r-WEAK cb SPACES

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Abstract. The purpose of the paper is to introduce the notion of r-weak cb spaces as a generalization of weak cb spaces. The r-weak cb property of a space is defined with the help of a stronger form of normal upper semi continuous functions viz. strongly normal upper semi continuous (s-nusc) function. A stronger form of regular closed subsets called strongly regular (s-regular) closed subsets turns out to be the natural tool for defining the new function. The r-weak cb spaces are characterized by a increasing cover of s-regular open (i.e. complement of s-regular closed) subsets and a decreasing sequence of s-regular closed subsets. Some of the properties of the newly introduced spaces and its interrelationship with other spaces are investigated.

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1. Introduction

Throughout by a space we mean a completely regular (Hausdorff) space. For basic definitions of zero-sets, cozero-sets, F_σ-sets, G_δ-sets, regular open and regular closed sets, refinement, cover etc. we refer [9] and [13].

The concept of regular G_δ-subsets was introduced by J. Mack [14]. A subset H of a topological space X is called a regular G_δ-subset if H is an intersection of a sequence of closed sets whose interiors contain H.

Equivalently, if $H = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} Cl_X G_n$, where each G_n is open subset of X, then H is a regular G_δ-subset. The complement of a regular G_δ-subset is called regular F_σ, i.e., a subset V of a space X is said to be a regular F_σ if $V = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} Int_X F_n$, where each F_n is closed subset of X. Properties of regular G_δ and regular F_σ-subsets have been studied in [3]. The intersection (resp. union) of two regular F_σ (resp. regular G_δ)-subsets is a regular F_σ (resp. regular G_δ)-subset. The countable union (resp. intersection) of regular F_σ (resp. regular G_δ)-subsets is a regular F_σ (resp. regular G_δ)-subset. Also every zero-set is regular G_δ and every cozero-set is regular F_σ.

A real valued function on a topological space X is said to be locally bounded if each point has a neighbourhood (nbd) on which the function is bounded. Let $C(X)$ denotes the set of all real valued continuous functions on X. A collection of subsets $\{A_\alpha\}$ of a topological space is said to be locally finite (resp. σ-locally finite) if each point $x \in X$ has a nbd which has non-empty intersection only with a finite (resp. countable) number of members of the collection. Also for two families of sets $\mathcal{U}$ and $\mathcal{V}$, $\mathcal{V}$ is said to be a refinement of $\mathcal{U}$, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. A family of continuous functions $\mathcal{F}$ is locally finite (resp. subordinate to a cover $\mathcal{U}$) if the collection of cozero-sets associated with $\mathcal{F}$ is locally finite collection of sets (resp. is a refinement of $\mathcal{U}$). A family $\mathcal{F} \subseteq C(X)$ is a partition of unity of a space X if each function in $\mathcal{F}$ maps X into $[0, 1]$ and $\sum f(x) = 1$ at each point $x \in X$, where the summation is taken over all $f \in \mathcal{F}$. The constant function, on any set, whose constant value is the real number r is denoted by $\mathbf{r}$, e.g. $\mathbf{1}$ is used to represent the constant function whose constant value is 1, i.e., $\mathbf{1}(x) = 1$ for all x. For all real valued function f on a set, the symbols f^+ and f^- are respectively defined as follows:

$$f^+(x) = \begin{cases} f(x) & \text{for } f(x) > 0 \\ 0 & \text{for } f(x) \leq 0 \end{cases}$$

and

$$f^-(x) = \begin{cases} f(x) & \text{for } f(x) < 0 \\ 0 & \text{for } f(x) \geq 0 \end{cases}$$

Thus with these notations the function $h_n = [(n+1)f - 1]^+ \times [(n-1)f - 1]^-$ is explained as $h_n(x) = [(n+1)f(x) - 1]^+ \times [(n-1)f(x) - 1]^-$. For real valued functions f and g, the functions $f \vee g$ and $f \wedge g$ are defined by the formulae $(f \vee g)(x) = max\{f(x), g(x)\}$ and $(f \wedge g)(x) = min\{f(x), g(x)\}$ respectively. If $f, g \in C(X)$ then the functions $f \vee g$, $f \wedge g$, f^+ and f^- are also the members of $C(X)$.

Real valued semi continuous functions play an important role in topology. Lower (resp. upper) semi continuous functions abbreviated as lsc (resp. usc) functions and their stronger forms are found in the literature. A function $f : X \to R$ is lsc (resp. usc) at a point $x_0 \in X$, if for each $\epsilon > 0$, there exists an nbd of x_0, say $N(x_0)$, such that for all $x \in N(x_0)$, $f(x) > f(x_0) - \epsilon$ (resp.

$f(x) < f(x_0) - \epsilon$). Equivalently, a function $f : X \to R$ is lsc (resp. usc) iff for all $r \in R$, the set $\{x : f(x) > r\}$ (resp.$\{x : f(x) < r\}$) is an open set [13]. A lsc (resp. usc) function $f : X \to R$ is normal lower (resp. upper) semi continuous or nlsc (resp. nusc) at a point $x_0 \in X$, if $f(x_0) < r$ (resp. $f(x_0) > r$) and any open set U containing x_0, there exists a non-empty open set V, such that $Cl_X V \subseteq U$ and $f(y) < r$ (resp. $f(y) > r$) for all $y \in V$. Equivalently, a lsc (resp. usc) function $f : X \to R$ is nlsc (resp. nusc) iff for all $r \in R$, the set $\{x : f(x) < r\}$ (resp. $\{x : f(x) > r\}$) is a union of regular closed sets [8].

A Tychonoff space X is Oz iff every regular closed set of X is a zero-set [5]. Equivalently, a Tychonoff space X is Oz iff every regular closed set is the intersection of a countable collection of regular closed nbds [6]. A topological space X is said to be a cb space if for each locally bounded function h, there exists $f \in C(X)$ such that $f \geq |h|$ [12]. The cb property is stronger than countable paracompactness, but equivalent to it for normal spaces. A space X is said to be weak cb if each locally bounded lsc function is bounded above by a continuous function. The weak cb and countably paracompactness are independent properties in Tychonoff spaces [15]. Two important characterizations of cb (resp. weak cb) spaces are: a space X is cb (resp. weak cb) iff (i) given an usc (resp. nusc) function h on X there exists $f \in C(X)$ such that $f \geq h$, (ii) for each decreasing sequence $\{F_n\}$ of closed (resp. regular closed) subsets of X with empty intersection, there exists a sequence of zero-sets $\{Z_n\}$ with $Z_n \supseteq F_n$ and $\bigcap_n Z_n = \phi$.

It would be interesting to study whether a further generalization of weak cb spaces could be introduced characterized by equivalent conditions like (i) and (ii) of weak cb spaces, where the nusc function in (i) and the regular closed subsets in (ii) are respectively replaced by their stronger forms. Further it is known that the characteristic function of regular closed (resp. regular open) subset of a space is nusc (resp. nlsc). This motivates us to search for a class of subsets which is stronger than that of regular closed subsets and whose characteristic function is at the same time stronger than the nusc function. In this direction it is found that the desired characteristics are present in the class of s-regular closed subsets which is defined below.

2. s-Regular closed subset and s-*nusc* function

Definition 2.1 A subset F (resp. G) of X is said to be strongly regular closed or s-regular closed (resp. strongly regular open or s-regular open) if $F = Cl_X B$ (resp. $G = Int_X A$), where B (resp. A) is a regular F_σ (resp. regular G_δ)-subset of X.

Theorem 2.2 *The complement of an s-regular open subset is s-regular closed and vice versa.*

Proof. Let G be an s-regular open subset of X. Then $G = Int_X A$, where A is a regular G_δ-subset. Now $X - G = X - Int_X A = Cl_X(X - A) = Cl_X B$, where $B = X - A$ is a regular F_σ-subset of X. Conversely, the complement of an s-regular closed set is s-regular open. ■

From the definition it follows that every s-regular open (resp. s-regular closed) subset is regular open (resp. regular closed). Now we cite an example to show that a regular open (resp. regular closed) subset may not be s-regular open (resp. s-regular closed).

Example 2.3 Let us consider the spaces of ordinals, $\mathbf{W}=[0,\omega_1)$ and $\mathbf{W}^*=[0,\omega_1]$, the one point compactification of $\mathbf{W}$, where ω_1 denotes the first uncountable ordinal. Again let us consider a subset A of $\mathbf{W}$ which is cofinal in $\mathbf{W}$, but not a tail. Then A is open in $\mathbf{W}$ and hence in $\mathbf{W}^*$ and $B = Cl_{\mathbf{W}^*}A$ is a regular closed subset of $\mathbf{W}^*$, which contains ω_1 but not a tail of $\mathbf{W}^*$. Next we show that B is not a regular G_δ. Now any open set in $\mathbf{W}^*$ containing ω_1 must contain a tail $\mathbf{W} - W(\alpha) = \{\sigma : \sigma > \alpha\}$, α is an ordinal. If possible, let B is a regular G_δ-subset of $\mathbf{W}^*$. Then $B = \bigcap_n G_n = \bigcap_n Cl_{\mathbf{W}^*}G_n$, where each G_n is open in $\mathbf{W}^*$ and contains ω_1. Then for each $n \in N$, there exists an ordinal α_n such that $\{\sigma : \sigma > \alpha_n\} \subseteq G_n$. Let $\alpha^* > \sup_n \alpha_n$, then $\alpha^* \in \mathbf{W}$ and $\{\sigma : \sigma > \alpha^*\} \subseteq \bigcap_n G_n = B$, i.e., B contains a tail $\mathbf{W} - W(\alpha^*)$, which is a contradiction. Thus B is not even a G_δ and hence not a regular G_δ. Therefore any regular G_δ-subset containing ω_1 must contain a tail $\{\sigma : \sigma > \alpha\}$, α is an ordinal. Let C be a regular G_δ-subset of $\mathbf{W}^*$ containing ω_1. Then $B \neq C$ and also $Int_{\mathbf{W}^*}B = A$ and $IntC$ contains a tail, which in turn implies that $IntB \neq IntC$ and hence the regular open subset $IntB$ is not equal to the s-regular open subset $IntC$.

This example also shows that the regular closed subset $X - A = X - Int_X B = Cl_X(X - B)$, where $X = \mathbf{W}^*$, is not s-regular closed.

With the help of s-regular closed subsets next we define functions to be called strongly nlsc (in short s-nlsc) and strongly nusc (in short s-nusc) functions.

Definition 2.4 A lsc (resp. usc) function $f : X \to R$ is said to be s-nlsc (resp. s-nusc) if for each real number $r \in R$, the set $\{x : f(x) < r\}$ (resp. $\{x : f(x) > r\}$) is a countable union of s-regular closed subsets of X.

The rationale of taking countable unions of the above definition stems from the fact that closures of countable unions of s-regular closed subsets are s-regular closed (Lemma 3.4), unlike regular closed subsets where closures of arbitrary unions of regular closed subsets are again regular closed. The following theorem provides us with examples of such functions.

Theorem 2.5 *The characteristic function of an s-regular open subset is s-nlsc.*

Proof. Let G be an s-regular open subset in a space X. The characteristic function f of G is defined by

$$\begin{aligned} f(x) &= 1, \quad \text{for all } x \in G \\ &= 0, \quad \text{for all } x \in X - G \end{aligned}$$

It can be easily seen that f is lsc.

We prove that f is s-nlsc. For this we show that $\{x : f(x) < r\}$ is a countable union of s-regular closed subsets of X. We consider the following cases.

Case (i). $r < 0$, $\{x : f(x) < r\} = \phi = Cl_X\phi$, and ϕ is regular F_σ.

Case (ii). $0 \leq r < 1$, $\{x : f(x) < r\} = X - G$, which is s-regular closed and can be thought of as a countable union of s-regular closed subsets.

Case (iii). $r \geq 1$, $\{x : f(x) > r\} = X$, which is s-regular closed.

Hence f is s-nlsc. ∎

Dually we can prove that the characteristic function of an s-regular closed subset is s-nusc.

Remark 2.6 From Hardy and Woods [11] we quote here a similar result for nusc (resp. nlsc) function. The characteristic function of a regular closed (resp. open) subset is nusc (resp. nlsc).

Next, we find the relationship between nlsc and s-nlsc functions. That every s-nlsc function is nlsc follows directly from Definition 2.4 and the characterization of nlsc functions [8]. But the converse is not true which follows from the following example.

Example 2.7 In Example 2.3 we have seen that the subset A of $X = \mathbf{W}^*$ is regular open, but not s-regular open and $X - A = F$ is regular closed, but not s-regular closed. Thus, in view of Remark 2.6, it follows that the characteristic function χ_A of A is nlsc. We want to show that χ_A is not s-nlsc. Let us suppose the contrary. Then, by definition, $\{x : \chi_A(x) < r\}$ is a countable union of s-regular closed subsets of X. Then, for $r = \frac{1}{2}$, $\{x : \chi_A(x) < r\} = X - A$ and thus $X - A$ is a countable union of s-regular closed subsets. Hence, $X - A$ is an F_σ-subset, since each s-regular closed subset is closed. In Example 2.3, it has been shown that $Cl_X A$ is regular closed, but not a regular G_δ (not even a G_δ). Thus $X - Cl_X A = Int_X(X - A) = X - A$ (since $X - A$ is open in X) is not regular F_σ (not even a F_σ) and hence cannot be expressed as a countable union of closed subsets. Thus, we arrive at a contradiction. Therefore, χ_A is not s-nlsc.

Similarly, the characteristic function χ_F of F is nusc, but not s-nusc.

3. r-Weak cb spaces and its properties

In this section, we introduce r-weak cb spaces as a generalization of weak cb spaces and study its properties.

Definition 3.1 A space X is called *r-weak cb* space if for each locally bounded s-nusc function h in X, there exists $f \in C(X)$ such that $|h| \leq f$.

As remarked earlier, our aim is to characterize the r-weak cb space in terms of an increasing cover of s-regular open and a decreasing sequence of s-regular closed subsets. To achieve our goal, we first prove some results in the form of lemmas which will be utilized in the desired characterization.

Lemma 3.2 *If f is strictly positive s-nlsc function on X, then $\frac{1}{f}$ is s-nusc.*

Proof. Let f be an s-nlsc function on X which is strictly positive. Now, we show that $g = \frac{1}{f}$ is s-nusc. Clearly, $\{x : g(x) > 0\} = X$ is a countable union of s-regular closed subsets. Let $r > 0$. Now,

$$\{x : g(x) > r\} = \left\{x : \frac{1}{f(x)} > r\right\} = \left\{x : f(x) < \frac{1}{r}\right\},$$

which is a countable union of s-regular closed subsets, since f is s-nlsc. Thus, for each $r \in R$, the set $\{x : g(x) > r\}$ is a countable union of s-regular closed subsets. Hence $g = \frac{1}{f}$ is s-nusc. ∎

Lemma 3.3 *Let $\{B_n\}$ be a countable increasing cover of X consisting of s-regular open subsets and g be a function defined by $g(x) = 1$ if $x \in B_1$ and $g(x) = \frac{1}{n}$ for $x \in B_n - B_{n-1}$, $n \geq 2$, then g is s-nlsc function.*

Proof. Let us consider a countable increasing s-regular open cover $\{B_n\}$ of X. Now, let us define a function g by $g(x) = 1$ if $x \in B_1$ and $g(x) = \frac{1}{n}$ for $x \in B_n - B_{n-1}$, $n \geq 2$. Clearly, g is lsc. Now, we see that the function g is actually s-nlsc. For this, we need to consider the following cases only as $g > 0$ for all x.

Case (i). Let $0 < r \leq 1$, we can always find $n \in N$ such that $\frac{1}{n+1} \leq r < \frac{1}{n}$. Now, $\{x \in X : g(x) < r\} = \left\{x \in X : g(x) < \frac{1}{n}\right\} = X - B_n$, which is s-regular closed and can be thought of a countable union of s-regular closed subsets.

Case (ii). Let $r > 1$, then $\{x : g(x) < r\} = X$, which is a countable union of s-regular closed subsets. Thus, for each real number r, the set $\{x : g(x) < r\}$ is a countable union of s-regular closed subsets of X. Hence g is an s-nlsc function. ∎

Lemma 3.4 *If $A = \bigcup_n Cl_X G_n$, where each G_n is regular F_σ in X, $n \in N$, then $Cl_X A$ is s-regular closed.*

Proof. Let $V = \bigcup_n G_n$, which is regular F_σ. Now, we show that $Cl_X A = Cl_X V$. Clearly, $V \subseteq A$ and so $Cl_X V \subseteq Cl_X A$.

Again, $Cl_X V = Cl_X \left(\bigcup_n G_n \right) \supseteq \bigcup_n Cl_X G_n = A$, which implies $Cl_X Cl_X V \supseteq Cl_X A$, i.e., $Cl_X V \supseteq Cl_X A$. Therefore, $Cl_X A = Cl_X V$, which is an s-regular closed. ∎

We are now in a position to characterize the r-weak cb spaces in terms of an increasing cover of s-regular open subsets and a decreasing sequence of s-regular closed subsets having empty intersection.

Theorem 3.5 *Let X be any topological space. Then the following statements are equivalent.*

a) *X is r-weak cb.*

b) *For a given strictly positive s-nlsc function g on X, there exists $f \in C(X)$ such that $0 < f(x) \leq g(x)$ for all $x \in X$.*

c) *For each countable increasing s-regular open cover of X, there exists a locally finite partition of unity subordinate that cover.*

d) *Each countable increasing s-regular open cover of X has a locally finite cozero refinement.*

e) *Each countable increasing s-regular open cover of X has a σ-locally finite cozero refinement.*

f) *Each countable increasing s-regular open cover of X has a countable cozero refinement.*

g) *Given a decreasing sequence $\{P_n : n \in N\}$ of s-regular closed subsets of X with $\bigcap_{n\in N} P_n = \phi$, there exists a decreasing sequence $\{Z_n : n \in N\}$ of zero-sets with $\bigcap_{n\in N} Z_n = \phi$, such that $P_n \subseteq Z_n$ for every n.*

Proof. $(a) \Rightarrow (b)$. Let g be a strictly positive s-nlsc function on X. Then $\frac{1}{g}$ is s-nusc (Lemma 3.2). Again, $\frac{1}{g}$ being strictly positive and usc, it is easy to see that $\frac{1}{g}$ is locally bounded. Now, by (a), there exists $f_1 \in C(X)$ such that $0 < \frac{1}{g(x)} \leq f_1(x)$ for all $x \in X$. Then $f(x) = \frac{1}{f_1(x)}$ is such that $f \in C(X)$ and $0 < f(x) \leq g(x)$ for all $x \in X$.

$(b) \Rightarrow (c)$. Let us consider a countable increasing s-regular open cover $\{B_n\}$ of X. Now, let us define a function g by $g(x) = 1$ if $x \in B_1$ and $g(x) = \frac{1}{n}$

for $x \in B_n - B_{n-1}$, $n \geq 2$. Then, g is an s-nlsc function (Lemma 3.3), which is also strictly positive. So, there exists $f \in C(X)$ such that $0 < f(x) \leq g(x)$ for all $x \in X$. Now, let us define a sequence of continuous functions h_n on X as $h_n = [(n+1)f - 1]^+[(n-1)f - 1]^-$, $n \in N$, where $\mathbf{1}(x) = 1$ for all $x \in X$. We see that $h_n(x) = 0$ for $f(x) \leq \dfrac{1}{n+1}$ or $f(x) \geq \dfrac{1}{n-1}$ for $n > 1$. Thus

$$Z(h_n) = \left\{x : f(x) \leq \frac{1}{n+1}\right\} \cup \left\{x : f(x) \geq \frac{1}{n-1}\right\},$$

i.e.,

$$\begin{aligned} X - Z(h_n) &= \left\{x : \frac{1}{n+1} < f(x) < \frac{1}{n-1}\right\}, \quad \text{for all } n > 1, \text{ and} \\ X - Z(h_1) &= \left\{x : \frac{1}{2} < f(x)\right\}, \quad \text{for } n = 1. \end{aligned}$$

Now, we show that the sequence of cozero-sets $\{X - Z(h_n)\}$ is locally finite. Let us consider a point $y \in X$ such that $f(y) = r$, then $0 < r \leq 1$. Since f is continuous, the set $\{x : r - \delta < f(x) < r + \delta\}$, $\delta > 0$ is open and clearly $y \in \{x : r-\delta < f(x) < r+\delta\}$. Now, by the Archimedean property of real number, we can choose $m \in N$ such that $r - \delta > \dfrac{1}{m-1}$. Then $\{x : r - \delta < f(x) < r + \delta\}$ is a nbd of y which has empty intersection with $X - Z(h_n)$ for $n \geq m$. So the cozero-sets $\{X - Z(h_n)\}$ forms a locally finite family, i.e., $\{h_n\}$ is locally finite. Now, $\{h_n\}$ being a locally finite family of continuous functions, $h(x) = \sum h_n(x)$ is continuous. Next, we show that h is non-vanishing. For this let $p \in X$ and $f(p) = a$, $0 < a \leq 1$. If $a = 1$, then $p \in \left\{x : \dfrac{1}{2} < f(x)\right\} = X - Z(h_1)$, so $h_1(p) \neq 0$. On the other hand, if $0 < a < 1$, we can find $n \in N$ such that $p \in \left\{x : \dfrac{1}{n+1} < f(x) < \dfrac{1}{n-1}\right\} = X - Z(h_n)$. In this case, $h_n(p) \neq 0$. Thus $h(p) = \sum h_n(p) \neq 0$. Since h is non-vanishing, $\dfrac{1}{h}$ is defined and belongs to $C(X)$ and $\left\{\dfrac{h_n}{h}\right\}$ is a locally finite partition of unity. Since

$$\begin{aligned} X - Z\left(\frac{h_n}{h}\right) &= \left\{x : \frac{1}{n+1} < f(x) < \frac{1}{n-1}\right\}, \quad \text{for all } n > 1 \text{ and} \\ X - Z\left(\frac{h_n}{h}\right) &= \left\{x : \frac{1}{2} < f(x)\right\}, \quad \text{for } n = 1, \end{aligned}$$

we have

$$X - Z\left(\frac{h_n}{h}\right) \subseteq \left\{x : \frac{1}{n+1} < f(x)\right\} = B_n \quad \text{and} \quad \bigcup_n \left\{X - Z\left(\frac{h_n}{h}\right)\right\} = X.$$

$(c) \Rightarrow (d)$. Let us consider a countable increasing s-regular open cover $\{B_n\}$ of X, then there exists a locally finite partition of unity $\{F_\alpha\}$, say $\alpha \in \Lambda$, an index set, subordinate to the cover $\{B_n\}$. So by definition $\{X - Z(F_\alpha)\}$ forms a locally finite cozero refinement of the cover $\{B_n\}$.

$(d) \Rightarrow (e)$. Since every locally finite collection is σ-locally finite, the result is obvious.

$(e) \Rightarrow (f)$. Let $\mathcal{A}$ be a σ-locally finite cozero refinement of the countable increasing s-regular open cover $\{B_n\}$ of X. Then $\mathcal{A} = \cup \mathcal{A}_m$, where each $\mathcal{A}_m$ is locally finite collection of cozero-sets. Let $A_{m,n}$ be the union of the sets $\{A_{m,\alpha}\}$ in $\mathcal{A}_m$ such that $A_{m,\alpha} \subseteq B_n$. Since $A_{m,n}$ is the union of a locally finite family of cozero-sets, it is a cozero-set [1]. Thus $\{A_{m,n}\}$ is a countable cozero refinement of $\{B_n\}$.

$(f) \Rightarrow (g)$. Let us consider a decreasing sequence $\{F_n : n \in N\}$ of s-regular closed subsets of X with $\bigcap_n F_n = \phi$. Then $\{X - F_n\}$ is an increasing s-regular open cover of X. Let $\mathcal{M}$ be a countable subset of $C(X)$ such that $\{X - Z(f)\}$, $f \in \mathcal{M}$, is a refinement of the cover $\{X - F_n\}$. Let us denote by Z_n, the intersection of all those $Z(f)$, $f \in \mathcal{M}$ for which $Z(f) \supseteq F_n$. Each Z_n, being a countable intersection of zero sets, is a zero set [9]. Now $\bigcap_n Z_n \subseteq \bigcap_{f \in \mathcal{M}} Z(f) = \phi$, since $\{X - Z(f)\}$ is a cover of X. Thus $\{Z_n\}$ is the desired sequence of zero-sets.

$(g) \Rightarrow (a)$. Let h be a locally bounded s-nusc function on X and define $P_n = \{x : h(x) > n\}$, $n \in N$. The set P_n need not be countable but each P_n can be expressed as a countable union of s-regular closed subsets of X (Definition 2.4). Hence $F_n = Cl_X P_n$ ($\subseteq \{x : h(x) \geq n\}$) is an s-regular closed subset (Lemma 3.4). Then, clearly, $\{F_n\}$ is a decreasing sequence of s-regular closed subsets of X with $\bigcap_n F_n = \phi$. So, by the given condition, there exists a decreasing sequence $\{Z_n = Z(g_n) : n \in N\}$ of zero sets with $\bigcap_{n \in N} Z(g_n) = \phi$ such that $F_n \subseteq Z(g_n)$ for every n, $g_n \in C(X)$. Let us define $f_n = 1 - \left(\bigvee_{i \leq n} n|g_i| \right) \wedge \mathbf{1}$, $n \in N$. Since $\bigcap_{n \in N} Z(g_n) = \phi$ for each $x \in X$, there exists $i \in N$ such that $g_i(x) \neq 0$. So, we can find $j \in N$ such that $|g_i(y)| > \dfrac{1}{j}$ for all y in an nbd $N(x)$ of x by the continuity of the function g_i and the Archimedean property of real numbers. Thus $j|g_i(y)| > 1$ for all $y \in N(x)$. If $n \geq i$ and $n \geq j$, then $n|g_i(y)| > j|g_i(y)| > 1$ for all $y \in N(x)$. Choosing $n \geq max\{i, j\}$, $n|g_i(y)| > 1$ and thus $(\vee_{i \leq n} n|g_i|)(y) > 1$ and, consequently, $\left(\bigvee_{i \leq n} n|g_i| \right) \wedge \mathbf{1} = 1$. Thus, by the definition of f_n, $f_n(y) = 0$ for all $y \in N(x)$, provided $n \geq \max\{i, j\}$, i.e., $y \in X - Z(f_n)$, $y \in N(x)$ only for finite number of values of n. These show that $\{f_n\}$ is locally finite. Therefore, by the locally finite property of $\{f_n\}$, $\sum_n f_n \in C(X)$, it follows that $f = 1 + \sum f_n \in C(X)$.

On F_n, we have $g_i(x) = 0$ for $i \leq n$ and hence $f_i(x) = 1$ for $i \leq n$. Thus, $f(x) \geq 1 + n > h(x)$ on $F_n - F_{n+1}$, for each $n \in N$. Also, for $\{X - F_1\}$, $h(x) < 1$, but $f(x) \geq 1$. So, $f(x) \geq h(x)$, for all $x \in X$. This proves the theorem. ∎

Next, we study some properties of r-weak cb spaces. In the following theorems it will be shown that, like weak cb spaces, this generalized space also possesses the properties that a cozero subspace of an r-weak cb space and its product with a locally compact paracompact space are again r-weak cb spaces. To prove the first result we need the following lemma.

Lemma 3.6 *If h is s-nusc on X, then h^+ is also s-nusc.*

Proof. Let us consider the set $\{x : h^+(x) < \lambda\}$, $\lambda \in R$. Since $h^+ \geq 0$, we need only to consider the case $\lambda \geq 0$. Now, $\{x : h^+(x) < \lambda\} = \{x : h(x) < \lambda\}$, which is open. Thus h^+ is usc.

Now, consider the set $\{x : h^+(x) > \lambda\}$, $\lambda \in R$.

Case (i). $\lambda < 0$, $\{x : h^+(x) > \lambda\} = X$, which is s-regular closed and can be thought of a countable union of s-regular closed subsets.

Case (ii). $\lambda \geq 0$, $\{x : h^+(x) > \lambda\} = \{x : h(x) > \lambda\}$, which a countable union of s-regular closed subsets, h being s-nusc.

Hence h^+ is also s-nusc. ∎

Theorem 3.7 *Each cozero subspace of an r-weak cb space is r-weak cb.*

Proof. Let X be an r-weak cb space and Y be a cozero subspace of X. Then $Y = X - Z(g)$ for some $g \in C(X)$, $0 \leq g \leq 1$. Let $F_n = \left\{x \in X : \frac{1}{n} \leq g(x)\right\}$. For a given locally bounded s-nusc function h on Y, let us define a sequence of functions $\{h_n\}$, $n \in N$ on X as follows:

$$h_n(x) = h^+(x) \text{ on } F_n \subseteq Y \text{ and } h_n(x) = 0 \text{ on } X - F_n.$$

It can be easily seen that h_n is a locally bounded s-nusc on X for each $n \in N$. Since X is r-weak cb, there exists $f_n \in C(X)$ such that $h_n \leq f_n$ for each n.

Next, we define the continuous functions $g_n : X \to [0, 1]$ for $n > 2$, as $g_n = 1 - [(n-1)(n-2)g - (n-2)]^+ \wedge \mathbf{1}$, where $\mathbf{1}(x) = 1$ etc. for all $x \in X$. Now

$$\begin{aligned} F_{n-2} &= \left\{x \in X : \frac{1}{n-2} \leq g(x)\right\} \\ &= \{x \in X : (n-1)(n-2)g(x) - (n-1) \geq 0\} \\ &= \{x \in X : (n-1)(n-2)g(x) - (n-2) \geq 1\}. \end{aligned}$$

So, $g_n(x) = 1 - 1 = 0$ for all $x \in F_{n-2}$, which implies that $Z(g_n) \supseteq F_{n-2}$. Similarly, it can be checked that $g_n(x) = 1$ for all $x \in X - F_{n-1}$. Assuming $g_1 = g_2 = \mathbf{1}$, we set $f = \vee_n (f_n g_n)|Y$. Next, we show that $\{g_n|Y\}$ is a locally finite family in $C(Y)$.

If $y \in Y$, then $g(y) \neq 0$. So, we can find an integer $i \in N$ such that $g(y) > \frac{1}{i}$, i.e., $ig(y) - 1 > 0$. As $g \in C(X)$, the set $\{x : ig(x) - 1 > 0\} = M$, say, is open and $y \in M$. Then, M is a nbd of y which does not meet $X - Z(g_{n+2})$ for $n \geq i$, since $X - Z(g_{n+2}) \subseteq X - F_n = \{x : ng(x) - 1 < 0\}$. Thus, $\{g_n|Y\}$ is a locally finite family in $C(Y)$, which implies that $\{(f_n g_n)|Y\}$ is also locally finite. So $f \in C(Y)$.

Again, since $g_n(x) = 1$ for $x \in X - F_{n-1}$, $F_{n-1} \subseteq F_n$ and hence $g_n(x) = 1$ for $x \in F_n - F_{n-1}$. Therefore, on $F_n - F_{n-1} \subseteq F_n \subseteq Y$, $f = \vee_n f_n g_n = \vee_n f_n \geq f_n$ for each $n \in N$. Finally, on $F_n - F_{n-1}$, $h^+ = h_n \leq f_n \leq f$ for all $n \in N$, which implies that $h(x) \leq f(x)$ on Y. Hence Y is r-weak cb. ∎

Since every clopen subset is cozero, we have the following:

Corollary 3.8 *Every clopen subspace of r-weak cb space is r-weak cb.*

To prove that the product of an r-weak cb space with a locally compact paracompact space is again r-weak cb we first prove the result considering a special case when the other space is compact. To that end, the following lemma is needed.

Lemma 3.9 *Let $p : X \times Y \to X$ be the projection map and Y be compact (T_1-space). Then for each s-regular closed subset A of $X \times Y$, $p(A)$ is an s-regular closed subset of X.*

Proof. Let $A = Cl_{X\times Y}G$, where G is a regular F_σ-subset of $X \times Y$, be an s-regular closed subset of $X \times Y$. We are to show that $p(A)$ is an s-regular closed subset of X. It is known that the projection map is open, continuous and surjective. Further since Y is compact, the projection of $X \times Y$ onto X is a closed mapping [14]. Therefore, $p(A) = p(Cl_{X\times Y}G) = Cl_X(p(G))$. It remains to show that $p(G)$ is a regular F_σ-subset of X. Let $G = \bigcup_n F_n = \bigcup_n Int_{X\times Y}F_n$, where each F_n is closed in $X \times Y$. Let $K_n = p(F_n)$, then each K_n is closed in X. Also $p(G) = p\left(\bigcup_n F_n\right) = \bigcup_n p(F_n) = \bigcup_n K_n$. Finally we show that $\bigcup_n K_n = \bigcup_n Int_X K_n$. It suffices to show that $\bigcup_n K_n \subseteq \bigcup_n Int_X K_n$. Now, p being an open mapping, $p(Int_{X\times Y}F_n) \subseteq Int_X p(F_n) = Int_X K_n$, for each $n \in N$. Hence $\bigcup_n p(Int_{X\times Y}F_n) \subseteq \bigcup_n Int_X K_n$, i.e., $p\left(\bigcup_n Int_{X\times Y}F_n\right) \subseteq \bigcup_n Int_X K_n$, i.e., $p\left(\bigcup_n F_n\right) \subseteq \bigcup_n Int_X K_n$, i.e., $\bigcup_n p(F_n) \subseteq \bigcup_n Int_X K_n$, i.e., $\bigcup_n K_n \subseteq \bigcup_n Int_X K_n$. Therefore, $p(G) = \bigcup_n K_n = \bigcup_n Int_X K_n$ is a regular F_σ-subset of X. ∎

Theorem 3.10 *The product of an r-weak cb space and a compact (Hausdorff) space is r-weak cb.*

Proof. Let X be an r-weak cb space and Y be a compact (Hausdorff) space. Let $\{K_n\}$, $n \in N$ be a decreasing sequence of s-regular closed subsets of $X \times Y$ with $\bigcap_n K_n = \phi$. Let $p : X \times Y \to X$ be the projection map. By Lemma 3.9,

$p(K_n) = A_n$, say, is an s-regular closed subset of X for each $n \in N$ and $\{A_n\}$ is also decreasing. Next, we prove that $\bigcap_n A_n = \phi$. For any $x \in X$, $\{x\} \times Y$ is compact. Thus, the family $\{K_n \cap (\{x\} \times Y)\}$ of closed sets does not have the finite intersection property, in other words $\{x\} \times Y$ fails to meet K_n for some n. Therefore $\{A_n\}$ has empty intersection. Now, X being r-weak cb, there exists a decreasing sequence $\{Z_n\}$ of zero-sets having empty intersection such that $Z_n \supseteq A_n$. Lastly, $\{Z_n \times Y\}$ is a sequence of zero-sets in $X \times Y$ with empty intersection such that $Z_n \times Y \supseteq K_n$ for each $n \in N$. Hence, in view of Theorem 3.5 (g), $X \times Y$ is r-weak cb. ∎

The conditions of the above theorem could be weakened by assuming locally compact paracompact (Hausdorff) property of the space Y. This will be shown in Theorem 3.17. For this purpose, we require some basic results regarding embedding properties of s-regular closed subsets, which are discussed first. In this direction, we consider an example to illustrate the difficulty in embedding an s-regular closed subset of a space to a subspace of it.

Example 3.11 Let $X = [0,1] \cup [2,3]$ and $Y = \{0\} \cup [2,3] \subseteq X$. Now $G = \left(0, \frac{1}{2}\right) \cup (2, 2.5)$ is regular F_σ in X and hence $F = Cl_X G = \left[0, \frac{1}{2}\right] \cup [2, 2.5]$ is s-regular closed in X. Then, $V = G \cap Y = (2, 2.5)$ is also a regular F_σ in Y. But $0 \notin Cl_Y V$, whereas $0 \in Cl_X G$. Thus $F \cap Y = \{0\} \cup [2, 2.5]$ and $Cl_Y V = [2, 2.5]$, which shows that $F \cap Y \neq Cl_Y V$.

Thus, we see that if $F = Cl_X G$, where G is regular F_σ in X, be an s-regular closed subset of a space X and $Y \subseteq X$, then for $V = G \cap Y$ the equality $F \cap Y = Cl_Y V$ may not be true always.

Now, we investigate the condition under which $F \cap Y = Cl_Y V$ holds.

Theorem 3.12 *If Y be a subset of X with the property that $Cl_X G \cap Y \subseteq Cl_Y V$, where G is regular F_σ in X and $V = G \cap Y$, then, for each s-regular closed subset $F = Cl_X G$ of X, $F \cap Y = Cl_Y V$ is s-regular closed in Y.*

Proof. Let $F = Cl_X G$, where G is regular F_σ-subset of X. Then, $G \cap Y$ is a regular F_σ-subset of Y [4]. Let $V = G \cap Y$. Then $Cl_Y V$ is s-regular closed subset of Y. We are to show that $F \cap Y = Cl_Y V$. Now, $Cl_Y V \subseteq Cl_X V \subseteq Cl_X G = F$, i.e., $Cl_Y V \cap Y \subseteq F \cap Y$, i.e., $Cl_Y V \subseteq F \cap Y$ as $Cl_Y V \subseteq Y$.

Conversely, let $p \in F \cap Y = Cl_X G \cap Y$. Then, by our assumption, $p \in Cl_Y V$. Hence, $F \cap Y \subseteq Cl_Y V$. Therefore, $F \cap Y = Cl_Y V$ is s-regular closed subset of Y. ∎

The assumption made in the above theorem is fulfilled by wide category of subspaces, e.g. open subspaces, dense subspaces [9, Art. 0.12] etc.

Next, we investigate the relationship between the properties of preservation of trace of an s-regular closed subset and the restriction of an s-nlsc (or s-nusc) function of a space to a subspace of it. In this direction first we see that the preservation of trace of an s-regular closed subset implies that the restriction of an s-nlsc (resp. s-nusc) function to a subspace is s-nlsc (resp. s-nusc).

Theorem 3.13 *The restriction of an s-nusc* (*resp. s-nlsc*) *function of a space X to a subset Y of X is also s-nusc* (*resp. s-nlsc*), *provided that the trace of each s-regular closed subset of X to Y is an s-regular closed subset of Y.*

Proof. First we prove the theorem for the s-nusc function. The proof for s-nlsc case is similar. Let f be an s-nusc function on X and Y is a subset of X. Let $f|Y = g$. For each real number r, $S_r(f) = \{x \in X : f(x) > r\}$ is a countable union of s-regular closed subsets of X, since f is s-nusc on X. Now, $S_r(f) \cap Y$ is a countable union of s-regular closed subsets of Y, by our assumption. Hence, $S_r(f) \cap Y = \{y \in Y : g(y) > r\}$. Hence, g is s-nusc function on Y, since $\{y \in Y : g(y) > r\} = S_r(f) \cap Y$. ∎

But it is interesting to see that the restriction of an s-nlsc (resp. s-nusc) function of a space to a closed subspace is again s-nlsc (resp. s-nusc).

Theorem 3.14 *The restriction of an s-nlsc* (*resp. s-nusc*) *function of a space X to a closed subset C having Lindelöff frontier is also s-nlsc* (*resp. s-nusc*).

Proof. Let us prove the theorem for the s-nlsc function. The s-nusc case may be dealt dually. Let f be an s-nlsc function on X and let C be a closed subset of X. Clearly, $g = f|C$ is lsc. Now consider the set $F_r = \{x : f(x) < r\}$, $r \in R$. By definition, F_r is a countable union of s-regular closed subsets of X, i.e. $F_r = \bigcup_n Cl_X U_n$, where U_n is a regular F_σ-subset of X for each $n \in N$. Now, $\{t \in C : g(t) < r\} = F_r \cap C = \bigcup_n (Cl_X U_n \cap C)$. We propose to show that the trace $Cl_X U_n \cap C$ of s-regular closed subsets $Cl_X U_n$ generated by the s-nlsc function f on X is again s-regular closed in C. Let $V_n = U_n \cap C$, then V_n is a regular F_σ-subset of C [4] and there exists a regular F_σ-subset W_n, $n \in N$, $W_n \supseteq V_n$ such that $\bigcup_n Cl_X U_n \cap C = \bigcup_n Cl_C W_n$. Since $Cl_C V_n \subseteq Cl_X U_n \cap C$ for each n, so we have $\bigcup_n Cl_C V_n \subseteq \bigcup_n (Cl_X U_n \cap C)$. Now let $p \in \bigcup_n (Cl_X U_n \cap C)$. Then, $p \in Cl_X U_n \cap C$ for some n, and every nbd of p in X meets U_n. Now, we claim that either every nbd of p in C meets V_n, i.e., $p \in Cl_C V_n$ or else p is an isolated point in C. Clearly, $f(p) = g(p) < r$. Now, f being lsc, the sets $\{x \in X : f(x) > f(p) - \epsilon\}$, $\epsilon > 0$ form basic nbds of p in X. Then, each such nbd meets $V_n = U_n \cap C$ and hence $p \in Cl_X V_n = Cl_C V_n$ provided $\{x \in C : r > f(x) > f(p) - \epsilon\} - \{p\} \neq \phi$. Otherwise, p would be an isolated point of C. As C is closed, all such points p satisfying $p \in Cl_X U_n \cap C$, but $p \notin Cl_X V_n$ for some n, must be a boundary point of C. Let G be the collection of all such p's. Now, we see that G is indeed a regular F_σ-subset of C. By our assumption, $Fr_X C$ is Lindelöff and $G \subseteq Fr_X C$ consists of isolated points. Hence G must be a countable set, which in turn implies that G is a regular F_σ subset. Now, it is plain to see that $\left(\bigcup_n Cl_C V_n\right) \cup G = \bigcup_n (Cl_X U_n \cap C)$ and $\{t \in C : g(t) < r\} = F_r \cap C = \left(\bigcup_n Cl_C V_n\right) \cup G = \bigcup_n (Cl_X U_n \cap C)$ is an s-regular closed of C and hence g is s-nlsc. ∎

Corollary 3.15 *Let $X \times C \subseteq X \times Y$, where C is compact in Y. Then for each s-nlsc (resp. s-nusc) function f of $X \times Y$, $f|(X \times C)$ is s-nlsc (resp. s-nusc).*

Proof. Since compact subsets are closed in Hausdorff spaces and $Fr_{X\times Y}(X \times C)$ is compact and hence Lindelöff, the corollary at once follows from Theorem 3.14. ∎

Finally, before going to the proof of the main theorem we quote a result from [2].

Result 3.16 *Let $S \subseteq X$ be open in X and f and g are respectively members of $C(S)$ and $C(X)$. Then, the function h on X defined by $h = f + g$ on S and $h = g$ on $X - S$ is continuous.*

Theorem 3.17 *The product of a r-weak cb space and a locally compact paracompact (Hausdorff) space is r-weak cb.*

Proof. Let X be a r-weak cb space and Y be a locally compact paracompact (Hausdorff) space. Since Y is locally compact, each point of Y has a compact nbd. Consider a compact nbd K_y containing $y \in Y$. Then there exists an open set O_y in Y such that $y \in O_y \subseteq K_y$. Again Y being paracompact, the cover of Y by the open sets $\{O_y : y \in Y\}$ has a locally finite open refinement $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$, say. Then, clearly, the members of the family $\overline{\mathcal{U}} = \{ClU_\alpha : \alpha \in \Lambda\}$ are compact sets. Since Y is normal (as it is paracompact Hausdorff) and the open cover $\mathcal{U}$ of Y is point finite (as it is locally finite), there is an open cover $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ such that $ClV_\alpha \subseteq U_\alpha$ for each $\alpha \in \Lambda$ [7].

Now, let h be a locally bounded s-nusc function on $X \times Y$. By Theorem 3.10, $X \times ClU_\alpha$ is r-weak cb. The function $h|X \times ClU_\alpha$ being s-nusc on $X \times ClU_\alpha$ (Corollary 3.15) and $X \times ClU_\alpha$ is r-weak cb, the function $h|X \times ClU_\alpha$ is bounded above by a continuous function f in $X \times ClU_\alpha$. The disjoint sets ClV_α and $Y - U_\alpha$ are such that ClV_α is compact and $Y - U_\alpha$ is a closed set of the completely regular space Y. Hence ClV_α and $Y - U_\alpha$ are completely separated [9] and this implies that $X \times V_\alpha \subseteq X \times ClV_\alpha$ and $X \times (Y - U_\alpha)$ are completely separated. Thus, there exists a function e_α in $C(X \times Y)$ such that $e_\alpha = 1$ on $X \times V_\alpha$ and $e_\alpha = 0$ on $X \times (Y - U_\alpha)$. Finally, define a function g_α on $X \times Y$ as $g_\alpha = e_\alpha + f$ on $X \times V_\alpha$ and $g_\alpha = e_\alpha$ elsewhere. Since $X \times V_\alpha$ is open in $X \times Y$, it can be seen that $g_\alpha \in C(X \times Y)$ (Result 3.16) and $g_\alpha(X \times Y) \geq |h(x, y)|$ on $X \times V_\alpha$ and $g_\alpha = 0$ on $X \times (Y - U_\alpha)$. Again, since $\{U_\alpha : \alpha \in \Lambda\}$ is locally finite, $g = \vee_{\alpha\in\Lambda} g_\alpha$ exists in $C(X \times Y)$ and clearly $g \geq h$. Hence the theorem. ∎

Next we study the relationship of r-weak cb space with other spaces.

Theorem 3.18 *Every weak cb space is r-weak cb.*

Proof. Let X be a weak cb space and $\{P_n : n \in N\}$ be a decreasing sequence of s-regular closed subsets of X with $\bigcap_{n\in N} P_n = \phi$. Now every s-regular closed subset is regular closed. Hence $\{P_n\}$ is a decreasing sequence of regular closed subsets with $\bigcap_{n\in N} P_n = \phi$. By weak cb property there exists a decreasing sequence

$\{Z_n : n \in N\}$ of zero-sets with $\bigcap_{n \in N} Z_n = \phi$, such that $P_n \subseteq Z_n$ for every n. Therefore X is r-weak cb. ∎

Corollary 3.19 *Every completely regular (Hausdorff) pseudocompact space is r-weak cb.*

Proof. Since every completely regular (Hausdorff) pseudocompact space is weak cb [15], hence the result follows from Theorem 3.18. ∎

Theorem 3.20 *Every Oz space is r-weak cb.*

Proof. Since every Oz space is weak cb, the result follows immediately from Theorem 3.18. ∎

That an r-weak cb space may fail to be countably paracompact, can be shown by the following example.

Example 3.21 The Tychonoff plank T, being pseudocompact [9], is an r-weak cb space by Corollary 3.19. But T is not countably paracompact [15].

Conclusion

In this paper, a generalized weak cb space called r-weak cb space has been introduced with the help of s-nlsc function and its various properties have been studied. This space includes the well known cb, weak cb spaces, but fails to imply countably paracompactness. The results obtained are interesting in the sense that various properties possessed by the cb and weak cb spaces are also possessed by the generalized r-weak cb spaces. This study opens up the future scope of investigation for the properties of the newly introduced s-nusc (s-nlsc) functions, and interrelationship between r-weak cb and nd-spaces [10], where it is known that every nd-space is countably paracompact and together with weak cb it implies cb property.

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