 5, 31\}$ and $\{2, 3, 5, 31\}$.

Case a. $\pi(G) = \{2, 31\}$.

Since exp(P_{31})=31, then by Lemma 2.1, $|P_{31}| \mid 1+s_{31}$ and so $|P_{31}|$=31. Since $n_{31} = s_{31}/\phi(31) = 4000$, then 5 belongs to $\pi(G)$, a contradiction.

Case b. $\pi(G) = \{2, 3, 31\}$.

The proof is the same as Case a.

Case c. $\pi(G) = \{2, 5, 31\}$.

Since exp(P_{31})=31, then by Lemma 2.1, $|P_{31}| \mid 1 + s_{31}$ and so $|P_{31}| = 31$.

If $2.31 \in \omega(G)$, set P and Q are Sylow 31-subgroups of G, then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are conjugate in G. Then $s_{2.31} = \phi(2.31).n_{31}.k$, where k is the number of cyclic subgroups of order 2 in $C_G(P_{31})$, and so $n_{31} \mid s_{2.31}$. So $s_{2.31} = n_{31}.t$ for some integer t. Since $n_{31} = s_{31}/\phi(31)$, $s_{2.31} = 4000t$ for some integer t and so $s_{2.31} = s_{31}$. On the other hand, $2.31 \mid 1 + s_2 + s_{31} + s_{2.31}(=240776)$, a contradiction. Thus $2.31 \notin \omega(G)$. It follows that the Sylow 2-subgroup of G acts fixed point freely on the set of elements of order 31 and so $|P_2| \mid s_{31}$. So $|P_2| \mid 2^6$. Also by (3.1), $5.31 \notin \omega(G)$ and $|P_5| \mid 5^4$.

We know that $\exp(P_5) = 5, 5^2, 5^3, 5^4$.

Let $\exp(P_5) = 5$.

- If $|P_5| = 5$, then $n_5 = s_5/\phi(5) = 3906$ and so $3 \in \pi(G)$, a contradiction.
- If $|P_5| = 5^2$, then $372000+15500k_1+15624k_2+18600k_3+24800k_4+31000k_5+37200k_6 + 62000k_7 + 120000k_8 = 2^m.5^2.31$ where $k_i, i = 1, 2, ..., 8$ and m are non-negative integers, and $0 \leq \sum_{i=1}^{8} k_i \leq 3$. Since $372000 \leq |G| = 2^m.5^2.31 \leq 37200 + 3.120000$, the equation has no solution since m is at most 6.
- If $|P_5| = 5^3$, then similarly, the equation has no solution.
- If $|P_5| = 5^4$, then similarly $m = 6, 5$. Let G be a group such that $|G| = 2^6.5^4.31$ or $|G| = 2^5.5^4.31$ and nse(G)=nse$(L_3(5))$. Then by programme of [7], there is no such group.

Let $\exp(P_5) = 25$. Then by (3.1), $s_{5^2} \in \{15500, 18600, 24800, 31000, 37200, 62000, 120000\}$

- If $|P_5| = 5^2$ and $s_{5^2} \in \{15500, 18600, 37200, 120000\}$, then $n_5 = s_{5^2}/\phi(5^2)$ and $3 \in \pi(G)$, a contradiction.

  If $|P_5| = 5^2$ and $s_{5^2} \in \{24800, 31000\}$, then $n_5 = 1240, 1550$. But by Sylow's theorem, $n_5 = 5k + 1$ for some integer k, the equation has no solution in $\mathbb{N}$.
- If $|P_5| \geq 5^3$, then similarly as the proof of "$\exp(P_5) = 5$", we can rule out this case.

Let $\exp(P_5) = 5^3$. Then by (3.1), $s_{5^3} \in \{15500, 24800, 31000, 37200, 120000\}$.

- If $s_{5^3} \in \{ 37200, 120000\}$ and $|P_5| = 5^3$, then $n_5 = s_{5^3}/\phi(5^3)$ and $3 \in \pi(G)$, a contradiction.

  If $s_{5^3} \in \{15500, 24800, 31000\}$ and $|P_5| = 5^3$, then $n_5 = 155, 248, 310$. On the other hand, by Sylow's theorem, $n_5 = 5k + 1$ for some integer k, the equation has no solution in $\mathbb{N}$.
- If $|P_5| = 5^4$, then similarly as the proof of "$\exp(P_5) = 5$", we can rule out this case.

Let $\exp(P_5) = 5^4$. Then similarly as the proof of "$\exp(P_5) = 5$", we can rule out this case.

Case d. $\pi(G) = \{2, 3, 5, 31\}$.

From the above, $|P_{31}|$=31.

If $3.31 \in \omega(G)$, set P and Q are Sylow 31-subgroups of G, then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are also conjugate in G. Therefore we have $s_{3.31} = \phi(3.31).n_{31}.k$, where k is the number of cyclic subgroups of order 3 in $C_G(P_{31})$. As $n_{31} = s_{31}/\phi(31) = 120000/30 = 40000$, $240000 \mid s_{3.31}$ and so $s_{3.31} = 240000t$ for some integer t, but the equation has no solution in $\mathbb{N}$. So $3.31 \notin \omega(G)$. It follows that the Sylow 3-subgroup P_3 of G acts fixed point freely on the set of elements of order 31 and so $|P_3| \mid s_{31}$. Therefore $|P_3| \mid 3$. Similarly we can prove that $3.5 \notin \omega(G)$ and $|P_5| \mid 5^3$.

Therefore, $|G| = 2^m.3.5^p.31$. Since $372000 = 2^5.3.5^3.31 \leq |G| = 2^m.3.5^p.31$, then $(m, p) = (5, 3), (6, 3)$.

In the following, first show that here is no group such that $|G| = 2^6.3.5^3.31$ and nse(G)=nse($L_3(5)$), second get the desired result.

Since $|\pi(G)| \geq 3$ and $\{3.31, 5.31, 3.5\} \cap \omega(G) = \Phi$, then by Lemma 2.5 of [3], G is insoluble.

Therefore, we can suppose that G has a normal series $1 \lhd K \lhd L \lhd G$ such that L/K is isomorphic to a simple K_i-group with $i = 3, 4$ as 9 and 961 do not divide order of G.

If L/K is isomorphic to a K_3-simple group, then from [6], $L/K \cong A_5, U_4(2)$. From [2], $n_5(L/K) = n_5(A_5) = 6$, and so $n_5(G) = 6t$ and $5 \nmid t$. Hence the number of elements of order 5 in G is: $s_5 = 6t \cdot 4 = 24t$ for some integer t. Since

$s_5 \in$nse(G), then s_5=15624 and $t = 651$. Hence $3.7.31 \mid |K| \mid 2^3.5^3 \cdot 31$, which is a contradiction. Similarly the other group $U_4(2)$ can be ruled out as the group "A_5".

If L/K is isomorphic to a K_4-group, then by Lemma 2.9, $L/K \cong L_2(31)$ or $L/K \cong L_3(5)$.

Let $L/K \cong L_2(31)$.

Let $\overline{G} = G/K$ and $\overline{L} = L/K$. Then

$$L_2(31) \leq \overline{L} \cong \overline{L}C_{\overline{G}}(\overline{L})/C_{\overline{G}}(\overline{L}) \leq \overline{G}/C_{\overline{G}}(\overline{L}) = N_{\overline{G}}(\overline{L})/C_{\overline{G}}(\overline{L}) \leq \mathrm{Aut}(\overline{L})$$

Set $M = \{xK \mid xK \in C_{\overline{G}}(\overline{L})\}$, then $G/M \cong \overline{G}/C_{\overline{G}}(\overline{L})$ and so $L_2(31) \leq G/M \leq$Aut$(L_2(31))$. Therefore $G/M \cong L_2(31)$, or $G/M \cong 2.L_2(31)$.

If $G/M \cong L_2(31)$, then order consideration $|M| = 2.5^2$. Then there exist a group M such that M is a Frobenius subgroup with a complement of order 2 and a Frobenius kernel of order 5^2. So there exists an element of order 3.5^2, which is a contradiction.

If $G/M \cong 2.L_2(31)$, then $|M| = 25$ and M is a normal subgroups generated by 5-central elements or with exponent of 5^2. If the former, then there is an element of order 5.31, a contradiction. If the latter, also we have a contradiction.

If $L/K \cong L_3(5)$, then similarly we have that $G/M \cong L_3(5)$ or $G/M \cong 2.L_3(5)$.

If $G/M \cong L_3(5)$, then $|M| = 2$. It follows that M is a normal subgroup generated by a 2-central element and so there exists an element of order 2.31, a contradiction.

$G/M \cong 2.L_3(5)$, then $M = 1$. Since nse$(2.L_3(5)) \neq$nse(G), then we rule out this case.

Hence $|G| = 2^5.3.5^3.31$ and by assumption, nse(G)=nse$(L_3(5))$, so by [14], $G \cong L_3(5)$.

This completes the proof of the theorem. ∎

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