ON LAPLACE TRANSFORMS OF GENERALIZED WHITTAKER FUNCTION OF MULTI-VARIABLES $M_{\lambda,\mu_1\cdots\mu_k}(x_1,...,x_k)$

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Abstract. In this paper, a Laplace transform of generalized Whittaker function is derived which is used to obtain further some partly bilateral and partly unilateral generating function and series expansion. Some special cases are also discussed.

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1. Introduction and definition

A Whittaker function $M_{\lambda,\mu}$ was introduced by Whittaker [2] (see also Whittaker and Watson [3]) in terms of the confluent hypergeometric function ${}_1F_1$ (or Kummer's function)

$$M_{\lambda,\mu}(x) = x^{\mu+1/2}e^{(-1/2)x}{}_1F_1\left(\frac{1}{2}+\mu-\lambda, 2\mu+1; x\right). \tag{1.1}$$

Further generalization of the Whittaker function $M_{\lambda,\mu}$ was introduced by Humbert [6; p.63 (15)] in the following form

$$\begin{aligned} M_{\lambda,\mu_1\cdots\mu_k}(x_1,...,x_k) &= x_1^{\mu+1/2}\cdots x_k^{\mu_k+1/2}\exp\left[\frac{-1}{2}(x_1+\cdots+x_k)\right] \\ &\cdot\Psi_2^{(k)}[\mu_1+\cdots+\mu_k-\lambda+k/2;\ 2\mu_1+1,...,2\mu_k+1;\ x_1,...,x_k], \end{aligned} \tag{1.2}$$

where $\Psi_2^{(k)}$ denotes Humbert's confluent hypergeometric function of n-variables [6; p. 62(11)]

$$\begin{aligned}&\Psi_2^{(k)}[a;c_1,...,c_k;x_1,...,x_k]\\ &=\sum_{m_1\cdots m_k=0}^{\infty}\frac{(a)_{m_1+\cdots+m_k}}{(c_1)_{m_1}\cdots(c_k)_{m_k}}\frac{x_1^{m_1}\cdots x_k^{m_k}}{m_1!\cdots m_k!}(\max\{|x_1|,|x_2|,...,|x_k|\}<\infty).\end{aligned}\tag{1.3}$$

A Lauricella function $F_C^{(k)}$ [6; p. 60] generalized the Appell function F_4 to a function of k variables which is defined as

$$\begin{aligned}F_C^{(k)}[a,b;c_1,...,c_k;x_1,...,x_k] &= F_{0:1;...;1}^{2:0;...;0}\left[\begin{array}{llll} a,b & :\_;...; & \_ & ; \\ & & & x_1,...,x_k \\ \_\_\_\_ & :c_1;...; & c_k & ; \end{array}\right]\\ &=\sum_{m_1,m_2,...,m_k=0}^{\infty}\frac{(a)_{m_1+\cdots+m_k}(b_1)_{m_1+\cdots+m_k}}{(c_1)_{m_1}\cdots(c_k)_{m_k}}\frac{x_1^{m_1}\cdots x_k^{m_k}}{m_1!\cdots m_k!}\\ &\qquad(\sqrt{|x_1|}+\cdots+\sqrt{|x_k|}<1).\end{aligned}\tag{1.4}$$

2. Integral transforms

We first establish the following integral

$$\begin{aligned}I &= \int_0^\infty u^{\nu-1}e^{-pu}M_{\lambda,\mu_1\cdots\mu_k}(x_1u,...,x_ku)du\\ &=\frac{x_1^{\mu_1+1/2}\cdots x_k^{\mu_k+1/2}\,\Gamma(b)}{(p+X)^b}F_C^{(k)}\left[a,b;2\mu_1+1,...,2\mu_k+1;\frac{x_1}{p+X},...,\frac{x_k}{p+X}\right],\end{aligned}\tag{2.1}$$

where $a=\mu_1+\cdots+\mu_k+\frac{k}{2}-\lambda$, $b=\mu_1+\cdots+\mu_k+\frac{k}{2}+\nu$, $X=\frac{x_1+\cdots+x_k}{2}$ and $Re(p+X)>0$.

For $k=2$, equation (2.1) reduces to

$$\begin{aligned}I &= \int_0^\infty u^{\nu-1}e^{-pu}M_{\lambda,\mu_1,\mu_2}(x_1u,x_2u)du\\ &=\frac{x_1^{\mu_1+1/2}x_2^{\mu_2+1/2}\,\Gamma(\mu_1+\mu_2+1+\nu)}{(p+X)^{\mu_1+\mu_2+1+\nu}}\\ &\cdot F_4\left[\mu_1+\mu_2+1-\lambda,\mu_1+\mu_2+1+\nu;2\mu_1+1,2\mu_2+1;\frac{x_1}{p+X},\frac{x_2}{p+X}\right]\\ &\qquad(Re(p+X)>0)\end{aligned}\tag{2.2}$$

where as for $k=1$, equation (2.1) reduces to a known result [1; p. 215(11)].

Proof of result (2.1)

By using definitions (1.2) and (1.3), the L.H.S. of integral (2.1) is given by

$$
\begin{aligned}
I &= \int_0^\infty u^{\nu-1} e^{-pu} (x_1 u)^{\mu_1+1/2} \cdots (x_k u)^{\mu_k+1/2} \exp\left[\frac{-1}{2}(x_1 u + \cdots + x_k u)\right] \\
&\quad \cdot \Psi_2^{(k)}[\mu_1 + \cdots + \mu_k - \lambda + k/2; 2\mu_1 + 1, ..., 2\mu_k + 1; x_1 u, ..., x_k u] du \\
&= x_1^{\mu_1+1/2} \cdots x_k^{\mu_k+1/2} \sum_{m_1 \cdots m_k = 0}^{\infty} \frac{(a)_{m_1+\cdots+m_k}}{(2\mu_1+1)_{m_1} \cdots (2\mu_k+1)_{m_k}} \frac{(x_1)^{m_1} \cdots (x_k)^{m_k}}{m_1! \cdots m_k!} \\
&\quad \cdot \int_0^\infty u^{\mu_1+\cdots+\mu_k+k/2+m_1+\cdots+m_k+\nu-1} e^{-(p+\frac{x_1+\cdots+x_k}{2})u} du,
\end{aligned}
$$

and then, using the integral transform [1, p.137(1)], we get the main result (2.1).

In view of integral (2.1), we can also establish the following integral

$$
\begin{aligned}
I &= \int_0^\infty u^{\nu-1} e^{-pu-zu^2} M_{\lambda,\mu_1\cdots\mu_k}(x_1 u, ..., x_k u) du \\
&= x_1^{\mu_1+1/2} \cdots x_k^{\mu_k+1/2} \sum_{q=0}^{\infty} \frac{(-1)^q z^q \Gamma(b+2q)}{q!(p+X)^{b+2q}} \\
&\quad \cdot F_C^{(k)}\left[a, b+2q; 2\mu_1+1, ..., 2\mu_k+1; \frac{x_1}{p+X}, ..., \frac{x_k}{p+X}\right].
\end{aligned} \tag{2.3}
$$

where $a = \mu_1 + \cdots + \mu_k + \frac{k}{2} - \lambda$, $b = \mu_1 + \cdots + \mu_k + \dfrac{k}{2} + \nu$, $X = \dfrac{x_1 + \cdots + x_k}{2}$ and $Re(p+X) > 0$.

Clearly, if $z = 0$ in (2.3), the above result reduces to the result (2.1).

3. Generating relations

A well known modified generating relation of Exton is given by [5; p. 147(3)]

$$
exp\left(s + t - \frac{xt}{s}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m\ t^n}{m!\ n!} {}_1F_1\left(-n;\ m+1\ ; x\right) \tag{3.1}
$$

where ${}_1F_1\left(-n;\ m+1\ ; x\right)/m!n! = L_n^{(m)}(x)/(m+n)!$, ${}_1F_1$ and $L_n^{(m)}$ are confluent hypergeometric function and Laguerre polynomials, respectively (see [4; p. 200(1)]). Pathan and Yasmeen [7] modified the above result (3.1) of Exton, by defining $m^* = max[0, -m]$ and

$$
\begin{aligned}
\frac{L_n^{(n)}(x)}{(m+n)!} &= \frac{1}{n!} \sum_{r=m^*}^{n} \frac{(-n)_r x^r}{(m+r)!r!} \quad \text{if} \quad n \geq m^* \\
&= 0 \quad \text{if} \quad 0 \leq n < m^*.
\end{aligned}
$$

A set of expansions

$$
x^r = 2^r \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(-n)_r (x/2)^{m+n}}{m!n!} {}_1F_1\left(-n;\ m+1\ ; x\right) \tag{3.2}
$$

for $r = 0, 1, 2, ...,$ has been obtained by Exton [5, p. 148(8)] from (3.1) by taking successive partial derivatives with respect to t and letting $s = t = x/2$.

On replacing s, t and x by su, tu and xu, respectively, in (3.1), multiplying both sides by

$$u^{\nu-1}e^{-pu}M_{\lambda,\mu_1\cdots\mu_k}(x_1u, ..., x_ku)$$

integrating the multiple series with respect to u between the limits zero and infinity. Using integral (2.1) and definition (1.2) and adjusting the parameters, we get

$$\begin{aligned}(3.3)\quad & F_C^{(k)}[a, b;\ 2\mu_1{+}1, ..., 2\mu_k{+}1; \frac{x_1}{p{-}s{-}t+\frac{xt}{s}{+}X}, ..., \frac{x_k}{p{-}s{-}t{+}\frac{xt}{s}{+}X}] \\ &= \frac{(p{-}s{-}t{+}\frac{xt}{s}{+}X)^b}{(p{+}X)^b}\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m\ t^n(b)_{m+n}}{m!\ n!(p{+}X)^{m+n}} \\ &\sum_{l=0}^{n}\frac{(-n)_l\ (b{+}m{+}n)_l}{(m{+}1)_l\ l!}\left(\frac{x}{p{+}X}\right)^l \\ &F_C^{(k)}\left[a, b{+}m{+}n+l; 2\mu_1{+}1, ..., 2\mu_k{+}1; \frac{x_1}{p{+}X}, ..., \frac{x_k}{p{+}X}\right]\end{aligned}$$

where $a = \mu_1 + \cdots + \mu_k + \frac{k}{2} - \lambda$, $b = \mu_1 + \cdots + \mu_k + \frac{k}{2} + \nu$, $X = \frac{x_1 + \cdots + x_k}{2}$ and $Re(p) > 0$, $Re(p + X) > 0$.

Similarly, using the result (3.2) in place of (3.1), we get

$$\begin{aligned}(3.4)\quad & F_C^{(k)}[a, b + r;\ 2\mu_1 + 1, ..., 2\mu_k + 1; \frac{x_1}{p + X}, ..., \frac{x_k}{p + X}] \\ &= \frac{2^r}{(b)_rx^r}\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(-n)_r(b)_{m+n}(x/2)^{m+n}}{m!n!(p + X)^{m+n-r}} \\ &\cdot\sum_{l=0}^{n}\frac{(-n)_l\ (b + m + n)_l}{(m + 1)_l\ l!}\left(\frac{x}{p + X}\right)^l \\ &\cdot F_C^{(k)}\left[a, b + m + n + l\ ; 2\mu_1 + 1, ..., 2\mu_k + 1; \frac{x_1}{p + X}, ..., \frac{x_k}{p + X}\right]\end{aligned}$$

where $a = \mu_1 + \cdots + \mu_k + \frac{k}{2} - \lambda$, $b = \mu_1 + \cdots + \mu_k + \frac{k}{2} + \nu$, $X = \frac{x_1 + \cdots + x_k}{2}$ and $Re(p) > 0$, $Re(p + X) > 0$.

By setting $s = t = x/2$ in (3.3), the result reduces to

$$\begin{aligned}(3.5)\quad & F_C^{(k)}[a, b;\ 2\mu_1 + 1, ..., 2\mu_k + 1; \tfrac{x_1}{p+X}, ..., \tfrac{x_k}{p+X}] \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(x/2)^{m+n}(b)_{m+n}}{m!\ n!(p + X)^{m+n}}\sum_{l=0}^{n}\frac{(-n)_l\ (b + m + n)_l}{(m + 1)_l\ l!}\left(\frac{x}{p + X}\right)^l \\ &\cdot F_C^{(k)}\left[a, b + m + n + l; 2\mu_1 + 1, ..., 2\mu_k + 1; \frac{x_1}{p + X}, ..., \frac{x_k}{p + X}\right]\end{aligned}$$

where $a=\mu_1+\cdots+\mu_k+\frac{k}{2}-\lambda$, $b=\mu_1+\cdots+\mu_k+\frac{k}{2}+\nu$, $X=\frac{x_1+\cdots+x_k}{2}$ and $Re(p)>0$, $Re(p+X)>0$. which can also be obtained from (3.4) by taking $r\ =\ 0$.

4. Special cases

On setting $x\ =\ 0$ in (3.3), the result reduces to

$$\begin{aligned}&F_C^{(k)}[a,b;2\mu_1+1,...,2\mu_k+1;\frac{x_1}{p-s-t+X},...,...,\frac{x_k}{p-s-t+X}]\\ &=\frac{(p-s-t+X)^b}{(p+X)^b}\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m\ t^n(b)_{m+n}}{m!\ n!\ (p+X)^{m+n}}\\ &\cdot F_C^{(k)}\left[a,b+m+n;2\mu_1+1,...,2\mu_k+1;\frac{x_1}{p+X},...,...,\frac{x_k}{p+X}\right]\end{aligned}\tag{4.1}$$

where $a=\mu_1+\cdots+\mu_k+\frac{k}{2}-\lambda$, $b=\mu_1+\cdots+\mu_k+\frac{k}{2}+\nu$, $X=\frac{x_1+\cdots+x_k}{2}$ and $Re(p)>0$, $Re(p+X)>0$.

For $k=2$, equation (4.1) reduces to

$$\begin{aligned}&F_4\left[a,b;\ 2\mu_1+1,2\mu_2+1;\frac{x_1}{p-s-t+X},\frac{x_2}{p-s-t+X}\right]\\ &\frac{(p-s-t+X)^b}{(p+X)^b}\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m\ t^n(b)_{m+n}}{m!\ n!\ (p+X)^{m+n}}\\ &\cdot F_4\left[a,b+m+n\ ;2\mu_1+1,2\mu_2+1;\frac{x_1}{p+X},\frac{x_2}{p+X}\right]\end{aligned}\tag{4.2}$$

where $a=\mu_1+\mu_2+1-\lambda$, $b=\mu_1+\mu_2+1+\nu$, $X=\frac{x_1+x_2}{2}$ and $Re(p)>0$, $Re(p+X)>0$.

For $k=2$, equation (3.5) reduces to

$$\begin{aligned}&F_4\left[a,b;\ 2\mu_1+1,2\mu_2+1;\frac{x_1}{p+X},\frac{x_2}{p+X}\right]\\ &=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(x/2)^{m+n}(b)_{m+n}}{m!\ n!(p+X)^{m+n}}\sum_{l=0}^{n}\frac{(-n)_l\ (b+m+n)_l}{(m+1)_l\ l!}\left(\frac{x}{p+X}\right)^l\\ &\cdot F_4\left[a,b+m+n+l\ ;2\mu_1+1,2\mu_2+1;\frac{x_1}{p+X},\frac{x_2}{p+X}\right]\end{aligned}\tag{4.3}$$

where $a=\mu_1+\mu_2+1-\lambda$, $b=\mu_1+\mu_2+1+\nu$, $X=\frac{x_1+x_2}{2}$ and $Re(p)>0$, $Re(p+X)>0$.

For $k=1$, equation (3.5) reduces to

$$
\begin{aligned}
&{}_2F_1[a,b;2\mu_1+1;\frac{x_1}{p+X}]\\
(4.4)\qquad &=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(x/2)^{m+n}(b)_{m+n}}{m!\,n!(p+X)^{m+n}}\sum_{l=0}^{n}\frac{(-n)_l\,(b+m+n)_l}{(m+1)_l\,l!}\left(\frac{x}{p+X}\right)^l\\
&\cdot{}_2F_1\left[a,b+m+n+l;2\mu_1+1;\frac{x_1}{p+X}\right]
\end{aligned}
$$

where $a=\mu_1+\frac{1}{2}-\lambda$, $b=\mu_1+\frac{1}{2}+\nu$, $X=\frac{x_1}{2}$ and $Re(p)>0$, $Re(p+X)>0$.

By putting $x_1 = 0$ and $x_2 = 0$ in (4.3), the above result reduces to a known result of Pathan and Yasmeen [7; p. 242(2.3)]

$$
(4.5)\qquad 1=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(x/2p)^{m+n}\,(c)_{m+n}}{(m+n)!}P_n^{(m,c-1)}\left(\frac{p-2x}{p}\right)
$$

where the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ is defined by [4; p. 254(1)]

$$
P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n!}\;{}_2F_1\left[\begin{array}{cc}-n,1+\alpha+\beta+n & ;\\ 1+\alpha & ;\end{array}\frac{1-x}{2}\right].
$$

References

[1] Erdelyi, A. et al., *Table of Integral Transfoms*, vol. 1, McGraw-Hill New York, 1954.

[2] Whittaker, E.T., *An expression of certain known function as generalized hypergeometric functions*, Bull. Amer. Math. Soc., 10 (1903), 125-134.

[3] Whittaker, E.T., Watson, G.N., *A Course in Modern Analysis*, 4th ed. Cambridge University Press. Cambridge, England, 1990.

[4] Rainville, E.D., *Special functions*, The Macmillan Co., New York, 1960.

[5] Exton, H., *A new generating functions for associated Laguerre polynomials and resulting expansions*, Jnanabha, 13 (1983), 147-149.

[6] Srivastava, H.M., Manocha, H.L., *A treatise on generating functions*, Halsted Press (Ellis Horwood Ltd., Chichester), John Wiley and Sons, New York, 1984.

[7] Pathan, M.A., Yasmeen, *On partly bilateral and partly unilateral generating functions*, J. Austr. Math. Soc., Ser. B., 28 (1986), 240-245.

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