SOME CLASSES OF GENERALIZED MinMax POLYNOMIALS

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Abstract. The so-called MinMax numbers $\{M_n\}$ and their subsidiary numbers $\{N_n\}$ for the Pell numbers $\{P_n\}$ were studied by (for example) A.F. Horadam [3]. These MinMax numbers $\{M_n\}$ are positive integers which are the minimal and maximal representations by means of the Pell numbers. Analogous results for the MinMax numbers $\{D_n\}$ and their subsidiary numbers $\{R_n\}$, and for the modified Pell numbers $\{q_n\}$, are obtained in [3], $Q_n := 2q_n$ being the Pell-Lucas numbers. A.F. Horadam [4], on the other hand, expanded this MinMax number system to the algebraic polynomials $\{M_n(x)\}$, $\{N_n(x)\}$, $\{D_n(x)\}$ and $\{R_n(x)\}$. Our aim in this paper is to investigate results, which are similar to those in [4], but which hold true instead for the following generalized sequences of polynomials:

$$\{P_{n,m}(x)\} \qquad \text{and} \qquad \{Q_{n,m}(x)\} \qquad (m \in \mathbb{N};\ n \in \mathbb{N} \cup \{0\}),$$

$\mathbb{N}$ being the set of positive integers.

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1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of *positive* integers. We also write

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, ...\}.$$

The generalized Pell polynomials $\{P_{n,m}(x)\}$, and the generalized Pell-Lucas polynomials $\{Q_{n,m}(x)\}$ and $\{q^*_{n,m}(x)\}$, are defined by (see [1])

$$P_{n,m}(x) = 2xP_{n-1,m}(x) + P_{n-m,m}(x) \qquad (n \geqq m;\ m, n \in \mathbb{N}_0) \tag{1.1}$$

with $P_{0,m}(x) = 0$ and $P_{n,m}(x) = (2x)^{n-1}$ $(n = 1, ..., m-1)$;

$$Q_{n,m}(x) = 2xQ_{n-1,m}(x) + Q_{n-m,m}(x) \qquad (n \geqq m;\ m, n \in \mathbb{N}_0) \tag{1.2}$$

with $Q_{0,m}(x) = 2$ and $Q_{n,m}(x) = (2x)^n$ $(n = 1, ..., m-1;\ m \in \mathbb{N} \setminus \{1\};\ n \in \mathbb{N})$ and

$$q^*_{n,m}(x) = 2xq^*_{n-1,m}(x) + q^*_{n-2m,m}(x) \qquad (n \geqq 2m;\ m, n \in \mathbb{N}_0) \tag{1.3}$$

with $q^*_{0,m}(x) = 1$ and $q^*_{n,m}(x) = (2x)^{n-1}$ $(n = 1, ..., 2m-1;\ m, n \in \mathbb{N})$.

In their special case when $m = 2$, the polynomials

$$P_{n,2}\left(\frac{x}{2}\right) \qquad \text{and} \qquad Q_{n,2}\left(\frac{x}{2}\right)$$

are the same as the familiar Fibonacci and Lucas polynomials, respectively (see [2]; see also [6]).

By using the recurrence relations (1.1), (1.2) and (1.3), and their corresponding initial values, we can compute the first few members of the polynomials

$$\{P_{n,2}(x) \equiv P_n(x)\}, \quad \{Q_{n,2}(x) \equiv Q_n(x)\} \quad \text{and} \quad \{q^*_{n,1}(x) \equiv q^*_n(x)\},$$

which are given in Table 1, Table 2 and Table 3 below.

Table 1

$$\begin{aligned}
&P_0(x) = 0\\
&P_1(x) = 1\\
&P_2(x) = 2x\\
&P_3(x) = 4x^2 + 1\\
&P_4(x) = 8x^3 + 4x\\
&P_5(x) = 16x^4 + 12x^2 + 1\\
&P_6(x) = 32x^5 + 32x^3 + 6x\\
&P_7(x) = 64x^6 + 80x^4 + 24x^2 + 1\\
&P_8(x) = 126x^7 + 192x^5 + 80x^3 + 8x\\
&P_9(x) = 256x^8 + 448x^6 + 240x^4 + 40x^2 + 1\\
&P_{10}(x) = 512x^9 + 1024x^7 + 672x^5 + 160x^3 + 10x.
\end{aligned}$$

Table 2

$$Q_0(x) = 2$$
$$Q_1(x) = 2x$$
$$Q_2(x) = 4x^2 + 2$$
$$Q_3(x) = 8x^3 + 6x$$
$$Q_4(x) = 16x^4 + 16x^2 + 2$$
$$Q_5(x) = 32x^5 + 40x^3 + 10x$$
$$Q_6(x) = 64x^6 + 96x^4 + 36x^2 + 2$$
$$Q_7(x) = 128x^7 + 224x^5 + 112x^3 + 14x$$
$$Q_8(x) = 256x^8 + 512x^6 + 320x^4 + 64x^2 + 2$$
$$Q_9(x) = 512x^9 + 1152x^7 + 864x^5 + 240x^3 + 18x$$
$$Q_{10}(x) = 1024x^{10} + 2560x^8 + 2240x^6 + 800x^4 + 100x^2 + 2.$$

Table 3

$$q_0^*(x) = 1$$
$$q_1^*(x) = 1$$
$$q_2^*(x) = 2x + 1$$
$$q_3^*(x) = 4x^2 + 2x + 1$$
$$q_4^*(x) = 8x^3 + 4x^2 + 4x + 1$$
$$q_5^*(x) = 16x^4 + 8x^3 + 12x^2 + 4x + 1$$
$$q_6^*(x) = 32x^5 + 16x^4 + 32x^3 + 12x^2 + 6x + 1$$
$$q_7^*(x) = 64x^6 + 32x^5 + 80x^4 + 32x^3 + 24x^2 + 6x + 1$$
$$q_8^*(x) = 128x^7 + 64x^6 + 192x^5 + 80x^4 + 80x^3 + 24x^2 + 8x + 1$$
$$q_9^*(x) = 256x^8 + 128x^7 + 448x^6 + 192x^5 + 240x^4 + 80x^3 + 40x^2 + 8x + 1.$$

Our present investigation is motivated essentially by the earlier works of Horadam (see, for details, [3]) dealing with the so-called MinMax numbers $\{M_n\}$ and their subsidiary numbers $\{N_n\}$ for Pell numbers $\{P_n\}$, and also with the MinMax numbers $\{D_n\}$ and their subsidiary numbers $\{R_n\}$ and the modified Pell numbers $\{q_n\}$, $Q_n := 2q_n$ being the Pell-Lucas numbers. These MinMax numbers $\{M_n\}$ are positive integers which are the minimal and maximal representations by means of the Pell numbers. Analogous results for the MinMax numbers $\{D_n\}$ and their subsidiary numbers $\{R_n\}$, and also for the modified Pell numbers $\{q_n\}$. Subsequently, Horadam [4] expanded this MinMax number system to the algebraic polynomials $\{M_n(x)\}$, $\{N_n(x)\}$, $\{D_n(x)\}$ and $\{R_n(x)\}$. The main object of this paper is to present results, which are similar to those in [4], but which hold true instead for the following generalized sequences of polynomials:

$$\{P_{n,m}(x)\} \qquad \text{and} \qquad \{Q_{n,m}(x)\} \qquad (m \in \mathbb{N};\ n \in \mathbb{N} \cup \{0\}),$$

which are defined here by (1.1) and (1.2), respectively.

2. The MinMax polynomials $\{M_{n,m}(x)\}$

The polynomials $\{M_{n,m}(x)\}$ $(m \in \mathbb{N};\ n \in \mathbb{N}_0)$ are defined by means of the following recurrence relation:

$$M_{n,m}(x) = 2xM_{n-1,m}(x) + M_{n-2m,m}(x) + l \qquad (n \geqq 2m;\ m,n \in \mathbb{N}) \tag{2.1}$$

with $M_{0,m}(x) = 0$ and $M_{n,m}(x) = (2x)^{n-1}$ $(n = 1, ..., m)$ and $M_{m+1,m}(x) = (2x)^m + 1$ and $M_{n,m}(x) = 2xM_{n-1,m}(x)$ $(n = m+2, ..., 2m-1)$, where

$$l = \begin{cases} 0 & (n \neq mk+1;\ k \in \mathbb{N}_0) \\ 1 & (n = mk+1). \end{cases}$$

One of our main results is asserted by Theorem 1 below.

Theorem 1. *For $m \in \mathbb{N}$ and $n \geqq m$, each of the following equalities holds true*:

$$M_{n,m}(x) = \sum_{i=0}^{[n/m]} P_{n-mi,2m}(x) \tag{2.2}$$

and

$$M_{n,m}(x) = \frac{P_{n+1,2m}(x) + P_{n+1-m,2m}(x) - l}{2x}, \tag{2.3}$$

where

$$l = \begin{cases} 0 & (n \neq mk;\ k \in \mathbb{N}_0) \\ 1 & (n = mk). \end{cases}$$

Proof. We make use of the principle of mathematical induction on n in order to prove equalities (2.2) and (2.3). First of all, it is easy to prove equality (2.2) for

$$n = 0, 1, ..., 2m-1.$$

Suppose now that (2.2) holds true for $n = mn$ $(n \in \mathbb{N})$. Then, for $n = mn+1$, we apply the recurrence relation (2.1) to get

$$\begin{aligned} M_{mn+1,m}(x) &= 2xM_{mn,m}(x) + M_{mn+1-2m,m}(x) + 1 \\ &= 2x\sum_{i=0}^{n} P_{mn-mi,2m}(x) + \sum_{i=0}^{n-2} P_{mn+1-2m-mi,2m}(x) + 1 \\ &= \sum_{i=0}^{n-2}(2xP_{mn-mi,2m} + P_{mn+1-2m-mi,2m}) \\ &\qquad + 2x(P_{m,2m}(x) + P_{0,2m}(x)) + 1 \\ &= \sum_{i=0}^{n-2} P_{mn+1-mi,2m}(x) + P_{m+1,2m}(x) + P_{1,2m}(x) \\ &= \sum_{i=0}^{n} P_{mn+1-mi,2m}(x), \end{aligned}$$

which proves the the first assertion (2.2) of Theorem 1.

Next, clearly, equality (2.3) holds true for $n = 0, 1, \cdots, 2m-1$, by means of the recurrence relations (1.1) and (2.1). Suppose that (2.3) holds true for $n = mn$. Then, for $n = mn + 1$, we find that

$$\begin{aligned} M_{mn+1,m}(x) &= 2xM_{mn,m}(x) + M_{mn+1-2m,m}(x) + 1 \\ &= 2x\frac{P_{mn+1,2m}(x) + P_{mn+1-m,2m}(x) - 1}{2x} \\ &\qquad + \frac{P_{mn+2-2m,2m}(x) + P_{mn+2-3m,2m}(x)}{2x} + 1 \\ &= \frac{P_{mn+2,2m}(x) + P_{mn+2-m,2m}(x)}{2x}, \end{aligned}$$

which, by the principle of mathematical induction on n, proves the second assertion (2.3) of Theorem 1. Our proof of Theorem 1 is thus completed. ∎

3. The polynomials $\{N_{n,m}(x)\}$

The polynomials $\{N_{n,m}(x)\}$ $(m \in \mathbb{N};\ n \in \mathbb{N}_0)$ are defined by

$$N_{n,m}(x) = M_{n+1,m}(x) + M_{n+1-2m,m}(x) \qquad (n \geqq 2m), \tag{3.1}$$

which, in conjunction with (2.1), shows that the polynomials $\{N_{n,m}(x)\}$ satisfy the following recurrence relation:

$$N_{n,m}(x) = 2xN_{n-1,m}(x) + N_{n-2m,m}(x) + l \tag{3.2}$$

with $N_{0,m}(x) = 1$ and $N_{n,m}(x) = (2x)^n$ $(n = 1, ..., m-1)$ and $N_{m,m}(x) = (2x)^m + 1$ and $N_{n,m}(x) = 2xN_{n-1,m}(x)$ $(n = m+1, ..., 2m-1)$, where

$$l = \begin{cases} 0 & (n \neq mk;\ k \in \mathbb{N}_0) \\ 2 & (n = mk). \end{cases}$$

Theorem 2. *The polynomials* $\{N_{n,m}(x)\}$ *satisfy the following relationship*:

$$N_{n,m}(x) = \frac{Q_{n+1,2m} + Q_{n+1-m,2m}(x) - l}{2x}, \tag{3.3}$$

where

$$l = \begin{cases} 0 & (n \neq mk + m - 1;\ k \in \mathbb{N}_0) \\ 2 & (n = mk + m - 1). \end{cases}$$

Proof. By using (1.2) and (3.2), we readily obtain

$$N_{m-1,m}(x) = \frac{Q_{m,2m}(x) + Q_{0,2m}(x) - 2}{2x},$$

which shows that relationship (3.3) holds true for $n = m - 1$. If we assume that (3.3) is holds true for $n \neq mk$, then, for $n = mk$, we find by using the recurrence relation (3.2) that

$$\begin{aligned} N_{mk,m}(x) &= 2xN_{mk-1,m}(x) + N_{mk-2m,m}(x) + 2 \\ &= 2x\left(\frac{Q_{mk,2m}(x) + Q_{mk-m,2m}(x) - 2}{2x}\right) \\ &\qquad + \frac{Q_{mk+1-2m,2m}(x) + Q_{mk+1-3m,2m}(x)}{2x} + 2 \\ &= \frac{Q_{mk+1,2m}(x) + Q_{mk+1-m,2m}(x)}{2x}, \end{aligned}$$

which evidently completes our proof of Theorem 2. ∎

By some suitable algebraic manipulations based upon (3.3), we get the following result.

Theorem 3. *For* $m \in \mathbb{N}$ *and* $n \geqq m$, *the following relations hold true*:

$$N_{n,m}(x) - N_{n-m,m}(x) = Q_{n,2m}(x) \tag{3.4}$$

and

$$N_{n,m}(x) - N_{n-2m,m}(x) = Q_{n,2m}(x) + Q_{n-m,2m}(x). \tag{3.5}$$

Proof. Upon setting $n = m$ in (3.4), we get

$$N_{m,m}(x) - N_{0,m}(x) = Q_{m,2m}(x),$$

which, by virtue of (3.2) and Table 2, coincides with the identity:

$$(2x)^m + 1 - 1 = (2x)^m.$$

If (3.4) holds true for some fixed positive integer n ($n \geqq m$), then

$$\begin{aligned} N_{n+1,m}(x) &- N_{n+1-m,m}(x) \\ &= 2xN_{n,m}(x) + N_{n+1-2m,m}(x) \\ &\qquad - 2xN_{n-m,m}(x) - N_{n+1-3m,m}(x) \\ &= 2x\left[N_{n,m}(x) - N_{n-m,m}(x)\right] \\ &\qquad + \left[N_{n+1-2m,m}(x) - N_{n+1-3m,m}(x)\right] \\ &= 2xQ_{n,2m}(x) + Q_{n+1-2m,2m}(x) \\ &= Q_{n+1,2m}(x), \end{aligned} \tag{3.6}$$

which implies that relation (3.4) is holds true when n is replaced by $n+1$.

The second assertion (3.5) of Theorem 3 is a *direct* consequence of the first assertion (3.4). ∎

4. The subsidiary MinMax polynomials $\{D_{n,m}(x)\}$

Instead of the MinMax polynomials for the generalized Pell polynomials $\{P_{n,m}(x)\}$ defined by (1.1), we now consider the analogous polynomials for the polynomials $\{Q_{n,m}(x)\}$ and $\{q^*_{n,m}(x)\}$, which are defined by means of the recurrence relations (1.2) and (1.3), respectively. We thus define the MinMax polynomials $\{D_{n,m}(x)\}_{n\in\mathbb{N}_0}$ by means of the following recurrence relation:

$$D_{n,m}(x) = 2xD_{n-1,m}(x) + D_{n-2m,m}(x) + l \qquad (n \geqq 2m;\ m,n \in \mathbb{N}) \tag{4.1}$$

with $D_{0,m}(x) = 0$ and $D_{n,m}(x) = (2x)^{n-1}$ ($n = 1, \cdots, m$) and $D_{m+1,m}(x) = (2x)^m + 2$ and $D_{n,m}(x) = 2xD_{n-1,m}(x)$ ($n = m+2, \cdots, 2m-1$), where

$$l = \begin{cases} 0 & (n \neq mk+1;\ k \in \mathbb{N}_0) \\ 2 & (n = mk+1). \end{cases}$$

Theorem 4 below provides the connection between the classes of polynomials $\{M_{n,m}(x)\}$ and $\{D_{n,m}(x)\}$.

Theorem 4. *For $m \in \mathbb{N}$ and $n \geqq m$ the following relationships hold true*:

$$M_{n,m}(x) + M_{n-m,m}(x) = D_{n,m}(x) \qquad (n > m); \tag{4.2}$$

$$D_{n,m}(x) = \frac{q^*_{n+1,m}(x) + q^*_{n+1-m,m}(x) - l}{2x} \qquad (n \geqq m), \tag{4.3}$$

where

$$l = \begin{cases} 0 & (n \neq mk;\ k \in \mathbb{N}_0) \\ 2 & (n = mk); \end{cases}$$

$$D_{n,m}(x) - D_{n-m,m}(x) = q^*_{n,m}(x) \qquad (n \geqq m) \tag{4.4}$$

and

$$D_{n,m}(x) - D_{n-2m,m}(x) = q^*_{n,m}(x) + q^*_{n-m,m}(x) \qquad (n \geqq 2m). \tag{4.5}$$

Proof. We first assume that (4.2) holds true for some positive integer n. Then, for $n \mapsto n+1$, we get

$$\begin{aligned} M_{n+1,m}(x) + M_{n+1-m,m}(x) &= 2xM_{n,m}(x) + M_{n+1-2m,m}(x) + l \\ &\quad + 2xM_{n-m,m}(x) + M_{n+1-3m,m}(x) + l \\ &= 2x\left[M_{n,m}(x) + M_{n-m,m}(x)\right] + M_{n+1-2m,m}(x) \\ &\quad + M_{n+1-3m,m}(x) + 2l \\ &= 2xD_{n,m}(x) + D_{n+1-2m,m}(x) + 2l \\ &= D_{n+1,m}(x) \qquad (2l = 2 \quad \text{or} \quad 2l = 0). \end{aligned}$$

Next, we suppose that the relationship (4.3) holds true for $n = mk$. Then, for $n = mk + 1$, we get

$$\begin{aligned} D_{mk+1,m}(x) &= 2xD_{mk,m}(x) + D_{mk+1-2m,m}(x) + 2 \\ &= 2x\frac{q^*_{mk+1,m}(x) + q^*_{mk+1-m,m}(x) - 2}{2x} \\ &\quad + \frac{q^*_{mk+1-2m,m}(x) + q^*_{mk+1-3m,m}(x)}{2x} + 2 \\ &= \frac{1}{2x}\Big[2xq^*_{mk+1,m}(x) + q^*_{mk+1-2m,m}(x) \\ &\quad + 2xq^*_{mk+1-m,m}(x) + q^*_{mk+1-3m,m}(x)\Big] \\ &= \frac{q^*_{mk+2,m}(x) + q^*_{mk+2-m,m}(x)}{2x}. \end{aligned}$$

For $n = m$ in (4.4), we are led at once to the following obvious identity:

$$D_{m,m}(x) - D_{0,m}(x) = q^*_{m,m}(x),$$

so (4.4) holds true for $n = m$. Suppose that (4.4) holds true for some positive integer n $(n \geqq m)$. Then, for $n \mapsto n+1$, we get

$$\begin{aligned} D_{n+1,m}(x) - D_{n+1-m,m}(x) &= 2xD_{n,m}(x) + D_{n+1-2m,m}(x) + l \\ &\quad - 2xD_{n-m,m}(x) - D_{n+1-3m,m}(x) - l \\ &= 2x(D_{n,m,}(x) - D_{n-m,m}(x)) + D_{n+1-2m,m}(x) \\ &\quad - D_{n+1-3m,m}(x) \\ &= 2xq^*_{n,m}(x) + q^*_{n+1-2m,m}(x) \\ &= q^*_{n+1,m}(x). \end{aligned}$$

The last assertion (4.5) of Theorem 4 is an immediate consequence of assertion (4.4). The proof of Theorem 4 is thus completed. ∎

We now present another interesting result.

Theorem 5. *The polynomials* $\{D_{n,m}(x)\}$ *possess the following representation*:

$$D_{n,m}(x) = \sum_{i=0}^{[(n-1)/m]} q^*_{n-mi,m}(x) \qquad (m, n \in \mathbb{N}). \tag{4.6}$$

Proof. For $n = 1$ in (4.6), we have the following obvious special case:

$$D_{1,m}(x) = \sum_{i=0}^{0} q^*_{1-mi,m}(x),$$

which shows that that formula (4.6) holds true for $n = 1$.

Suppose now that (4.6) holds true for $n = mk$. Then, for $n = mk + 1$, we get

$$\begin{aligned}
D_{mk+1,m}(x) &= 2xD_{mk,m}(x) + D_{mk+1-2m,m}(x) + 2 \\
&= 2x \sum_{i=0}^{[(mk-1)/m]} q^*_{mk-mi,m}(x) + \sum_{i=0}^{[(mk-2m)/m]} q^*_{mk+1-2m,m}(x) + 2 \\
&= 2x \sum_{i=0}^{k-1} q^*_{mk-mi,m}(x) + \sum_{i=0}^{k-2} q^*_{mk+1-2m,m}(x) + 2 \\
&= \sum_{i=0}^{k-2} (2xq^*_{mk-mi,m}(x) + q^*_{mk+1-2m-mi,m}(x)) \\
&\qquad + 2xq^*_{mk-m(k-1),m}(x) + 2 \\
&= \sum_{i=0}^{k-2} q^*_{mk+1-mi,m}(x) + 2xq^*_{m,m}(x) + 1 + 1 \\
&= \sum_{i=0}^{k-2} q^*_{mk+1-mi,m}(x) + q^*_{m+1,m}(x) + 1 \\
&= \sum_{i=0}^{k} q^*_{mk+1-mi,m}(x) \qquad \left(k = \left[\frac{n}{m}\right]\right).
\end{aligned}$$

For $n = mk + 2$, we find from the recurrence relation (4.1) that

$$\begin{aligned} D_{mk+2,m}(x) &= 2xD_{mk+1,m}(x) + D_{mk+2-2m,m}(x) \\ &= 2x\sum_{i=0}^{k} q^*_{mk+1-mi,m}(x) + \sum_{i=0}^{k-2} q^*_{mk+2-2m-mi,m}(x) \\ &= \sum_{i=0}^{k-2} q^*_{mk+2-mi,m}(x) + 2xq^*_{m+1,m}(x) + 2xq^*_{1,m}(x) \\ &= \sum_{i=0}^{k-2} q^*_{mk+2-mi,m}(x) + q^*_{m+2,m}(x) + q^*_{2,m}(x) \\ &= \sum_{i=0}^{k} q^*_{mk+2-mi,m}(x). \end{aligned}$$

This, evidently, completes our proof of Theorem 5. ∎

5. The subsidiary MinMax polynomials $\{R_{n,m}(x)\}$

In this section, we introduce and investigate the sequence of polynomials $\{R_{n,m}(x)\}_{m\in\mathbb{N}}$, which are the subsidiary MinMax polynomials of $\{D_{n,m}(x)\}$ for $\{q_{n,m}(x)\}$, where

$$2q_{n,2}(x) = Q_{n,2}(x).$$

Thus, by definition, we have

$$R_{n,m}(x) = 2xR_{n-1,m}(x) + R_{n-2m,m}(x) + l \qquad (n \geqq 2m;\ m \in \mathbb{N}) \tag{5.1}$$

with $R_{0,m}(x) = 0$ and $R_{n,m}(x) = (2x)^n$ $(n = 1, ..., m-1)$ and $R_{m,m}(x) = (2x)^m + 2$ and $R_{n,m}(x) = 2xR_{n-1,m}(x)$ $(n = m+1, ..., 2m-1)$, where

$$l = \begin{cases} 0 & (n \neq mk;\ k \in \mathbb{N}_0) \\ 2 & (n = mk). \end{cases}$$

The connection between the polynomials $\{R_{n,m}(x)\}$, $\{N_{n,m}(x)\}$ and $\{D_{n,m}(x)\}$ is given by Theorem 6 below.

Theorem 6. *Each of the following relationships holds true:*

$$R_{n,m}(x) = D_{n+1,m}(x) + D_{n+1-2m,m}(x) \qquad (n \geqq 2m), \tag{5.2}$$

$$R_{n,m}(x) = N_{n,m}(x) + N_{n-m,m}(x) \qquad (n \geqq m), \tag{5.3}$$

$$R_{n,m}(x) - R_{n-m,m}(x) = q^*_{n+1,m}(x) + q^*_{n+1-2m,m}(x) \qquad (n \geqq 2m) \tag{5.4}$$

and

$$R_{n,m}(x) - R_{n-m,m}(x) = N_{n,m}(x) - N_{n-2m,m}(x) \qquad (n \geqq 2m). \tag{5.5}$$

Proof. Suppose that (5.2) is correct for $n = mk$. Then, for $n = mk + 1$, the recurrence relation (5.1) yields

$$\begin{aligned} &R_{mk+1,m}(x) = 2xR_{mk,m}(x) + R_{mk+1-2m,m}(x) \\ &= 2x(D_{mk+1,m}(x) + D_{mk+1-2m,m}(x)) + D_{mk+2-2m,m}(x) + D_{mk+2-4m,m}(x) \\ &= D_{mk+2,m}(x) + D_{mk+2-2m,m}(x), \end{aligned}$$

which proves relationship (5.2).

By using the initial values for $R_{n,m}(x)$ and $N_{n,m}(x)$, we can easily prove relationship (5.3). Let relationship (5.3) hold true for $n \neq mk$. Then

$$\begin{aligned} &R_{mk,m}(x) = 2xR_{mk-1,m}(x) + R_{mk-2m,m}(x) + 4 \\ &= 2x(N_{mk-1,m}(x) + N_{mk-1-m,m}(x)) + N_{mk-2m,m}(x) + N_{mk-3m,m}(x) + 4 \\ &= 2xN_{mk-1,m}(x) + N_{mk-2m,m}(x) + 2 + 2xN_{mk-1-m,m}(x) + N_{mk-3m,m}(x) + 2 \\ &= N_{mk,m}(x) + N_{mk-m,m}(x), \end{aligned}$$

which proves relationship (5.3).

Suppose now that the relationship (5.4) holds true for some positive integer n. Then, for $n \mapsto n + 1$, we find from the recurrence relation (5.1) in conjunction with (3.1) that

$$\begin{aligned} &R_{n+1,m}(x) - R_{n+1-m,m}(x) \\ &= 2xR_{n,m}(x) + R_{n+1-2m,m}(x) + l - 2xR_{n-m,m}(x) - R_{n+1-3m,m}(x) - l \\ &= 2x\left[R_{n,m}(x) - R_{n-m,m}(x)\right] + R_{n+1-2m,m}(x) - R_{n+1-3m,m}(x) \\ &= 2xq^*_{n+1,m}(x) + 2xq^*_{n+1-2m,m}(x) + q^*_{n+2-2m,m}(x) + q_{n+2-4m,m}(x) \\ &= q^*_{n+2,m}(x) + q^*_{n+2-2m,m}(x), \end{aligned} \tag{5.6}$$

which proves relationship (5.4).

It is easily observed that the last assertion (5.5) of Theorem 6 is a *direct* consequence of relationship (5.3). ∎

6. A set of sequences of numbers

In their special cases when $x = 0$, the above-investigated MInMax polynomials $\{M_{n,m}(x)\}$, $\{N_{n,m}(x)\}$, $\{D_{n,m}(x)\}$ and $\{R_{n,m}(x)\}$ would lead us to some interesting sequences of MinMax numbers, which we denote by $\{M_{n,m}\}$, $\{N_{n,m}\}$, $\{D_{n,m}\}$ and $\{R_{n,m}\}$, respectively. Some of these sequences of MinMax numbers are given below:

$$
\begin{aligned}
M_{n,2} &: \{0,1,0,1,0,2,0,2,0,3,0,3,0,4,0,4,...\};\\
M_{n,3} &: \{0,1,0,0,1,0,0,2,0,0,2,0,0,3,0,0,3,...\};\\
M_{m,m} &: \{0,1,\underbrace{0,...,0}_{m-1},1,\underbrace{0,...,0}_{m-1},2,\underbrace{0,...,0}_{m-1},...\}.\\
N_{n,2} &: \{1,0,1,0,3,0,3,0,5,0,5,...\};\\
N_{n,3} &: \{1,0,0,1,0,0,3,0,0,3,0,0,5,0,0,5,...\};\\
N_{n,m} &: \{1,\underbrace{0,...,0}_{m-1},1,\underbrace{0,...,0}_{m-1},3,\underbrace{0,...,0}_{m-1},3,\underbrace{0,...,0}_{m-1},5,...\}.\\
D_{n,2} &: \{0,1,0,2,0,3,0,4,0,5,0,...\};\\
D_{n,3} &: \{0,1,0,0,2,0,0,3,0,0,4,0,0,5,...\};\\
D_{n,m} &: \{0,1,\underbrace{0,...,0}_{m-1},2,\underbrace{0,...,0}_{m-1},...\}.\\
R_{n,2} &: \{0,0,2,0,4,0,6,0,8,0,10,...\};\\
R_{n,3} &: \{0,0,0,2,0,0,4,0,0,6,0,0,8,0,0,10,...\};\\
R_{n,m} &: \{\underbrace{0,...,0}_{m-1},2,\underbrace{0,...,0}_{m-1},...\}.
\end{aligned}
$$

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References

[1] Djordjević, G.B., *On the kth order derivative sequences of generalized Fibonacci and Lucas polynomials*, Fibonacci Quart., 43 (2005), 290–298.

[2] Djordjević, G.B., *Generalizations of the Fibonacci and Lucas polynomials*, FILOMAT, 23 (2009), 287–297.

[3] Horadam, A.F., *MinMax sequences for Pell numbers*, in *Applications of Fibonacci Numbers*, vol. 6 (Pullman, Washington, 1994), 231–249, Kluwer Academic Publishers, Dordrecht, Boston and London, 1996.

[4] Horadam, A.F., *MinMax polynomials,* Fibonacci Quart., 34 (1996), 7–17.

[5] Horadam, A.F., Mahon, J.M., *Pell and Pell-Lucas polynomials*, Fibonacci Quart., 23 (1985), 7–20.

[6] Raina, R.K., Srivastava, H.M., *A class of numbers associated with the Lucas numbers*, Math. Comput. Modelling, 25 (7) (1997), 15–22.

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