A NOTE ON P-REGULAR SEMIRINGS

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Abstract. The notion of $P-$regular semiring is introduced and characterization of the same has been given. We also study the representation of the elements in terms of quasi-ideals of weak $P-$regular semirings relative to the right$k-$ideal P.

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1. Introduction and preliminaries

Throughout this paper S will denote a semiring. A semiring is a commutative monoid $(S, +, 0_S)$ having additive identity zero 0_S and a semigroup $(S, \cdot)$ which are connected by ring like distributivity and $0_S \cdot x = x \cdot 0_S = 0_S$ for all $x \in S$. A left (right) ideal of a semiring S is a non-empty subset I of S such that $a+b \in I$ and $ra(ar) \in I$ for all $a, b \in I$ and $r \in S$. An ideal of a semiring S is a non-empty subset I of S such that I is both left and right ideal of S. A left (right)ideal I of S is called a left (right)$k-$ideal if $x + y \in I$, $x \in S$, $y \in I$ imply that $x \in I$. An ideal I of a semiring S is called a $k-$ideal if it is both left $k-$ideal and right $k-$ideal. If A is an ideal(resp. left, right) of a semiring S then

$$\overline{A} = \{a \in S : a + x \in A \text{ for some } x \in A\}$$

is called $k-$closure of A. It can easily be verified that an ideal (resp. left, right) A of S is a $k-$ideal if and only if A=$\overline{A}$. An additive subsemigroup Q of a semiring S is called a quasi-ideal of Sif $QS \cap SQ \subseteq Q$. Clearly, every quasi-ideal of a semiring S is a subsemiring of S.

2. $P-$regular semirings

In this section, we study the concept of $P-$regular semiring and some properties related to the same.

Definition 2.1. Let S be a semiring and P be a right $k-$ ideal of S. Then semiring S is said to be a $P-$regular if for each $a \in S$, there exists $x \in S$ such that $a + p_1 = axa + p_2$ for some $p_1, p_2 \in P$ and $axP \subseteq P$ and a semiring S is said to be weak $P-$regular if for each $a \in S$, there exists $x \in S$ such that $a = axa + p$ for some $p \in P$ and $axP \subseteq P$.

Definition 2.2. Let S be a semiring and P be a right $k-$ ideal of S. An element $e \in S$ is called idempotent relative to P if $e + p' = e^2 + p''$ for some $p', p'' \in P$ and $eP \subseteq P$.

Clearly, it is easy to see that if $P = \{0,\}$ then $P-$regular semiring is a regular and idempotent element is same as in semiring theory. From definition it is also clear that $ax = e$ is an idempotent element relative to P. Consider the semiring $Z_4 = \{0, 1, 2, 3\}$ with respect to addition and multiplication modulo 4 and $I = \{0, 2\}$ is a right as well as $k-$ideal of Z_4. Clearly Z_4 is not regular because $2 \neq 2 \odot x \odot 2$ for $x \in Z_4$ but it is $P-$regular.

Theorem 2.3. *The semiring S with unity is $P-$regular if and only if every right ideal of S be of the form $aS + P$ where $a \in S$ has the form $aS + P = eS + P$, where e is an idempotent relative to P.*

Proof. Suppose S is $P-$regular. Therefore for each $a \in S$, there exists $x \in S$ such that $a+p_1 = axa+p_2$ for some $p_1, p_2 \in P$ and $axP \subseteq P$. Now $aS+P = axaS+P \subseteq axS+P = eS+P$. Therefore $aS+P \subseteq eS+P$. Again, $eS+P = axS+P \subseteq aS+P$. Form above conclusion, we have $eS + P = aS + P$. Conversely, suppose that $aS + P = eS + P$ and $e + p' = e^2 + p''$ for some $p', p'' \in P$ and $eP \subseteq P$. To show S is $P-$regular. From above, we can write $a + p_1 = ey + p_2$ and $e + p_3 = ax + p_4$. Now

$$ea + p_3a = axa + p_4a \tag{1}$$

Again, $ea + ep_1 = e^2y + ep_2$. Adding $p''y + p_2$ on both sides, we get $ea + ep_1 + p_2 + p''y = e^2y + ep_2 + p_2 + p''y = ey + ep_2 + p'y + p_2 = a + p_1 + p'y + ep_2$, which implies

$$ea + z_1 = a + z_2 \tag{2}$$

where $z_1 = ep_1 + p_2 + p''y, z_2 = p'y + ep_2 + p_1 \in P$. Adding z_1 in (1) and p_3a in (2), we get $a + z_2 + p_3a = axa + p_4a + z_1$. Thus $a + p_5 = axa + p_6$. Also, $e + p_3 = ax + p_4$ implies $ep + p_3p = axp + p_4p \in P$ (because $ep \in eP \subseteq P$ and $p_3p \subseteq P$) implies $axP \subseteq P$ (since P is a right $k-$ideal of S). Therefore S is a $P-$regular. ∎

Proposition 2.4. *Suppose P is a right $k-$ideal of S and I is an ideal of S such that $P \subseteq I$. If S is $P-$regular then I is $P-$regular.*

Proof. Suppose S is $P-$regular and I is an ideal of S such that $P \subseteq I$. Then for each $a \in I$ there exists $x \in S$ and $p_1, p_2 \in P$ such that $a + p_1 = axa + p_2$ and $axP \subseteq P$. Let $y = xax \in I$. Then $axa + p_1xa = axaxa + p_2xa$ implies $p_2 + axa + p_1xa = aya + p_2xa + p_2$ implies $a + p_1 + p_1xa = aya + p_2xa + p_2$ implies $a + p' = aya + p''$ for some $p', p'' \in P$. Also, $ayP = a(xax)p = ax(axp) \in axP \subseteq P$. Thus I is a $P-$regular. ∎

Theorem 2.5. *Let a semiring S be a $P-$regular. Then every right $k-$ ideal R and left $k-$ideal L of S has the form $(P + R) \cap \overline{(P + L)} \subseteq \overline{P + RL}$.*

Proof. Let S be a $P-$ regular and let $a \in (P + R) \cap \overline{(P + L)}$. Then $a \in S$ can be written as $a = p + r$ and $a + p_1 + l_1 = p_2 + l_2$ for some $p, p_1, p_2 \in P$, and $r \in R$ and $l_1, l_2 \in L$. Since S is a $P-$regular therefore for each $a \in S$ there exists $x \in S$ such that $a + p_3 = axa + p_4$ for some $p_3, p_4 \in P$ and $axP \subseteq P$. Now $(a)x(a + p_1 + l_1) = (p + r)x(p_2 + l_2)$ which implies that $axa + axp_1 + axl_1 = pxp_2 + pxl_2 + rxp_2 + rxl_2$ implies $axa + p_4 + axp_1 + axl_1 = p_4 + pxp_2 + pxl_2 + rxp_2 + rxl_2$ which gives $a + p_3 + axp_1 + axl_1 = pxp_2 + pxl_2 + rxp_2 + rxl_2 + p_4$. Since $axp_2 = pxp_2 + rxp_2 \in P$ and P is a right $k-$ideal of S. Therefore $rxp_2 \in P$ and also $axl_1 = pxl_1 + rxl_1 \in P + RL$. Therefore $a \in \overline{P + RL}$. Thus $(P + R) \cap \overline{(P + L)} \subseteq \overline{(P + RL)}$. ∎

Note. The equality holds if $P + R$ is a right $k-$ideal of S because suppose $a \in \overline{(P + RL)}$. This implies $a \in \overline{(P + R)}$ and $a \in \overline{(P + L)}$. Therefore, $a \in (P+R) \cap \overline{(P + L)}$ (as $P+R$ is a right $k-$ideal of S). Hence $(P+R) \cap \overline{(P + L)} = \overline{P + RL}$.

Theorem 2.6. *The semiring S is a weak $P-$regular if every right ideal R and left ideal L of S has the form $(P + R) \cap (P + L) = P + RL$.*

Proof. The proof is same as in Theorem 2.6. ∎

Proposition 2.7. *Let S be a weak$P-$regular semiring and R be any right ideal of S. Then the following holds:*

(i) $R + P = R^2 + P$

(ii) *if $R^2 \subseteq P$ then $R \subseteq P$.*

Proof. (i). Since S is a weak$P-$regular therefore for each $a \in S$ there exists $x \in S$ and some $p \in P$ such that $a = axa+p$ and $axP \subseteq P$. Suppose $a \in R$. Then $a = (ax)a + p \in R^2 + P$. This implies $R \subseteq R^2 + P$. This implies $R + P \subseteq R^2 + P$. But $R^2 + P \subseteq R + P$(always). Therefore $R + P = R^2 + P$.

(ii). Since $R^2 \subseteq P$ therefore $R^2 + P \subseteq P$. This gives $R + P \subseteq P$. Therefore $R \subseteq P$ (as P is a right a $k-$ideal of S). ∎

Theorem 2.8. *If a semiring S is weak $P-$regular, then every element of a quasi-ideal Q can be represented as the sum of two elements of P and Q.*

Proof. Let S be a weak $P-$regular semiring and Q be a quasi-ideal of S. Then any $q \in Q$ can be written as $q = qxq + p$ and $qxP \subseteq P$ for some $p \in P$ and $x \in S$. Since Q is a quasi-ideal of S therefore $qxq \in QSQ \subseteq QS \cap SQ \subseteq Q$ and therefore we have $q = p + qxq \in P + Q$. ∎

Theorem 2.9. *Let S be a weak$P-$regular semiring and $Q_1$and Q_2 are quasi-ideals of S. If $q \in Q_1 \cap Q_2$, then this element q can be expressed as $q = p + q_1xq_2$ for some $p \in P, x \in S, q_1 \in Q_1$ and $q_2 \in Q_2$ and also $q_1xq_2xP \subseteq P$.*

Proof. Let $q \in Q_1 \cap Q_2$. Since S is a weak $P-$regular then there exists $x \in S$ such that $q = qxq + p$ for some $p \in P$ and $qxP \subseteq P$. Since $Q_1 \cap Q_2$ is a quasi-ideal of S therefore the element $q \in Q_1 \cap Q_2$ can be written as both $q = p_1 + q_1 \in Q_1$ and $q = p_2 + q_2 \in Q_2$ for some $p_1, p_2 \in P, q_1 \in Q_1, q_2 \in Q_2$. Since S is a weak $P-$regular therefore the element $q \in S$ can be written as $q = p_3 + qxq$ for some $p_3 \in P$ and $qxP \subseteq P$. Now $q = p_3 + qxq = p_3 + (p_1 + q_1)x(p_2 + q_2) = p_3 + q_1xp_2 + q_1xq_2 + p_1xp_2 + p_1xq_2 = p_6 + q_1xq_2$ where $p_6 \in P$ because $qxp_2 = p_1xp_2 + q_1xp_2 \in P$ and P is a right $k-$ideal of S, therefore $q_1xp_2 \in P$. Again, $qxP = p_6xP + q_1xq_2xP \subseteq P$. Since P is a right $k-$ideal of S therefore $q_1xq_2xP \subseteq P$. ∎

Theorem 2.10. *Let S be a weak $P-$regular. Then every quasi-ideal Q of S can be written as $P + Q = P + QSQ$.*

Proof. Let S be a weak$P-$regular and Q be a quasi-ideal of S. Then $QSQ \subseteq (QS) \cap (SQ) \subseteq Q$ holds. Therefore $P + QSQ \subseteq P + Q$.

Conversely, let $a \in P + Q$. Then $a = p + q$ for some $p \in P$ and $q \in Q$. By weak $P-$regularity of S, we have $a = axa + p_1$, for some $p_1 \in P, x \in S$ and $axP \subseteq P$. This gives $a = (p + q)x(p + q) + p_1 = pxp + pxq + qxp + qxq + p_1$. Now $axp = (p + q)xp = pxp + qxp \in P$ and P is a right $k-$ideal of S. Therefore $qxp \in P$. Thus we have $a = (pxp + pxq + qxp + p_1) + qxq \in P + QSQ$. Thus $P + Q \subseteq P + QSQ$. Hence $P + Q = P + QSQ$. ∎

Theorem 2.11. *Let S be weak $P-$regular. If $Q_1$and $Q_2$are quasi-ideals of S, then $P + (Q_1 \cap Q_2) = P + (Q_1SQ_2) \cap (Q_2SQ_1)$.*

Proof. Let S be a weak $P-$regular and Q_1 and Q_2 be quasi-ideals of S. If $q \in P + Q_1 \cap Q_2$. Then q can be written as $q = p + q'$ for some $p \in P$ and $q' \in Q_1 \cap Q_2$. Also, by weak $P-$regularity of S, there exists $x \in S$ such that $q' = p' + q'xq'$, for some $p' \in P$. Hence we have $q = p + p' + q'xq' \in P + (Q_1SQ_2) \cap (Q_2SQ_1)$. Therefore, $P + Q_1 \cap Q_2 \subseteq P + (Q_1SQ_2) \cap (Q_2SQ_1)$. Conversely, since $(Q_1SQ_2) \cap (Q_2SQ_1) \subseteq (Q_1S \cap SQ_2) \cap (Q_2S \cap SQ_1) \subseteq Q_1 \cap Q_2$, then $P + (Q_1SQ_2 \cap Q_2SQ_1) \subseteq P + (Q_1 \cap Q_2)$. Hence $P + (Q_1SQ_2 \cap Q_2SQ_1) = P + (Q_1 \cap Q_2)$. ∎

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