FEKETE-SZEGŐ PROBLEM FOR CONCAVE UNIVALENT FUNCTIONS DEFINED BY SĂLĂGEAN OPERATOR

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Abstract. Let $C_0(\alpha)$ denote the class of concave univalent functions defined in the open unit disk $U$. In this paper, we investigate the sharp upper bounds of Fekete-Szegő functional with real and complex parameter $\lambda$ for the class of concave univalent functions defined by Sălăgean differential operator.

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1. Introduction

Let $S$ denote the class of all analytic and univalent functions

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]  \hspace{1cm} (1.1)

defined on the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$.

Denote by $S^*(\beta)$, $C(\beta)$ and $K(\alpha, \beta)$, the classes of starlike functions of order $\beta$, convex functions of order $\beta$ and close-to-convex functions of order $\alpha$ type $\beta$ respectively, which are analytically defined as follows:

(i) \( S^*(\beta) = \left\{ f \in A : \Re \left( \frac{zf'(z)}{f(z)} \right) > \beta, \ z \in U, \ 0 \leq \beta < 1 \right\} \),

(ii) \( C(\beta) = \left\{ f \in A : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta, \ z \in U, \ 0 \leq \beta < 1 \right\} \),

(iii) \( K(\alpha, \beta) = \left\{ f \in A : \Re \left( \frac{f'(z)}{g'(z)} \right) > \alpha, \ g(z) \in C(\beta), \ z \in U, \ 0 \leq \alpha < 1, 0 \leq \beta < 1 \right\} \).

In 1933, Fekete and Szegő [19] obtained the maximum value of \( |a_3 - \lambda a_2^2| \) as a function of the real parameter $\lambda$, namely

\[ |a_3 - \lambda a_2^2| \leq 1 + 2 \exp \left( \frac{-2\lambda}{1-\lambda} \right), \]

for the class $S$ of analytic and univalent functions given by (1.1). This inequality is sharp for each $\lambda \in [0, 1]$. In the literature, there exists a large number of results of the Fekete-Szegő functional \( |a_3 - \lambda a_2^2| \) for various subclasses of $S$, such as the class of $S^*(\beta)$, $C(\beta)$ and $K(\alpha, \beta)$. For instance, Keogh and Merkers [10], Kaplan [26], Koepf [27] solved the Fekete-Szegő problem for close-to-convex functions. Nasr and Gawad [20], Gawad and Thomas [12], Darus and Thomas [18], Ibrahim and Darus [4] and others generalized this result for the class of functions that are close-to-convex functions of order $\alpha$ and type $\beta$. Later, Avkhadiev et al. [8], [9] and Bhowmik et al. [5], [6], they gave another treatment of Fekete-Szegő problem by considering the class of concave univalent functions given by (1.1).

Also, there are several authors that proved this type of result for the Fekete-Szegő functional for the class of function defined by differential operator, see [16], [3], for example, by using the Sălăgean differential operator $D^k$ [11], for $f \in S$ which is defined by

(i) \( D^0 f(z) = f(z) \),

(ii) \( D^1 f(z) = D f(z) = z + \sum_{n=2}^{\infty} na_n z^n \),
(iii) \( D^k f(z) = D \left( D^{k-1} f(z) \right) = z + \sum_{n=2}^{\infty} n^k a_n z^n; \quad k = 1, 2, \ldots \).

Denote by \( S_k^* \), the class of \( k \)-starlike functions which is analytically defined as follows:

\[
S_k^* = \left\{ f(z) \in S : \Re \left( \frac{D^{k+1} f(z)}{D^k f(z)} \right) > 0, \quad k = 0, 1, 2, \ldots, \quad z \in U \right\}.
\]

In this paper, we investigated the sharp upper bounds of Fekete-Szegö functional \( |a_3 - \lambda a_2^2| \) for the class of concave univalent functions with real and complex parameter \( \lambda \), where the function of \( f \) is defined by Sălăgean differential operator (1.2).

2. Preliminary results

A function \( f : U \to \mathbb{C} \) is said to belong to the family \( C_0(\alpha) \) if \( f \) satisfies the following conditions:

(a) \( f \) is analytic in \( U \) with the standard normalization \( f(0) = f'(0) - 1 = 0 \). In addition it satisfies \( f(1) = \infty \).

(b) \( f \) maps conformally onto a set whose complement with respect to \( C \) is convex.

(c) The opening angle of \( f(U) \) at \( \infty \) is less than or equal to \( \pi \alpha, \alpha \in (1, 2] \).

The class \( C_0(\alpha) \) is referred to concave univalent functions and for a detailed discussion about concave functions we refer to [8], [9], [17] and the references therein. Recently, the class \( C_0(\alpha) \) of concave function was considered by Bhowmik et al. [5], [6].

We recall the analytic characterization for the functions in \( C_0(\alpha), \alpha \in (1, 2] \): \( f \in C_0(\alpha) \) if and only if \( \Re P_f(z) > 0, \quad z \in U \), where

\[
P_f(z) = \frac{2}{\alpha - 1} \left[ \left( \frac{\alpha + 1}{2} \right) \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right].
\]

In [5], [6] they used this characterization and proved the following theorem.

**Theorem 1** Let \( \alpha \in (1, 2] \). A function \( f \in C_0(\alpha) \) if, and only if, there exist a starlike function \( \phi \in S^* \) such that \( f(z) = \Lambda_\phi(z) \) where

\[
\Lambda_\phi(z) = \int_0^z \frac{1}{(1-t)^{\alpha+1}} \left( \frac{t}{\phi(t)} \right)^{(\alpha-1)/2} dt
\]

and \( S^* \) denote the family of starlike functions \( g \) defined by \( g \in S^* \) if and only if \( \Re \left( \frac{zg'(z)}{g(z)} \right) > 0.\)
The objective of the present paper is to give some generalizations of the result of Fekete-Szegő problem given by Bhowmik et al. [5] for the starlike function defined by Sălăgean differential operator $D_k f, k = 0, 1, 2, \ldots$, which is $f \in S_k^*$ is characterized by the condition (1.2).

In order to prove our main results, we need to recall the following lemma.

**Lemma 1** [27] Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*$. Then $|b_3 - \lambda b_2^2| \leq \max \{1, |3 - 4\lambda|\}$, which is sharp for the Koebe function $k$ if $|\lambda - 3/4| \geq 1/4$ and for $(k(z))^{1/2} = \frac{z}{1 - z^2}$ if $|\lambda - 3/4| \leq 1/4$.

3. Main result and its proof

We consider the Fekete-Szegő functional $|a_3 - \lambda a_2^2|$ for real and complex parameter $\lambda$. Our results are contained in the following theorems.

**Theorem 1** Let $f \in C_0(\alpha)$ have the expansion given by (1.1), $\alpha \in (1, 2]$, $k = 0, 1, 2, \ldots$. If $\lambda$ is real, then we have

$$12 |a_3 - \lambda a_2^2| \leq \begin{cases} 
(3 + 2^{2k}) (2 - 3\lambda) \alpha^2 + 3 (1 - 2^{2k}) (1 - 2\alpha) \lambda + 6 (1 - 3\alpha) \alpha + 2 (3^{k+1} - 2^{2k}) , & \text{if } \lambda \leq \lambda_0; \\
4 [(2 - 3\lambda) \alpha^2 + 1] , & \text{if } \lambda_0 \leq \lambda \leq \frac{2(\alpha - 1)}{3\alpha}; \\
\frac{4 [(10 - 9\lambda) \alpha + (2 - 3\lambda)]}{3 (2 - \lambda) - (2 - 3\lambda) \alpha} , & \text{if } \frac{2(\alpha - 1)}{3\alpha} \leq \lambda \leq \frac{2}{3}; \\
12 (1 - \lambda) \alpha \sqrt{\frac{12(1-\lambda)}{(4-3\lambda)^2 - (3\lambda - 2)^2\alpha^2}} , & \text{if } \frac{2}{3} \leq \lambda \leq \lambda_2; \\
4 [(3\lambda - 2) \alpha^2 - 1] , & \text{if } \lambda_2 \leq \lambda \leq \frac{2(\alpha + 2)}{3(\alpha + 1)}; \\
(3 + 2^{2k}) (3\lambda - 2) \alpha^2 + 3 (2^{2k} - 1) (1 - 2\alpha) \lambda + 6 (3^{k} - 1) \alpha + 2 (2^{2k} - 3^{k+1}) , & \text{if } \lambda \geq \frac{2(\alpha + 2)}{3(\alpha + 1)}; 
\end{cases}$$

where

$$\lambda_0 = \frac{2^{2k-2} (\alpha + 1) - 3^k}{3 (2^{2k-3}) (\alpha - 1)} \text{ and } \lambda_2 = \frac{2}{3} + \frac{1}{6\alpha^2} \left(\sqrt{8\alpha^2 + 1} - 1\right).$$

The inequalities are sharp.
Theorem 2 Let \( f \in C_0(\alpha) \) have the expansion given by (1.1), \( \alpha \in (1, 2], \ k = 0, 1, 2, \ldots \). If \( \lambda \) are complex numbers, then we have

\[
|a_3 - \lambda a_2|^2 \leq \max \left\{ 1, \frac{1}{12} (\alpha + 1) \nu (\alpha, \lambda) \right\},
\]

where

\[
\nu (\alpha, \lambda) = |(2 - 3\lambda)(\alpha + 1) + 2| + 2 (\alpha - 1) |3\lambda - 2| + \left(\frac{\alpha - 1}{\alpha + 1}\right) |6 + [2 - 3 (\alpha - 1) \lambda]|.
\]

Proof. We recall from Theorem 1 that for \( f \in C_0(\alpha) \) if and only if there exist a function \( \phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \in S_k^* \), \( k = 0, 1, 2, \ldots \) such that

\[
f'(z) = \frac{1}{(1-z)^{\alpha+1}} \left( \frac{z}{D^n \phi(z)} \right)^{(\alpha-1)/2},
\]

where \( f \) has the form given by (1.1) and \( D^n \) is the Sălăgean operator. Comparing the coefficients of \( z \) and \( z^2 \) on the both sides of the series expansion (3.1), we obtain that

\[
a_2 = \frac{(\alpha + 1)}{2} - 2^{k-2} (\alpha - 1) \phi_2
\]

and

\[
a_3 = \frac{1}{6} (\alpha + 1) (\alpha + 2) - \frac{2^{k-1}}{3} (\alpha^2 - 1) \phi_2
\]

\[
- \frac{3^{k-1}}{2} (\alpha - 1) \phi_3 + \frac{2^{2k-3}}{3} (\alpha - 1) \phi_2^2,
\]

respectively.

A computation yields that

\[
a_3 - \lambda a_2^2 = \frac{(\alpha + 1)^2}{4} \left[ \frac{2 (\alpha + 2)}{3 (\alpha + 1)} - \lambda \right]
\]

\[
+ 2^{k-2} (\alpha^2 - 1) \left( \frac{\lambda - \frac{2}{3}}{2} \right) \phi_2 - \frac{3^{k-1}}{2} (\alpha - 1)
\]

\[
\times \left[ \phi_3 - \left( \frac{2^{2k-2} (\alpha + 1) - 3\lambda \left( \frac{2^{2k-3}}{3^k} \right) (\alpha - 1)}{3^k} \right) \phi_2^2 \right]
\]

(3.2)

Now, we need to investigate the maximum values of the function \( |a_3 - \lambda a_2^2| \) by considering several cases of \( \lambda \).

Case 1: Consider the first case for all \( \lambda \leq \frac{2^{2k-2} (\alpha + 1) - 3^k}{3 \left( \frac{2^{2k-3}}{3^k} \right) (\alpha - 1)} \).

We observe that the assumption on \( \lambda \) is seen to be equivalent to

\[
\frac{1}{3^k} \left[ \frac{2^{2k-2} (\alpha + 1) - 3\lambda \left( \frac{2^{2k-3}}{3^k} \right) (\alpha - 1)}{3^k} \right] \geq 1
\]
and the first term in equation (3.2) is nonnegative. Hence, using the Lemma 1 for the last term in (3.2), we have

\[
\phi_3 - \left(\frac{2^{2k-2} (\alpha + 1) - 3\lambda (2^{2k-3}) (\alpha - 1)}{3^k}\right) \phi_2^2 \\
\leq \frac{2^{2k} (\alpha + 1) - 3\lambda (2^{2k-1}) (\alpha - 1) - 3}{3^k}
\]

and noticing that for \( \phi \in S_k^* \), \(|\phi_n| \leq n^{1-k} \), \( k = 2, 3, \ldots \), we have from the equality (3.2) that

\[
|a_3 - \lambda a_2^2| \leq \frac{(\alpha + 1)^2}{4} \left[\frac{2 (\alpha + 2)}{3 (\alpha + 1)} - \lambda\right] + 2^{k-2} (\alpha^2 - 1) \left(\frac{2}{3} - \lambda\right) |\phi_2| \\
+ \frac{3^{k-1}}{2} (\alpha - 1) \left|\phi_3 - \left(\frac{2^{2k-2} (\alpha + 1) - 3\lambda (2^{2k-3}) (\alpha - 1)}{3^k}\right) \phi_2^2\right| \\
= \frac{(\alpha + 1) (\alpha + 2)}{6} - \frac{\lambda}{4} (\alpha + 1)^2 + \frac{(\alpha^2 - 1)}{2} \left(\frac{2}{3} - \lambda\right) \\
+ \frac{3^{k-1}}{2} (\alpha - 1) \left(\frac{2^{2k} (\alpha + 1) - 3\lambda (2^{2k-1}) (\alpha - 1) - 3}{3^k}\right).
\]

It can be simplified to

\[
|a_3 - \lambda a_2^2| \leq \frac{1}{12} \left[\left(3 + 2^{2k}\right) (2 - 3\lambda) \alpha^2 + 3 \left(1 - 2^{2k}\right) (1 - 2\alpha) \lambda\right. \\
\left. + 6 (1 - 3^k) \alpha + 2 (3^{k+1} - 2^{2k})\right],
\]

for \( \lambda \in \left(\infty, \frac{2^{2k-2} (\alpha + 1) - 3^k}{3 (2^{2k-3}) (\alpha - 1)}\right) \).

**Case 2:** Let \( \lambda \geq \frac{2 (\alpha + 2)}{3 (\alpha + 1)} \).

For this case, the first term in (3.2) is nonnegative. The condition on \( \lambda \) in particular gives \( \lambda \geq \frac{2}{3} \) and therefore our assumption on \( \lambda \) implies that

\[
\frac{2^{2k-2} (\alpha + 1) - 3\lambda (2^{2k-3}) (\alpha - 1)}{3^k} \leq \frac{2^{2k}}{3^k} \left(\frac{1}{2}\right).
\]

Again, it follows from Lemma 1, that

\[
\left|\phi_3 - \frac{2^{2k-2} (\alpha + 1) - 3\lambda (2^{2k-3}) (\alpha - 1)}{3^k}\right| \phi_2^2 \leq 3 \left(\frac{2^{2k} (\alpha + 1) - 3\lambda (2^{2k-1}) (\alpha - 1)}{3^k}\right).
\]
In view of these observation and an use of the inequality that $|\phi_2| \leq 2^{1-k}$, equality (3.2) gives

$$|a_3 - \lambda a_2^2| \leq \frac{(\alpha + 1)^2}{4} \left[ \lambda - \frac{2(\alpha + 2)}{3(\alpha + 1)} \right] + 2^{k-2}(\alpha^2 - 1) \left( \lambda - \frac{2}{3} \right) (2^{1-k})$$

(3.3)

$$+ \frac{3^{k-1}}{2} (\alpha - 1) \left( 3 - \frac{2^{2k}(\alpha + 1) - 3\lambda(2^{2k-1})(\alpha - 1)}{3^k} \right).$$

Thus, simplifying the right hand side expression (3.3), we obtain that

$$|a_3 - \lambda a_2^2| \leq \frac{1}{12} \left[ (3 + 2^{2k}) (3\lambda - 2) \alpha^2 + 3 (2^{2k} - 1) (1 - 2\alpha) \lambda + 6 (3^k - 1) \alpha - 2 (3^{k+1} - 2^{2k}) \right],$$

for $\lambda \in \left[ \frac{2(\alpha + 2)}{3(\alpha + 1)}, \infty \right)$. 

**Case 3:** Consider $\lambda$, where

$$\lambda \in \left( \frac{2^{2k-2}(\alpha + 1) - 3^{k-2}(\alpha + 2)}{3 (2^{2k-3})(\alpha - 1) + 3(\alpha + 1)} \right).$$

Now we deal with the case by using the formulas (3.1) and (3.2) together with the representation formula for $\phi(z) \in S_k^*$. Let us define $w(z)$ by

$$\frac{D^{k+1}\phi(z)}{D^k\phi(z)} = \frac{1 + zw(z)}{1 - zw(z)}, \quad (w(z) \neq 1) \quad (3.4)$$

where $w : U \rightarrow \overline{U}$ is a function analytic in $U$ with the Taylor series

$$w(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Comparing the coefficients of $z$ and $z^2$ in (3.4), we get that

$$\phi_2 = 2^{1-k}c_0 \quad \text{and} \quad \phi_3 = \frac{1}{3^k} (c_1 + 3c_0^2). \quad (3.5)$$

Inserting these resulting formulas (3.5) into (3.2) yields

$$a_3 - \lambda a_2^2 \leq \frac{(\alpha + 1)^2}{4} \left[ \frac{2(\alpha + 2)}{3(\alpha + 1)} - \lambda \right]$$

$$+ 2^{k-2}(\alpha^2 - 1) \left( \lambda - \frac{2}{3} \right) (2^{1-k}c_0)$$

$$+ \frac{3^{k-1}}{2} (\alpha - 1) \left[ \frac{1}{3^k} (c_1 + 3c_0^2) \right.$$

$$- \left. \frac{2^{2k-2}(\alpha + 1) - 3\lambda(2^{2k-3})(\alpha - 1)}{3^k} \right] (2^{2-2k}) c_0^2$$

$$= A + Bc_0 + Cc_0^2 + Dc_1,$$

$$\text{for } \lambda \in \left[ \frac{2(\alpha + 2)}{3(\alpha + 1)}, \infty \right).$$
where
\[
A = \frac{1}{6} (\alpha + 2) (\alpha + 1) - \frac{\lambda}{4} (\alpha + 1)^2, \\
B = \frac{1}{6} (\alpha^2 - 1) (3\lambda - 2), \\
C = -\frac{1}{12} (\alpha - 1) [4 - 2\alpha + 3\lambda (\alpha - 1)], \\
D = -\frac{1}{6} (\alpha - 1).
\]
Hence, by using the well known inequalities that \(|c_0| \leq 1\) and \(|c_1| \leq 1 - |c_0|^2\), from (3.6) we obtain that
\[
|a_3 - \lambda a_2^2| \leq |A + Bc_0 + Cc_0^2| + \frac{1}{6} (\alpha - 1) (1 - |c_0|^2).
\]
Now, in order to determine the maximum value of (3.7), let \(c_0 = re^{i\theta}\), then we consider the quadratic expression
\[
f(r, \theta) = |A + Bc_0 + Cc_0^2|^2 \\
= (A - Cr^2)^2 + B^2r^2 + 2Br (A + Cr^2) \cos \theta + 4ACr^2 \cos^2 \theta,
\]
where \(\cos \theta \in [-1, 1]\), \(r \in (0, 1]\). For getting the upper bounds of \(|a_3 - \lambda a_2^2|\), we have to find the biggest value of (3.8) for \(r\) in the interval \((0, 1]\). So, let \(x = \cos \theta\), then from (3.8) we have
\[
h(x) = (A - Cr^2)^2 + B^2r^2 + 2Br (A + Cr^2) x + 4ACr^2 x^2.
\]
We have to determine the maximum value of (3.9) for \(x \in [-1, 1]\). So, for this, we need to consider the several subclasses of \(\lambda\), where \(\lambda \in \left(\frac{2^{2k-2} (\alpha + 1) - 3^k}{3 (2^{2k-3}) (\alpha - 1)}, \frac{2^{2k-2} (\alpha + 1) - 3^k}{3 (\alpha + 1)}\right)\).

**Case 3A:** First, consider
\[
\lambda \in \left(\frac{2^{2k-2} (\alpha + 1) - 3^k}{3 (2^{2k-3}) (\alpha - 1)}, \frac{2^{2k-2} (\alpha + 1) - 3^k}{3 (\alpha + 1)}\right).
\]
We observe that for \(\lambda\) in this interval, we have \(A > 0\), \(B < 0\), \(C > 0\) and \(A + Cr^2 > 0\) for \(r \in (0, 1]\), and (3.9) attains its maximum value at \(x = -1\). Therefore, it gives that
\[
|a_3 - \lambda a_2^2| \leq g(r) = A - Br + Cr^2 + \frac{1}{3} (\alpha - 1) (1 - r^2).
\]
By a simple calculation, we show that the maximum value of (3.10) attains at the boundary of \(r\), i.e. \(r = 1\). Therefore
\[
g(r) \leq g(1) = A - B + C = \frac{1}{3} [(2 - 3\lambda) \alpha^2 + 1].
\]
Case 3B: Let \( \lambda = \frac{2(\alpha - 2)}{3(\alpha - 1)} \).

In this case, we get \( C = 0 \), therefore \( h(x) \) becomes a linear function,
\[
h(x) = A^2 + B^2r^2 + 2BrAx.
\]
It is easy to show that the maximum value of (3.11) occurs at \( x = -1 \) and \( r = 1 \). Again we get the maximum value of \( |a_3 - \lambda a_2^2| \) as the previous case.

Case 3C: Let \( \lambda \in \left( \frac{2(\alpha - 2)}{3(\alpha - 1)} \right)^2 \).

In this interval, the quadratic function (3.9) has maximum value at
\[
x(r) = -\frac{B}{4} \left( \frac{1}{Cr} + \frac{r}{A} \right),
\]
where \( x(r) \) is monotonic increasing in \( r \in (0, 1) \) and \( x(1) < -1 \). Hence we get the upper bound as in Cases 3A and 3B. As conclusion, from the Cases 3A, 3B and 3C give us that
\[
|a_3 - \lambda a_2^2| \leq \frac{1}{3} \left[ (2 - 3\lambda) \alpha^2 + 1 \right]
\]
for
\[
\lambda \in \left( \frac{2^{2k-2}(\alpha + 1) - 3^k}{3(2^{2k-3})(\alpha - 1)}, \frac{2(\alpha - 1)}{3\alpha} \right).
\]

Case 3D: Let \( \lambda \in \left[ \frac{2(\alpha - 1)}{3\alpha}, \frac{2}{3} \right] \).

From the Case 3C, the inequality \( x(1) < -1 \) gives that
\[
\frac{2(3\lambda + 4\alpha^2 - 12\alpha^2\lambda + 9\alpha^2\lambda^2 - 4)}{[(3\lambda - 4) + \alpha(3\lambda - 2)] [\alpha(3\lambda - 2) - (3\lambda - 4)]} < 0,
\]

hence it shows that
\[
p(\lambda) = 9\alpha^2\lambda^2 + (3 - 12\alpha^2)\lambda + 4(\alpha^2 - 1) < 0
\]
where \( \lambda < \frac{2}{3} \). Factorizing \( p(\lambda) \), we have
\[
\lambda_1 = \frac{2}{3} - \frac{1}{6\alpha^2} \left( 1 + \sqrt{8\alpha^2 + 1} \right)
\]
and
\[
\lambda_2 = \frac{2}{3} - \frac{1}{6\alpha^2} \left( 1 - \sqrt{8\alpha^2 + 1} \right).
\]
It is clear that \( \lambda_1 < \lambda_2 \). Therefore, for \( \lambda \in \left[ \frac{2(\alpha - 1)}{3\alpha}, \lambda_1 \right) \), functions (3.9) and (3.10) have their maximum value at
\[
x = -1 \quad \text{and} \quad r_m = \frac{-3B}{-6C + \alpha - 1} \in (0, 1]
respectively. Hence the upper bound of Fekete-Szegő functional is given by

\[
|a_3 - \lambda a_2^2| \leq g(r_m) = A - Br_m + Cr_m^2 + \frac{1}{3} (\alpha - 1) (1 - r_m^2)
\]

Next, we consider for \( \lambda \in \left[ \lambda_1, \frac{2}{3} \right] \). In this interval, the quadratic equation (3.9) attains its maximum value at

\[
x(r) = \frac{-B (A + Cr^2)}{4ACr}
\]

with

\[
h(x(r)) = -\frac{1}{4AC} (B^2 - 4AC) (A - Cr)^2.
\]

Hence, the Fekete-Szegő functional satisfies the following inequality

\[
|a_3 - \lambda a_2^2| \leq \sqrt{h(x(r))} + \frac{(\alpha - 1)}{6} (1 - r^2)
\]

The maximum value of \( g(r) \),

\[
g(r) = A - Br + Cr^2 + \frac{(\alpha - 1)}{6} (1 - r^2)
\]

and (3.15) occurs at

\[
r_m = \frac{-B}{-2C + \frac{(\alpha - 1)}{3}} \text{ and } r_0 = \frac{B}{2C + \sqrt{1 - \frac{B^2}{4AC}}}
\]

respectively. It is easy to show that (3.15) is monotonic decreasing for \( r \geq r_0 \). Hence, the maximum value of \( |a_3 - \lambda a_2^2| \) is given by (3.14).

For \( \lambda = \frac{2}{3} \), we get \( B = 0 \) and \( C = \frac{1}{6} (1 - \alpha) \). Thus, the maximum value occurs at \( x = \cos \theta = 0 \) and \( r \in (0, 1) \).

From (3.14), (3.15) and (3.16) we concluded that

\[
|a_3 - \lambda a_2^2| \leq \frac{4 [(10 - 9\lambda) \alpha + (2 - 3\lambda)]}{3 (2 - \lambda) - (2 - 3\lambda) \alpha}
\]

for \( \lambda \in \left[ \frac{2 (\alpha - 1)}{3\alpha}, \frac{2}{3} \right] \).
Case 3E: Let $\lambda \in \left(\frac{2}{3}, \lambda_2\right]$, where $\lambda_2$ is given by (3.13).

In this interval, we have $B > 0$. So that (3.9) attains its maximum value at $x = 1$. Then, we consider the function

$$l(r) = h(1) = A + Br + Cr^2 + \frac{(\alpha - 1)}{6} (1 - r^2).$$

Again, by a simple calculation shows that the maximum value of $l(r)$ to be ocurred at $r_n = B - 2C + \frac{A}{3}$, hence the maximum of the function (3.15) to be attained at

$$r_1 = \frac{B}{-2C + \frac{(\alpha - 1)}{3}} \in (0, 1].$$

It is easily to prove that $r_1 < r_n \leq 1$. Since $k(r)$ is monotonic increasing function, then

$$k(r) \leq k(1) = (A - C) \sqrt{1 - \frac{B^2}{4AC}},$$

which gives that

$$|a_3 - \lambda a_2^2| \leq k(1) = (1 - \lambda) \alpha \sqrt{\frac{12 (1 - \lambda)}{(4 - 3\lambda)^2 - (3\lambda - 2)^2 \alpha^2}}$$

for $\lambda \in \left(\frac{2}{3}, \lambda_2\right]$.

Case 3F: Finally, we consider the case for $\lambda \in \left(\lambda_2, \frac{2(\alpha + 2)}{3(\alpha + 1)}\right)$.

For these $\lambda$, we see that $A < 0$, $B > 0$, $C < 0$, $A + Cr^2 < 0$ and the maximum value of function (3.7) is attained for $x = -1$, i.e.

$$\eta(x) = -A + Br - Cr^2 + \frac{(\alpha - 1)}{6} (1 - r^2).$$

We get $\eta(r) \leq \eta(1)$ for all $\lambda$ in these interval and hence

$$|a_3 - \lambda a_2^2| \leq -A + B - C = \frac{1}{3} [(3\lambda - 2) \alpha^2 - 1].$$

Thus, the proof of Theorem 1 is complete.

Further, substitute (3.5) into (3.2) yields

$$12 (a_3 - \lambda a_2^2) = (\alpha + 1) [(2 - 3\lambda) (\alpha + 1) + 2] + 2 (\alpha^2 - 1) (3\lambda - 2) c_0 + (\alpha - 1) (6 + [2 - 3 (\alpha - 1) \lambda]) c_0^2 + 2 (1 - \alpha) c_1.$$
Hence for $\lambda$ complex numbers, we have

$$12 |a_3 - \lambda a_2^2| \leq (\alpha + 1) |(2 - 3\lambda)(\alpha + 1) + 2| + 2(1 - \alpha) |c_1| + 2(\alpha^2 - 1) |3\lambda - 2| |c_0| + (\alpha - 1) |6 + \left[2 - 3(\alpha - 1)\lambda\right]| |c_0|^2.$$  \hfill (3.17)

Using the well known inequality that $|c_0| \leq 1$ and $|c_1| \leq 1 - |c_0|^2$, then from (3.17) we get

$$12 |a_3 - \lambda a_2^2| \leq \frac{1}{12} (\alpha + 1) \nu (\alpha, \lambda)$$

for $\text{Re} \{\nu (\alpha, \lambda)\} > 0$, where

$$\nu (\alpha, \lambda) = |(2 - 3\lambda)(\alpha + 1) + 2| + 2(1 - \alpha) |3\lambda - 2| + \frac{(\alpha - 1)}{\alpha + 1} |6 + \left[2 - 3(\alpha - 1)\lambda\right]|.$$

Thus, the proof of Theorem 2 is complete.

**Remark 1** Taking $k = 0$ and $\lambda$ real numbers, we deduce a result of Bhowmik et al. [5].

Other problems related to Fekete-Szegö functional for further reading can be found in ([1], [2], [7], [13], [14], [15], [21], [22], [23], [24], [25]).

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**References**


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