$(0,2;0)$-INTERPOLATION ON THE UNIT CIRCLE

Swarnima Bahadur

Department of Mathematics & Astronomy
University of Lucknow
Lucknow 226007
India
e-mail: swarnimabahadur@ymail.com

Abstract. In this paper, we study the explicit representation and convergence of Pál-type weighted $(0,2;0)$-interpolation on two pairwise disjoint sets of nodes on the unit circle, which are obtained by projecting vertically the zeros of $\left(1-x^2\right)P_n(x)$ and $P_n'(x)$ respectively on the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial.

Keywords: Legendre polynomial, weighted $(0,2)$-interpolation, Pál-type interpolation, regularity, explicit forms, convergence.

2000 Mathematics Subject Classification: 41A05, 30E10.

1. Introduction

P. Turán initiated the study of $(0,2)$-interpolation in order to get approximate solution of the differential equation $y" + f \cdot y = 0$. The first result was published by J. Surányi and P. Turán [19] in 1955. J. Balázs [2] introduced a generalization of this problem and considered the weighted $(0,2)$-interpolation on the zeros of ultraspherical polynomial $P_n^{(\lambda)}(x)$, $\lambda > -1$. Then several mathematicians [5], [8], [9], [13], [15], [16], [22], [23], [24] considered the weighted $(0,2)$-interpolation on various set of nodes. O. Kiŝ [11] was first to consider the interpolation problem on the unit circle and then a lot of mathematicians [17], [18], [20] studied different interpolation problems on the unit circle. In paper [1], author proved the convergence theorem of weighted $(0,2)^*$-interpolation on the projected nodes on the unit circle.

In case of Pál-type interpolation, P. Mathur and S. Datta [14] considered the weighted Pál-type $(0,2;0)$ $-$interpolation on $(-\infty,\infty)$. Recently M. Lenard [12] has considered the $(0,2;0)$ and $(0;0,2)$-type interpolation problem on the zeros of Legendre polynomial $P_n(x)$ and give the explicit formulae. H.P. Dikshit [7] considered the existence of Pál-type interpolation on the non-uniformly distributed nodes on the unit circle. Later on, in [2], [3], author considered the convergence of $(0,1;0)$ and $(0;0,1)$-interpolation on the sets obtained by projecting vertically the zeros of $(1-x^2)\,P_n\,(x)$ and $P_n'\,(x)$ respectively on the unit circle, where $P_n\,(x)$ stands for n^{th} Legendre polynomial. In paper [4], author considered the $(0;0,2)$-interpolation on the projected nodes of $(1-x^2)\,P_n\,(x)$ and $P_n'\,(x)$ respectively on the unit circle and proved a convergence theorem. The aim of this paper is to consider explicit representation and convergence of weighted Pál-type $(0,2;0)$-interpolation on the unit circle.

Let $Z_n = \{z_k = k = 0,1,...,2n+1\}$ satisfying

$$\begin{cases} z_0 = 1, z_{2n+1} = -1 \\ z_k = \cos\theta_k + i\sin\theta_k, \\ z_{n+k} = -z_k,\ k = 1\,(1)\,n \end{cases} \tag{1.1}$$

and $T_n = \{\omega_k = k = 1,...,2n-2\}$ such that

$$\begin{cases} \omega_k = \cos\phi_k + i\sin\phi_k, \\ \omega_{n+k} = -\omega_k,\ k = 1\,(1)\,n-1 \end{cases} \tag{1.2}$$

be two set of nodes such that the weighted $(0,2)$-interpolation is prescribed on the one set of nodes, whereas Lagrange interpolation on the points of other one

In Section 2 we give some preliminaries, in Section 3 we describe the problem, in Section 4 we represent the explicit forms of interpolatory polynomials, and in Sections 5 and 6 we give the estimates and convergence of interpolatory polynomials.

2. Preliminaries

In this section, we shall give some well known results, which we shall use in our present paper.

The differential equation satisfied by $P_n\,(x)$ is

$$(1-x^2)\ P_n''\,(x) \quad -2x\ P_n'\,(x) + n\,(n+1)\ P_n\,(x) = 0 \tag{2.1}$$

$$W(z) \ = \prod_{k=1}^{2n} (z-z_k) = K_n P_n\left(\frac{1+z^2}{2z}\right)\ z^n \tag{2.2}$$

$$H(z) = \prod_{k=1}^{2n-2} (z-\omega_k) \quad = K_n^* P_n'\left(\frac{1+z^2}{2z}\right)\ z^{n-1} \tag{2.3}$$

We shall require the fundamental polynomials of Lagrange interpolation based on Z_n and T_n

$$L_k\ (z) = \frac{W(z)}{W'\,(z_k)\,(z-z_k)} \tag{2.4}$$

$$l_k\ (z) = \frac{H(z)}{H'(\omega_k)(z-\omega_k)} \tag{2.5}$$

$$J_k(z) = \int_0^z z^{2n+2}\ L_k\ (z)\,dz \tag{2.6}$$

$$J_{1j}(z) = \int_0^z z^{2n+j}\ W\ (z)\,dz,\ j = 0, 1, \tag{2.7}$$

which satisfies

$$J_{1j}(-1) = (-1)^{j+1}\,J_{1j}(1)\,. \tag{2.8}$$

For $-1 \le x \le 1$,

$$|P_n(x)| \le 1 \tag{2.9}$$

$$(1-x^2)^{\frac{1}{4}}\,|P_n(x)| \le \left(\frac{2}{n\pi}\right)^{\frac{1}{2}} \tag{2.10}$$

$$(1-x^2)^{\frac{3}{4}}\,|P_n'(x)| \le \sqrt{2n} \tag{2.11}$$

$$|P_n'(x)| \le \frac{n(n+1)}{2}. \tag{2.12}$$

If u_k be the zeros of $P_n'(x)$, then by [6]

$$P_n(u_k) > \frac{1}{\sqrt{8\pi k}}. \tag{2.13}$$

Let $x_k = \cos\theta_k$, $(k = 1, 2, .., n)$ be the zeros of the n^{th} Legendre polynomial $P_n(x)$, with $1 > x_1 > x_2 > ... > -1$, then

$$\begin{cases} (1-x_k^2)^2 \ge k^2 n^{-2}, & k = 1, .., \left[\frac{n}{2}\right] \\ (1-x_k^2)^2 \ge (n-k+1)^2\, n^{-2}, & k = \left[\frac{n}{2}\right]+1, .., n \end{cases} \tag{2.14}$$

$$\begin{cases} |P_n'(x_k)| \ge k^{-\frac{3}{2}} n^2, & k = 1, .., \left[\frac{n}{2}\right] \\ |P_n'(x_k)| \ge (n-k+1)^{-\frac{3}{2}}\, n^2, & k = \left[\frac{n}{2}\right]+1, .., n \end{cases} \tag{2.15}$$

For more details, see [20], [21].

3. The Problem

Let $\{z_k\}_{k=0}^{2n+1}$ and $\{\omega_k\}_{k=1}^{2n-2}$ be two disjoint set of nodes obtained by projecting vertically the zeros of $(1-x^2)\,P_n(x)$ and $P_n'(x)$ respectively on the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial and $w(z) \in \mathbb{C}$ be a given function, called weight function. Here we are interested to determine the convergence of interpolatory polynomials satisfying the conditions:

$$\tag{3.1} \begin{cases} R_n(z_k) = \alpha_k, & k = 0,1,2,..,2n+1 \\ \{w(z)\,R_n(z)\}''_{z=z_k} = \beta_k, & k = 1,2,..,2n \\ R_n(\omega_k) = \gamma_k, & k = 1,2,..,2n-2 \end{cases}$$

where $w(z) = (1-z^2)^{\frac{5}{2}}$ and α_k, β_k and γ_k are complex constants.

4. Explicit representation of interpolatory polynomials

We shall write $R_n(z)$ satisfying (3.1) as

$$\tag{4.1} R_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=1}^{2n} \beta_k B_k(z) + \sum_{k=1}^{2n-2} \gamma_k C_k(z),$$

where $A_k(z)$, $B_k(z)$ and $C_k(z)$ are unique polynomials each of degree $\leq 6n-1$ determined by the following conditions:

For $k = 0\,(1)\,2n+1$

$$\tag{4.2} \begin{cases} A_k(z_j) = \delta_{kj}, & j = 0\,(1)\,2n+1 \\ \left\{(1-z^2)^{\frac{5}{2}} A_k(z)\right\}''_{z=z_j} = 0, & j = 1\,(1)\,2n \\ A_k(\omega_j) = 0, & j = 1\,(1)\,2n-2 \end{cases}$$

For $k = 1\,(1)\,2n$

$$\tag{4.3} \begin{cases} B_k(z_j) = 0, & j = 0\,(1)\,2n+1 \\ \left\{(1-z^2)^{\frac{5}{2}} B_k(z)\right\}''_{z=z_j} = \delta_{kj}, & j = 1\,(1)\,2n \\ B_k(\omega_j) = 0, & j = 1\,(1)\,2n-2 \end{cases}$$

For $k = 1\,(1)\,2n-2$

$$\tag{4.4} \begin{cases} C_k(z_j) = 0, & j = 0\,(1)\,2n+1 \\ \left\{(1-z^2)^{\frac{5}{2}} C_k(z)\right\}''_{z=z_j} = 0, & j = 1\,(1)\,2n \\ C_k(\omega_j) = \delta_{kj}, & j = 1\,(1)\,2n-2 \end{cases}$$

Theorem 4.1. *For* $k = 1\,(1)\,2n-2$

$$C_k(z) = \frac{z^{-2n}(z^2-1)\,W^2(z)L_k(z)}{\omega_k^{-2n}(\omega_k^2-1)\,W^2(\omega_k)L_k(\omega_k)}\;l_k(z) \tag{4.5}$$
$$+\frac{z^{-2n-1}W(z)H(z)}{\omega_k^{-2n}(\omega_k^2-1)\,W^2(\omega_k)H'(\omega_k)\,L_k(\omega_k)}\left\{G_k(z)+c_1J_{10}(z)+c_2J_{11}(z)\right\}$$

where

$$G_k(z) = \int_0^z \left(1-z^2\right)\frac{zW'(z)+c_kW(z)}{z-\omega_k}\;L_k\;(z)\,dz \tag{4.6}$$

with

$$c_k = -\omega_k\frac{W'(\omega_k)}{W(\omega_k)}$$
$$c_1 = -\frac{G_k(1)-G_k(-1)}{2J_{10}(1)} \quad \textit{and} \tag{4.7}$$
$$c_2 = -\frac{G_k(1)+G_k(-1)}{2J_{11}(1)}.$$

Proof. Consider (4.5).

Clearly, $C_k(z_j)=0$ for $j=1\,(1)\,2n$ and $C_k(\omega_j)=\delta_{kj}$ for $j=1\,(1)\,2n-2$.

Also, for $z=\pm1$, we get (4.7).

Further, from $\left\{(1-z^2)^{\frac{5}{2}}\,C_k(z)\right\}''_{z=z_j} = 0$ for $j = 1\,(1)\,2n$, we get $G_k'(z)$, which on integration gives (4.6).

Theorem 4.2. *For* $k = 1\,(1)\,2n$

$$B_k(z) = z^{-2n-1}W(z)H\;(z)\left\{b_kJ_k(z)+b_1J_{10}(z)+b_2J_{11}(z)\right\} \tag{4.8}$$

where

$$b_k = \frac{1}{2z_k\,(z_k^2-1)^{\frac{5}{2}}\,W'(z_k)H\;(z_k)} \tag{4.9}$$

$$b_1 = -b_k\frac{J_k(1)-J_k(-1)}{2J_{10}(1)} \quad \textit{and} \quad b_2 = -b_k\frac{J_k(1)+J_k(-1)}{2J_{11}(1)}. \tag{4.10}$$

Proof. Consider (4.8).

Clearly, $B_k(z_j) = 0$ for $j = 1(1)2n$ and $B_k(\omega_j) = \delta_{kj}$ for $j = 1(1)2n-2$.

Again, for $z = \pm 1$, we get (4.10).

Further, from $\left\{(1-z^2)^{\frac{5}{2}} B_k(z)\right\}''_{z=z_j} = \delta_{kj}$ for $j = 1(1)2n$, we get (4.9) for $j = k$, owing to (2.7) and (2.14)-(2.15).

For $j \neq k$, one can verify the result.

Theorem 4.3. *For $k = 1(1)2n$,*

$$\begin{aligned} A_k(z) &= \frac{(z^2-1)\,H(z)L_k^2\ (z)}{(z_k^2-1)\,H(z_k)} \\ &+\frac{z^{-2n-1}W(z)H\ \ (z)}{(z_k^2-1)\,W'(\omega_k)H(z_k)}\left\{T_k(z)+a_1J_{10}(z)+a_2J_{11}(z)\right\}+a_kB_k(z) \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} a_k \;=\; &-\left(z_k^2-1\right)^{\frac{5}{2}}\left\{\frac{H''(z_k)}{H'(z_k)}+\frac{2z_k}{(z_k^2-1)}\frac{H'(z_k)}{H(z_k)}+\frac{35z_k}{\left(z_k^2-1\right)^2}\right. \\ &\left.+\frac{7}{(z_k^2-1)}+\frac{28z_k}{(z_k^2-1)}L_k'(z_k)+4\frac{H'(z_k)}{H(z_k)}L_k'(z_k)\right\} \end{aligned} \tag{4.12}$$

$$T_k(z) = \int_0^z z^{2n+1}\left(1-z^2\right)\frac{L_k\ \ (z)-L_k'(z_k)\,L_k(z)}{z-z_k}\,dz \tag{4.13}$$

$$a_1 = -\frac{T_k(1)-T_k(-1)}{2J_{10}(1)} \quad \textit{and} \quad a_2 = -\frac{T_k(1)+T_k(-1)}{2J_{11}(1)}. \tag{4.14}$$

Further, for $k = 0, 2n+1$

$$A_k(z) = z^{-2n-1}W(z)H\ \ (z)\left\{a_1J_{10}(z)+a_2J_{11}(z)\right\} \tag{4.15}$$

where

$$a_1 = \frac{1}{2W(1)H(1)\,J_{10}(1)} \tag{4.16}$$

$$a_2 = \frac{1}{2W(1)H(1)J_{11}(z)}. \tag{4.17}$$

Proof. One can check that in this theorem (4.11) and (4.15) are polynomials of required degree satisfying (4.2), so we omit the details of proof.

5. Estimation of interpolatory polynomials

For convergence of interpolatory polynomial, we need the estimates of fundamental polynomials:

Lemma 1. *Let $A_k(z)$ be defined in* (4.11)*, then for* $|z| \leq 1$*, we have*

$$\sum_{k=1}^{2n} |A_k(z)| \leq cn^{\frac{3}{2}} \log n \tag{5.1}$$

$$|A_0(z)| \leq c, \quad |A_{2n+1}(z)| \leq c, \tag{5.2}$$

where c is a constant independent of n and z.

Lemma 2. *Let $B_k(z)$ be defined in* (4.8)*, then for* $|z| \leq 1$*, we have*

$$\sum_{k=1}^{2n-2} |B_k(z)| \leq c\frac{\log n}{n^{\frac{1}{2}}} \tag{5.3}$$

where c is a constant independent of n and z.

Lemma 3. *Let $C_k(z)$ be defined in* (4.5)*, then for* $|z| \leq 1$*, we have*

$$\sum_{k=1}^{2n-2} |C_k(z)| \leq cn^{\frac{3}{2}} \log n \tag{5.4}$$

where c is a constant independent of n and z.

Proof of Lemma 3. Consider (4.5) and using (2.10)-(2.15), we get (5.4), owing to results from [20], [21]. So we can omit the details of the proof.

Proof of Lemma 2. Consider (4.8), we have

$$|B_k(z)| \leq |P_n(x) P_n'(x)| \{|b_k| |J_k(z)| + |b_1| |J_{10}(z)| + |b_2| |J_{11}(z)|\}, \tag{5.5}$$

where

$$|b_k| \leq \frac{1}{(1-x_k^2)^{\frac{7}{4}} |P_n'(x_k)|^2} \tag{5.6}$$

$$|J_k(z)| \leq \max_{|z|\leq 1} |L_k(z)| \int_0^z t^{2n+2} dt, \tag{5.7}$$

by substituting $z = e^{i\theta}$ $(0 \leq \theta < 2\pi)$.

Combining (5.5)-(5.8) and using (2.10)-(2.15), we get (5.3).

Similarly one can prove Lemma 1.

6. Convergence

In this section we prove the following:

Theorem 6.1. *Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Let the arbitrary numbers β_k's be such that*

$$|\beta_k| = o\left(n^{\frac{3}{2}}\omega_2(f, n^{-1})\right), \quad k = 1\,(1)\,2n. \tag{6.1}$$

Then the sequence $\{R_n\}$ be defined by

$$R_n(z) = \sum_{k=0}^{2n+1} f(z_k)A_k(z) + \sum_{k=1}^{2n} \beta_k B_k(z) + \sum_{k=1}^{2n-2} \gamma_k C_k(z) \tag{6.2}$$

satisfies the relation

$$|\,R_n(z) - f(z)| = o\left(n^{\frac{3}{2}} \log n.\omega_2(f, n^{-1})\right) \tag{6.3}$$

where $\omega_2(f, n^{-1})$ is the modulus of smoothness of $f(z)$.

Remark. Let $f(z)$ be continuous in $|z| \leq 1$ and $f' \in$ *Lip* α. $\alpha > \frac{1}{2}$, then the sequence $\{R_n\}$ converges uniformly to $f(z)$ in $|z| \leq 1$, follows from (6.3) provided

$$\omega_2(f, n^{-1}) = o\left(n^{-1-\alpha}\right). \tag{6.4}$$

To prove Theorem 6.1, we shall need the following:

Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Then there exists a polynomial $F_n(z)$ of degree $2n-2$ satisfying Jackson's inequality

$$|f(z) - F_n(z)| \leq c\omega_2\left(f, n^{-1}\right), \quad z = e^{i\theta}\,(0 \leq \theta < 2\pi) \tag{6.5}$$

and also an inequality due to O. Kiš [11] *viz.* :

$$F_n^{(m)}(z) \leq cn^m \omega_2\left(f, \frac{1}{n}\right), \tag{6.6}$$

for positive integer m.

Proof. Let $z = e^{i\theta}$ $(0 \leq \theta < 2\pi)$, using (6.1), (6.2), (6.4)–(6.6) and Lemmas 1, 2 and 3, we get the result.

References

[1] Bahadur, S., Mathur, K.K., *Weighted* $(0,2)^*-$ *interpolation on the unit circle*, Int. J. Appl. Math. Stat., 20 (M11) (2011), 73-78.

[2] Swarnima Bahadur, *Pál-type* $(0,1;0)$*-interpolation on the unit circle*, Advances in Theo. and Appl. Math., 6 (1) 2011, 35-39.

[3] Swarnima Bahadur, *Pál-type* $(0;0,1)-$*interpolation on the unit circle*, Int. Journal of Math. Analysis, vol. 5 (2011), no. 29, 1429-1434

[4] Swarnima Bahadur, *Pál-type* $(0;0,2)-$*interpolation on the unit circle*, Int. J. Math. Sci., vol. 11, no. 1-2 (2012), 103-111.

[5] Eneduanya, S.A.N, *The weighted* $(0,2)$ *interpolation I*, Demonstracio Mathematica, 18 (1) (1985), 9-13.

[6] Balázs , J., Turán , P., *Notes on Interpolation* III, Acta Math. Sci. Hungar., 9 (1958), 195-214.

[7] Baláz s, J., *Súlyozott* $(0,2)$*-interpoláció ultraszférikus polinomok gyôkein*, MTA III. oszt. Közl., 11 (1961), 305-338.

[8] Dikshit, H.P., *Pál-type interpolation on non-uniformly distributed nodes on the unit circle, Journal of computational and Applied Mathematics*, 155 (2) (15) (2003), 253-261.

[9] Joó, I., *On weighted* $(0,2)$ *interpolation*, Annales Univ. Sci. Budapest., Sect. Math., 38 (1995), 185-222.

[10] Joó, I., Szili, L., *On weighted* $(0,2)$*-interpolation on the roots of Jacobi polynomials*, Acta Math. Hung., 66 (1-2) (1995), 25-50.

[11] Kiŝ, O., *Remarks on interpolation* (Russian), Acta Math. Sci. Hungar., 11 (1961), 49-64.

[12] Lénard, Margit, *On weighted* (0.2)- *type interpolation*, Elect. Trans. Num. Anal., 25 (2006), 206-223.

[13] Lénard, Margit, *Weighted* $(0,2)$*-Interpolation with interpolatory boundary conditions*, Annales Univ. Sci. Budapest., Sect. Comp. 24 (2004), 253-273.

[14] Mathur, P., Datta, S., *On pal type weighted lacunary* $(0,2;0)$*-interpolation on the infinite interval* $(-\infty,\infty)$, Approx. Theo. & its Appl., 17 (4) (2001), 1-10.

[15] Prasad, J., *Balázs -type interpolation on Laguerre abscissas*, Math. Japonica, 13 (1967), 47-53.

[16] Prasad, J., *On the weighted* $(0,2)$*-interpolation*, SIAM J. Numer. Anal., 7 (1970), 428-446.

[17] Sharma, A., *Lacunary interpolation in the roots of unity*, Z. Angew. Math., 46 (1964), 41-49.

[18] Sharma, A., Riemanschneider, S.D., *Birkhoff interpolation at the n^{th} roots of unity, Convergence*, Cand. J. Math., 33 (2) (1981), 362-371.

[19] Su rányi, J., Turán, P., *Notes on interpolation. I. On some interpolatorical properties of the ultraspherical polynomials*, Acta Math. Acad. Sci. Hung., 6 (1955), 66-79.

[20] Xie, S., $(0,1,3)^*$*-interpolation on the unit circle*, Acta Math. Sinica, 39 (5) (1996), 690-700.

[21] Szegö, G., *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ., New York, 1959.

[22] Szili, L., *Weighted* $(0,2)$*-interpolation on the roots of Hermite polynomials*, Annales Univ. Sci. Budapest., Sect. Math., 27 (1985), 153-166.

[23] Szili, L., *Weighted* $(0,2)$ *interpolation on the roots of the classical orthogonal polynomials*, Bull. Allahabad Math. Soc., 8-9 (1993-94), 111-120.

[24] Szili, L., *A survey on* $(0,2)$ *interpolation*, Annales Univ. Sci. Budapest., Sect. Comp., 16 (1996), 377-390.

Accepted: 10.12.2011