ON AUTOMATIC BOUNDEDNESS OF LINEAR OPERATORS ON CONVEX BORNOLOGICAL SPACES

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Abstract. In this paper, using the notion of separating space of a linear operator defined on a bornological vector space introduced in [5], we give some useful criteria to study the automatic boundedness of operators. In particular, we give necessary and sufficient conditions in order that operators should be bounded (Theorem 3.1 and Theorem 4.1).

Keywords and phrases: bornological vector space, separating space, Mackey convergence, linear operator, automatic boundedness.

1. Introduction

In [4], Sinclair studied the necessary conditions for continuity of homomorphisms, derivations and pair of operators acting on a Banach space.

The aim of the present paper is to extend some of this results in case of bornological vector space (bvs) and consequently obtains some techniques to answer the boundedness problem for linear operators.

We extend naturally the notion of separating space of some linear operator S between (bvs) X and (bvs) Y (see Definition 3.1 below). The notion of separating space characterizes the continuity of linear operators

The nice properties of the notion of separating space is for an linear operator T acting on (bvs) X then, T is bounded if, and only if, its separating space is reduce a zero (see Theorem 3.1 below).

In the following we introduce some techniques which are necessary to study the boundedness of homomorphisms, derivations and pair of linear operators. Particulary, we give the characterization of bounded operators acting in bornological quotient (see Theorem 4.1 below).

2. Preliminaries

Recall that a bornology on a set X is a family $\mathcal{B}$ of subset X such that $\mathcal{B}$ is a covering of X, hereditary under inclusion and stable under finite union. The pair $(X, \mathcal{B})$ is called bornological set.

A subfamily $\mathcal{B}'$ of $\mathcal{B}$ is said to be base of bornology $\mathcal{B}$, if every element of $\mathcal{B}$ is contained in an element of $\mathcal{B}'$.

Let X and Y be two bornological set, a map X of Y is called bounded if the image of every bounded subset of X is bounded in Y.

A bornology $\mathcal{B}$ on $\mathbb{K}$-vector space E is said to be vector bornology on E if the maps $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda.x$ are bounded.

We called a bornological vector space(b.v.s) any pair $(X, \mathcal{B})$ consisting of a vector space E and a vector bornology $\mathcal{B}$ on E.

A vector bornology on a vector space is called a convex vector bornology if it is stable under the formation of convex hull.

A bornological vector space is said a convex bornological vector space (cbvs) if it bornology is convex.

A (b.v.s) space is called of type M_1 if, it satisfies the countability condition of Mackey:

For every sequence of bounded $(B_k)_k$ in E, there exists a sequence of scalars $(\lambda_k)_{k\geq 0}$ such that $\bigcup\limits_{k=0}^{\infty} \lambda_k B_k$ is bounded in E.

Observe that every Banach algebra is a multiplicative convex bornological complete algebra of type M_1. Also, if E is unital topological algebra with continuous inverse such that it is F-space, then E is a multiplicative convex bornological complete algebra of type M_1.

A sequence $(x_n)_{n\geq 0}$ in bornological vector space (b.v.s) E is said Mackey-convergent to 0 (or converge bornological to 0) if there exists a bounded set $B \subset E$ such that

$$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ n \geq n_0 \text{ implies } x_n \in \varepsilon B.$$

If E is (cbvs), then $(x_n)_{n\geq 0}$ is Mackey-convergent to 0 if there exists a bounded disk $B \subset E$ such that $(x_n)_{n\geq 0} \subset E_B$ and $(x_n)_{n\geq 0}$ converges to 0 in E_B, where (E_B, p_B) is the vector space spanned by B and endowed with the semi-norm p_B gauge of B.

Let E be a (bvs), E is said separated if there is not a non-zero bounded line in E, equivalently, every sequence Mackey-convergent its limit is unique.

A (bvs) E is separated if, and only if, for every bounded disk B the space (E_B, p_B) is a normed space.

A set B in a (bvs) E is said M-closed (or b-closed) if every sequence $(x_n)_{n\geq 0} \subseteq B$ Mackey convergent in E its limit belongs in B.

Let E a separated (bvs) and let F be a subspace of E, the bornological quotient space E/F is separated, if and only if, F is b-closed in E.

Let E be a (cbvs) and A is a disk in E. A is called a completant disk if the space (E_A, p_A) spanned by A and semi-normed by the gauge of A is a Banach space.

A (cbvs) E is called a complete convex bornological vector space if its bornology has a base consisting of completant disks.

Let E be a (bvs) and $A \subset E$, the bornological closure (briefly b-closure or M-closure) of A denoted $\overline{A}$ is the intersection of all bornological closed subsets of E containing A. A (cbvs) E satisfies M-closed properties if for any subset A of E we have $\overline{A} = A^{(1)}$ where $A^{(1)}$ is the set of limits in Mackey-sense of the sequences belonging in A.

For bornology with a nets see [1], [2]. Recall that every bornology of a (cbvs) having a countable base has a net.

Theorem 2.1 (bornological closed graph theorem) *Let* $(E, \mathcal{B})$ *be a complete* (cbvs) *and* $(E', \mathcal{B}')$ *be a* (cbvs) *such that* $\mathcal{B}'$ *has a net. Then, every linear map* $u : E \to E'$ *with a bornological closed graph in* $E \times E'$ *is bounded.*

Theorem 2.2 (bornological isomorphism) *Let* E *be a complete* (cbvs) *and let* F *be a* (cbvs) *where its bornology has a net. Then every bijective bounded linear map* $u : F \to E$ *is a bornological isomorphism.*

For details, see [1], [2], [3], or [6].

3. Separating space

Definition 3.1 [5] Let X and Y two (bvs), let T a linear map between X and Y. We called separating space of T the subset of Y denotes by $\sigma(T)$

$$\sigma(T) = \{y \in Y / \exists (x_n)_n \subset X : x_n \longrightarrow^M 0 \text{ and } T(x_n) \longrightarrow^M y\}.$$

Proposition 3.1 *Let* X *and* Y *two* (cbvs) *of type* M_1. *Then, every separating space of linear map* $T : X \longrightarrow Y$ *is a b-closed subspace in* Y.

Proof. Evidently, $\sigma(T)$ is a subspace vector of Y. $\sigma(T)$ is b-closed, indeed.

Let $(y_k)_{k\geq 0}$ a sequence in $\sigma(T)$ converging to y in Y. We prove that $y \in \sigma(T)$. For every $k \in \mathbb{N}^*$, $y_k \in \sigma(T)$. Then, there is a sequence $(x_{n,k})_n \subset X$ such that

$$x_{n,k} \longrightarrow^M 0 \text{ and } T(x_{n,k}) \longrightarrow^M y.$$

Since $x_{n,k} \longrightarrow^M 0$, then there is a bounded disc B_k in X:

$$(x_{n,k})_n \subset X_{B_k} \text{ and } \lim_{n\longrightarrow+\infty} p_{B_k}(x_{n,k}) = 0.$$

$(B_k)_{k\geq 0}$ is a sequence of bounded disc in X which is of type M_1, hence there is $(\lambda_k)_k \subset \mathbb{R}^{*+}$ and there is circled bounded B such that $B_k \subset \lambda_k B$, therefore $p_B \leq p_{B_k}$, for any $k \in \mathbb{N}$. (B can be disked, if not we take the disked hull of B).

Consequently, there exists a bounded disked B in X such that

$$(x_{n,k})_{n,k} \subset X_B,$$

we have

$$\forall k \geq 1 \ \exists N_k \in \mathbb{N}, \ \ n \geq N_k \Longrightarrow p_B(x_{n,k}) \leq \frac{1}{k}.$$

On other hand, $(T(x_{n,k}))_n$ converges bornological to y_k.

Again, by the same argument, there exists a bounded disked B' in Y such that $[T(x_{n,k}) - y_k]_n \subset Y_{B'}$, we have

$$\forall k \geq 1 \ \exists N'_k \in \mathbb{N}, \ \ n \geq N'_k \Longrightarrow p_{B'}\left(T(x_{n,k}) - y_k\right) \leq \frac{1}{k}.$$

Consequently, there exists a sequence $(z_k)_k \subset X_B$ such that $\left(T(z_k) - y_k\right)_k \subset Y_{B'}$ and, for every $k \in \mathbb{N}^*$, we have

$$p_B(z_k) < \frac{1}{k} \ \text{ and } p_{B'}\left(T(z_k) - y_k\right) < \frac{1}{k}.$$

Since $y_k \to^M y$, then there is a bounded disked C in Y such that

$$(y_k - y)_{k\geq 1} \subset Y_C \ \text{ and } p_C(y_k - y) < \frac{1}{k}.$$

Let D be the disked hull of $B' \cup C$. We have $p_D \leq p_{B'}$, $p_D \leq p_C$, $Y_{B'} \subset Y_D$ and $Y_C \subset Y_D$.

On other hand, for every $k \in \mathbb{N}^*$, we have

$$T(z_k) - y = (T(z_k) - y) + (y_k - y).$$

This gives

$$(T(z_k) - y)_{k\geq 1} \subset Y_D \quad \text{and} \quad \lim_{k\longrightarrow +\infty} p_D(T(z_k) - y) = 0.$$

We conclude that, $y \in \sigma(T)$. ■

Proposition 3.2 *Let X and Y two* (cbvs) *and let $T : X \to Y$ be a linear map. Then, we have*

i) $\sigma(T) = \{0\}$ *if, and only if, the graph of T is b-closed.*

ii) *Let R and S be two linear operators of X into Y, if $TR = ST$, then*

$$S(\sigma(T)) \subset \sigma(T).$$

Proof. Let $G(T)$ the graph of T.

i) Suppose that $\sigma(T) = \{0\}$.

Let $(x_n, T(x_n))_{n\geq 0} \subset G(T)$ such that $(x_n, T(x_n)) \longrightarrow^M (x, y) \in X \times Y$. Then, $x_n \longrightarrow^M x$ and $T(x_n) \longrightarrow^M y$. Since, $T(x_n - x) = T(x_n) - T(x)$, then $T(x_n - x) \longrightarrow^M y - T(x)$.

Consequently, $(y - T(x)) \in \sigma(T)$. Hence, $y = T(x)$.

Conversely, suppose that $G(T)$ is b-closed, and let $y \in \sigma(T)$. There exists $(x_n)_{n\geq 0} \subset X$ such that:

$$x_n \longrightarrow^M 0 \text{ and } T(x_n) \longrightarrow^M y.$$

Since, $(x_n, T(x_n))_{n\geq 0} \subset G(T)$ and $(x_n, T(x_n))_n \longrightarrow^M (0, y)$. Then, $(0, y) \in G(T)$. Therefore, $y = 0$.

ii) Let $y \in \sigma(T)$. There exists $(x_n)_{n\geq 0} \subset X$ such that $x_n \longrightarrow^M 0$ and $T(x_n) \longrightarrow^M y$. R being bounded, then $R(x_n) \longrightarrow^M 0$.

On other hand, $TR(x_n) = ST(x_n)$ and S is bounded then $TR(x_n) \longrightarrow^M S(y)$.

Consequently, $S(y) \in \sigma(T)$, i.e., $S(\sigma(T)) \subset \sigma(T)$. ∎

Theorem 3.1 *Let X be a complete* (cbvs) *and Y a* (cbvs) *such that its bornology has a net. Let $T : X \to Y$ be a linear map. Then T is bounded if, and only if, $\sigma(T) = \{0\}$.*

Proof. Applique the b-closed graph theorem's and Proposition 3.2. ∎

4. Characterization of bounded operators

Proposition 4.1 *Let X and Y two* (cbvs) *of type M_1 and Z a separated* (bvs)*. Suppose that X is complete and the bornology of Y has a net. Let $S : X \to Y$ be linear and $R : Y \to Z$ be bounded linear map. Then*

i) *RS is bounded if, and only if, $R(\sigma(S)) = \{0\}$.*

ii) *$[R\sigma(S)]^{(1)} = \sigma(RS)$.*

For the proof. we shall need the following lemma.

Lemma 4.1 *Let X be a* (bvs) *and F be a vector subspace of E. Let $\varphi : E \longrightarrow E/F$ the canonical surjection. Then, For every sequence $(x_n)_{n\geq 0} \subset E$ such that $(\varphi(x_n))_{n\geq 0}$ bornological converges to 0 in E/F, there exists a sequence $(y_n)_{n\geq 0} \subset F$ such that $(x_n - y_n)_{n\geq 0}$ bornological converges to* 0 *in E.*

Proof. Let $(x_n)_{n\geq 0} \subset E$ such that $(\varphi(x_n))_n$ borbological converges to 0 in E/F. The exists increasing sequence $(\epsilon_n)_n \subset \mathbb{R}^{*+}$ converging to 0 and a bounded discked B in E such that

$$\varphi(x_n) \in \epsilon_n \varphi(B), \ \forall\, n \in \mathbb{N}.$$

Let $(z_n)_n \subset B$ such that $\varphi(x_n) = \epsilon_n \varphi(z_n)$ for every n. Then, $(x_n - \epsilon_n z_n)_n \subset F$. We set $y_n = x_n - \epsilon_n z_n, \forall\, n \in \mathbb{N}$. Then

$$x_n - y_n = \epsilon_n z_n \in \epsilon_n B, \ \forall\, n \in \mathbb{N}.$$

Therefore, $(x_n - y_n)_n$ bornological converges to 0. ∎

Proof of Proposition 4.1

i) Suppose that RS is bounded.

Let $y \in \sigma(S)$. Then there exists $(x_n)_{n\geq 0} \subset X$ such that $x_n \longrightarrow^M 0$ and $S(x_n) \longrightarrow^M y$.

R being bounded, then $RS(x_n) \longrightarrow^M R(y)$.

On other hand, by hypothesis RS is bounded then, $RS(x_n) \longrightarrow^M 0$.

Z is separated, thus $R(y) = 0$, i.e: $R\sigma(S) = \{0\}$.

Conversely, suppose that $R(\sigma(S)) = \{0\}$.

Let $Q : Y \longrightarrow Y/\sigma(S)$ the canonical quotient map.

Consider $R_0 : Y/\sigma(S) \longrightarrow Z$ defined by: $R_0(y + \sigma(S)) = R(y)$.

Clearly, R_0 is well defining and we have $R = R_0Q$. Therefore, $R_0QS = RS$.

Since R_0 is bounded, it suffice to show that QS is bounded and for this we prove that $\sigma(QS) = \{0\}$ (Theorem 3.1).

Let $y + \sigma(S) \in \sigma(QS)$. Then there exists $(x_n)_{n\geq 0} \subset X$ such that

$$x_n \longrightarrow^M 0 \text{ and } QS(x_n) \longrightarrow^M Q(y) = y + \sigma(S).$$

Then, $Q(S(x_n) - y) \longrightarrow^M 0$ in $Y/\sigma(S)$.

By Lemma 4.1, there exists a sequence $(y_n)_{n\geq 0} \subset \sigma(S)$ such that:

$$S(x_n) - y - y_n \longrightarrow^M 0 \text{ in } Y.$$

Thus, for every $k \in \mathbb{N}^*$, there exists $(x_{n,k})_{n\geq 0} \subset X$ such that:

$$x_{n,k} \longrightarrow^M 0 \text{ and } S(x_{n,k}) \longrightarrow^M y_k.$$

Now, as already showed in proposition 3.2, we conclude that there exists two bounded disked B and B' in X and Y respectively and a sequence $(z_n)_{n\geq 0} \subset X_B$ such that

$$(S(x_n) - y_n)_{n\geq 0} \subset Y_{B'}$$

and, for every $k \in \mathbb{N}^*$, we have

$$p_B(z_k) < \frac{1}{k} \text{ and } p_{B'}(S(x_z) - y_k) < \frac{1}{k}.$$

Since

$$S(x_n - z_n) - y = (S(x_n) - y - y_n) + (y - S(z_n)),$$

the sequence $(S(x_n - z_n))_{n\geq 0}$ bornological converges to y. Since $(x_n - z_n)_{n\geq 0}$ bornological converges to 0. Then, $y \in \sigma(S)$.

Consequently, $\sigma(QS) = \{0\}$.

ii) We shows that $[R\sigma(S)]^{(1)} = \sigma(RS)$.

We have $R(\sigma(S)) \subset \sigma(RS)$, indeed.

Let $y \in \sigma(S)$. There exists $(x_n)_{n\geq 0} \subset X$ such that

$$x_n \longrightarrow^M y \text{ and } S(x_n) \longrightarrow^M y.$$

R being bounded, then $R(S(x_n)) \longrightarrow^M R(y)$. This gives, $R(y) \in \sigma(RS)$. Hence, $\sigma(RS)$ is b-closed (Proposition 3.2). Then

$$[R\sigma(S)]^{(1)} \subset \sigma(RS).$$

Next, to show the converse inclusion, consider the canonical quotient map:

$$Q_0 : Z \longrightarrow Z/[R\sigma(S)]^{(1)}$$

such that

$$Q_0(z) = z + [R\sigma(S)]^{(1)}.$$

Then, Q_0 is bounded. Thus Q_0R is bounded.

On other hand, $Q_0[R\sigma(S)] = \{\overline{0}\}$ where $\overline{0}$ is the class of 0. Then, by Proposition 3.2, Q_0RS is bounded. Therefore,

$$Q_0\sigma(RS) = \{\overline{0}\}.$$

Thus

$$\sigma(RS) \subset [R\sigma(S)]^{(1)}.$$

The proof is complete. ∎

Remark 1 In the conditions of Proposition 4.1, the subspace $S^{-1}[\sigma(S)]$ is b-closed in X.

Proof. $S^{-1}[\sigma(S)] = Ker(QS) = (QS)^{-1}(\{\overline{0}\})$. Since $\sigma(S)$ is b-closed and $Y/\sigma(S)$ is separeted, then, $S^{-1}[\sigma(S)]$ is b-closed in X. ∎

Theorem 4.1 *Let X and Y two* (cbvs) *of type M_1 and $S : X \to Y$ be linear map. Suppose that X is complete and the bornology of Y has a net.*

Let X_0 and Y_0 two subspaces b-closed of X and Y respectively such that $S(X_0) \subset Y_0$.

Let $S_0 : X/X_0 \longrightarrow Y/Y_0$ defined by:

$$S_0(x + X_0) = S(x) + Y_0.$$

Then, S_0 is bounded if, and only if, $\sigma(S) \subset Y_0$.

Proof. Suppose that S_0 is bounded. Let $y \in \sigma(S)$. There exists $(x_n)_{n\geq 0} \subset X$ such that : $x_n \longrightarrow^M 0$ and $S(x_n) \longrightarrow^M y$. Then

$$S_0(x_n + X_0) = S(x_n) + Y_0 \longrightarrow^M y + Y_0 \text{ and } S_0(x_n + X_0) \longrightarrow^M S_0(X_0).$$

Y_0 being b-closed, then Y/Y_0 is separated. Consequently, $y + Y_0 = S_0(X_0) \subset Y_0$. Thus, $y \in Y_0$.

Conversely, suppose that $\sigma(T) \subset Y_0$. Consider the canonical quotient map $Q : Y \longrightarrow Y/Y_0$. Q is bounded by the definition of the quotient bornology, and we have

$$Q(\sigma(S)) \subset Q(Y_0) = \{\overline{0}\}.$$

Then, QS is bounded, (by Proposition 3.1). Since $S(X_0) \subset Y_0$, we have:

$$QS(X_0) = \{\bar{0}\}.$$

On other hand,

$$S_0(x + X_0) = S(x) + Y_0 = QS(x).$$

Then, S_0 is bounded. ∎

Remark 2 In the conditions of Proposition 4.2, we have

$$\sigma(S|X_0) \subset Y_0 \cap \sigma(S),$$

where S/X_0 is the restriction of S in X_0.

Proposition 4.2 *Let X and Y two* (cbvs) *of type M_1 such that X is complete and the bornology of Y has a net. Let $Z_1, Z_2, ..., Z_n$ the complete* (cbvs) *and $T_1, T_2, ..., T_n$ the bounded linear maps of Z_j into X such that $X = \sum_{j=1}^{n} T_j Z_j$. Let $S : X \longrightarrow Y$ be bounded linear map. Then, we have*

$$[\sigma(ST_1) + \cdots + \sigma(ST_n)]^{(1)} = \sigma(S).$$

Proof. The proof is establish in two parts.

1°) Suppose that, for every $j \in \{1, 2, ..., n\}$, ST_j is bounded.

Let $T : Z \longrightarrow X$ defined by

$$T(z_1, ..., z_n) = T_1(z_1) + \cdots + T_n(z_n)$$

such that

$$Z = Z_1 \oplus Z_2 \oplus \ldots \oplus Z_n.$$

Since for $j = 1, ..., n$, T_j is bounded, by definition the bornology of Z, T is bounded.

On other hand we have

$$ST(z) = S(T_1(z_1) + \cdots + T_n(z_n)) = ST_1(z_1) + \cdots + ST_n(z_n).$$

Then, ST is bounded.

Now, we prove again that S is bounded:

Let $x \in X$. Then $x = T(z)$, where $z \in Z$. Thus, $S(x) = ST(z)$.

Let B a bounded in $X = \sum_{j=1}^{n} T_j Z_j$. Then, there exists $B_1, B_2, ..., B_n$ bounded in Z_j such that

$$B \subset T_1 B_1 + T_2 B_2 + \cdots + T_n B_n$$

Therefore,

$$S(B) \subset ST_1(B_1) + ST_2(B_2) + \cdots + ST_n(B_n).$$

Since ST_j is bounded, for every $j \in \{1,2,...,n\}$, $\sum\limits_{j=1}^{n} ST_j(B_j)$ is bounded in Y. Thus, $S(B)$ is bounded in Y. Consequently, S is bounded. Then, $\sigma(S) = \{0\}$.

Since, $\sigma(ST_j) = \{0\}$, $\forall j = 1,2,...,n$. Thus :

$$\left[\sum_{j=1}^{n} \sigma(ST_j)\right]^{(1)} = \sigma(S)$$

2°) General Case:

We have:

$$\sigma(ST_j) \subset \sigma(S), \ \forall j = 1,2,\ldots,n.$$

Indeed, let $y \in \sigma(ST_j)$. There is a sequence $(z_j^n)_{n\geq 0} \subset Z_j$ such that

$$z_j^n \longrightarrow^M 0 \ \text{ et } ST_j(z_j^n) \longrightarrow^M y.$$

Since, by hypothesis, T_j are bounded, then

$$T_j(z_j^n) \longrightarrow^M 0 \ \text{ and } S[T_j(z_j^n)] \longrightarrow^M y.$$

Then, $y \in \sigma(S)$, i.e., $\sigma(ST_j) \subset \sigma(S)$. This gives

$$\sum_{j=1}^{n} \sigma(ST_j) \subset \sigma(S).$$

Since $\sigma(S)$ is b-closed, we have

$$\left[\sum_{j=1}^{n} \sigma(ST_j)\right]^{(1)} \subseteq \sigma(S).$$

Next, we shows the converse inclusion.

Let $Q : Y \longrightarrow Y/W$, where $W = \left[\sum\limits_{j=1}^{n} \sigma(ST_j)\right]^{(1)}$. Then, Q is bounded.

On other hand, we have

$$Q[\sigma(ST_j)] \subset Q(W) = \{\overline{0}\}.$$

Then,

$$Q[\sigma(ST_j)] = \{\overline{0}\}.$$

By Proposition 3.2, we conclude that QST_j are bounded. Therefore, by the first case it results that QS is bounded. Then, $Q[\sigma(S)] = \{\overline{0}\}$, i.e: $\sigma(S) \subset W$. ■

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