SOME UNIQUENESS RESULTS ON MEROMORPHIC FUNCTIONS SHARING TWO SETS

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Abstract. Using the notion of weighted sharing, we prove some uniqueness theorems of meromorphic functions that share two sets. The results in this paper improve and supplement some recent ones of the first author and consequently provide a better answer to the famous question of Gross than that was given previously.

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1. Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane $\mathbb{C}$. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \qquad (r \longrightarrow \infty, r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S)=\bigcup_{a\in S}\{z:f(z)-a=0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity, the set $\bigcup_{a\in S}\{z : f(z)-a=0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S)=E_g(S)$, we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S)=\overline{E}_g(S)$, we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values.

Let m be a positive integer or infinity and $a\in\mathbb{C}\cup\{\infty\}$. We denote by $E_{m)}(a;f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. If, for some $a\in\mathbb{C}\cup\{\infty\}$, $E_{\infty)}(a;f)=E_{\infty)}(a;g)$, we say that f, g share the value a CM. For a set S of distinct elements of $\mathbb{C}$, we define $E_{m)}(S,f)=\bigcup_{a\in S}E_{m)}(a,f)$. The condition $E_{m)}(S,f)=E_{m)}(S,g)$ obviously implies $E_{j)}(S,f)=E_{j)}(S,g)$ for all $1\leq j\leq m$.

Inspired by the Nevanlinna's three and four values theorems, in 1970s F. Gross and C.C. Yang started to study the similar but more general questions of two functions that share sets of distinct elements instead of values. For instance, they proved that if f and g are two non-constant entire functions and S_1, S_2 and S_3 are three distinct finite sets such that $f^{-1}(S_i)=g^{-1}(S_i)$ for $i=1,\,2,\,3$, then $f\equiv g$. In 1976, F. Gross proposed the following question in [8]:

Question A *Can one find two finite sets S_j $(j=1,2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2$ must be identical ?*

In [8], Gross wrote: *If the answer of* Question A *is affirmative it would be interesting to know how large both sets would have to be ?*

Yi [21] and independently Fang and Xu [7] gave the same answer in this direction.

In 2003, Lin and Yi posed the following question.

Question B ([19]) *Can one find two finite sets S_j $(j=1,2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2$ must be identical ?*

Gradually, the research on *Question B* gained pace and today it has become one of the most prominent branches of the uniqueness theory. For the last two decades several attempts have been made by different authors to consider the shared value problems relative to a meromorphic function sharing two sets and at the same time give affirmative answers to *Question B* under weaker hypothesis {see [1]-[7], [10], [14]-[21], [28]-[28], [29]-[30]}.

In 1994, Yi [20] gave an affirmative answer to *Question B* and proved that there exist two finite sets S_1 (with 2 elements) and S_2 (with 9 elements) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical.

Inspired by *Question B*, we will consider the uniqueness of two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$. According to what we know by now, perhaps this type of results were first obtained by Li and Yang in [17], where they proved that there exists one finite set S with 15 elements such that any two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ must be identical. In 1995, Yi [21] and independently Li and Yang [17] proved that there exists a set S of 11 elements such that any two non-constant meromorphic functions with $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ must be identical. In 1997 Fang and Guo in [6] exhibited a set S of nine elements with this property.

In 2002, Yi [25] proved the following result in which he not only reduced the cardinalities of the set S but also relaxed the sharing of the poles from CM to IM.

Theorem A. ([25]; see also [29]) *Let n be a positive integer such that $n \geq 8$, and let a, b be two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. Then the polynomial*

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2 \tag{1.1}$$

has only simple zeros. Let $S = \{w \mid P(w) = 0\}$. If f and g are two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $\overline{E}_f(\{\infty\}) = \overline{E}_g(\{\infty\})$ then $f \equiv g$.

Dealing with the question of Gross in [6], Fang and Lahiri obtained a unique range set S with smaller cardinalities than that obtained previously imposing some restrictions on the poles of f and g.

Theorem B. ([6]) *Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where n (≥ 7) is an integer and a and b are two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g are two non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

Let $S = \{z : z^7 - z^6 - 1 = 0\}$ and

$$f = \frac{e^z + e^{2z} + \ldots + e^{6z}}{1 + e^z + \ldots + e^{6z}}, \quad g = \frac{1 + e^z + \ldots + e^{5z}}{1 + e^z + \ldots + e^{6z}}$$

Obviously, $f = e^z g$, $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ but $f \not\equiv g$. So, for the validity of *Theorem B*, f and g must not have any simple pole.

In 2001, an idea of gradation of sharing known as weighted sharing has been introduced by I. Lahiri in [12], [13] which measures how close a shared value is to being shared CM or to being shared IM. In the following definition we explain the notion.

Definition 1.1 [12], [13] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2 [12] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.

Clearly, $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

Using the notion of weighted sharing of sets, Lahiri [14] proved the following theorem, which improved *Theorem B*.

Theorem C. ([14]) *Let S be defined as in* Theorem B *and n (≥ 7) be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) > 1$, $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ then $f \equiv g$.*

Recently, the first author [3] has generalized *Theorem C* by investigating the problem of further relaxation of the nature of sharing the set $\{\infty\}$ and obtained the following result.

Theorem D. ([3]) *Let S be defined as in* Theorem B *and n (≥ 7) be an integer. If, for two non-constant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) > 1 + \frac{29}{6nk + 6n - 5}$, $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, k) = E_g(\{\infty\}, k)$, where $0 \le k < \infty$, then $f \equiv g$.*

In the mean time, the first author [1] has improved *Theorem B* by relaxing the nature of sharing the set S and proved the following result.

Theorem E. *Let S be defined as in* Theorem B. *If for two non-constant meromorphic functions f and g, $\Theta(\infty; f) > \frac{1}{2}$, $\Theta(\infty; g) > \frac{1}{2}$ and $E_{3)}(S, f) = E_{3)}(S, g)$, $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$, then $f \equiv g$.*

In the paper, we consider a new range set different from those mentioned earlier and with the help of the same we will improve and supplement *Theorems D* and *E*.

The following theorems are the main results of the paper.

Theorem 1.1 *Let*

$$S=\left\{z:\frac{(n-1)(n-2)}{4}\ z^{n}-\frac{n(n-2)}{2}\ z^{n-1}+\frac{n(n-1)}{4}\ z^{n-2}-1=0\right\},$$

where n (≥ 7) *is an integer. If, for two non-constant meromorphic functions* f *and* g, $E_f(S,2)=E_g(S,2)$ *and* $E_f(\{\infty\},k)=E_g(\{\infty\},k)$, *where* $0\leq k<\infty$ *and* $\min\{\Theta_f,\Theta_g\}>4-\frac{n}{2}+\frac{3}{nk+n-1}$, *then* $f\equiv g$, *where* $\Theta_f=2\ \Theta(0;f)+2\ \Theta(1;f)+\Theta(\infty;f)$ *and* Θ_g *can be similarly defined.*

Corollary 1.1 *Let* S *be given as in* Theorem 1.1. *If, for two non-constant meromorphic functions* f *and* g, $E_f(S,2)=E_g(S,2)$, $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ *and* $\min\{\Theta_f,\Theta_g\}>4-\frac{n}{2}$, *where* Θ_f *and* Θ_g *have the same meaning as in* Theorem 1.1, *then* $f\equiv g$.

Theorem 1.2 *Let*

$$S=\left\{z:\frac{(n-1)(n-2)}{4}\ z^{n}-\frac{n(n-2)}{2}z^{n-1}+\frac{n(n-1)}{4}\ z^{n-2}-1=0\right\},$$

where n (≥ 8) *is an integer. If for two non-constant meromorphic functions* f *and* g $E_f(S,2)=E_g(S,2)$ *and* $E_f(\{\infty\},2)=E_g(\{\infty\},2)$, *then* $f\equiv g$.

Theorem 1.3 *Let* S *be defined as in* Theorem 1.1. *If for two non-constant meromorphic functions* f *and* g, $E_{3)}(S,f)=E_{3)}(S,g)$, $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ *and* $\min\{\Theta_f,\Theta_g\}>4-\frac{n}{2}$, *where* Θ_f *and* Θ_g *have the same meaning as in* Theorem 1.1, *then* $f\equiv g$.

Theorem 1.4 *Let* S *be defined as in* Theorem 1.1. *If for two non-constant meromorphic functions* f *and* g, $E_{m)}(S,f)=E_{m)}(S,g)$, $E_f(\{\infty\},k)=E_g(\{\infty\},k)$, *where* $0\leq k<\infty$, $m\geq 4$ *and* $\min\{\Theta_f,\Theta_g\}>4-\frac{n}{2}+\frac{3}{nk+n-1}$, *where* Θ_f *and* Θ_g *have the same meaning as in* Theorem 1.1 *then* $f\equiv g$.

It is assumed that the readers are familiar with the standard definitions and notations of the value distribution theory as those are available in [9]. We are still going to explain some notations as these are used in the paper.

Definition 1.3 [11] For a value a in the extended complex plane, we denote by $N(r,a;f\mid=1)$ the reduced counting function of simple a points of f in $|z|<r$. Denote by $N(r,a;f\mid\leq m)$ ($N(r,a;f\mid\geq m)$, respectively) the counting function of those a-points of f in $|z|<r$, where the multiplicity of each point is not greater (not less, respectively) than m, m is a positive integer, and each point is counted according to its multiplicity. Denote by $N(r,a;f\mid<m)$ ($N(r,a;f\mid>m)$, respectively) the counting function of those a-points of f in $|z|<r$, where the multiplicity of each point is less (greater, respectively) than m, and each point is counted according to its multiplicity. Denote by $\overline{N}(r,a;f\mid\leq m)$, $\overline{N}(r,a;f\mid\geq m)$, $\overline{N}(r,a;f\mid<m)$, $\overline{N}(r,a;f\mid>m)$ the reduced forms of $N(r,a;f\mid\leq m)$, $N(r,a;f\mid\geq m)$, $N(r,a;f\mid<m)$, $N(r,a;f\mid>m)$ respectively.

Definition 1.4 Let f and g be two non-constant meromorphic functions such that f and g share a value a IM where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ ($\overline{N}_L(r,a;g)$) the counting function of those a-points of f and g where $p > q$ ($q > p$), each a-point is counted only once.

Definition 1.5 Let f and g be two non-constant meromorphic functions and m be a positive integer such that $E_{m)}(a;f) = E_{m)}(a;g)$ where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity $p > 0$, an a-point of g with multiplicity $q > 0$. We denote by $\overline{N}_L^{m)}(r,a;f)$ ($\overline{N}_L^{m)}(r,a;g)$) the counting function of those common a-points of f and g, where $p > q$ ($q > p$), each a-point is counted only once.

Definition 1.6 [13] We denote by $N_2(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2)$.

Definition 1.7 Let m be a positive integer. Also let z_0 be a zero of $f(z) - a$ of multiplicity p and a zero of $g(z) - a$ of multiplicity q. We denote by $\overline{N}_{f\geq m+1}(r,a;f \mid g \neq a)$ ($\overline{N}_{g\geq m+1}(r,a;g \mid f \neq a)$) the reduced counting functions of those a-points of f and g, where $p \geq m+1$ and $q = 0$ ($q \geq m+1$ and $p = 0$).

Definition 1.8 [8], [9] Let f, g share $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly, $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$. For $E_{m)}(a;f) = E_{m)}(a;g)$ we can define $\overline{N}_*(r,a;f,g)$ in a similar manner and we note that here $\overline{N}_*(r,a;f,g) = \overline{N}_L^{m)}(r,a;f) + \overline{N}_L^{m)}(r,a;g) + \overline{N}_{f\geq m+1}(r,a;f \mid g \neq a)$ $+ \overline{N}_{g\geq m+1}(r,a;g \mid f \neq a)$.

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by H and V the following two functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Lemma 2.1 [13] *If F, G be two non-constant meromorphic functions such that they share $(1,1)$ and $H \not\equiv 0$, then*

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2 [19] *If F, G be two non-constant meromorphic functions such that $E_{1)}(1;F)=E_{1)}(1;G)$ and $H\not\equiv 0$ then*

$$N(r,1;F\mid=1)\le N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.3 *Let f and g be two non-constant meromorphic functions sharing $(1,m)$, where $1\le m<\infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\\&\quad\le\frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right]\end{aligned}$$

Proof. Let z_0 be a 1- point of f of multiplicity p and a 1-point of g of multiplicity q. Since f, g share $(1,m)$, we note that the 1-points of f and g up to multiplicity m are same. When $p=q=1$, z_0 is counted once, both in left and right hand side of the above inequality but when $2\le p=q\le m$, z_0 is counted 2 times in the left hand side of the above inequality whereas it is counted p times in the right hand side of the above inequality. If $p=m+1$, then the possible values of q are as follows. (i) $q=m+1$, (ii) $q\ge m+2$. When $p=m+2$, then q can take the following possible values (i) $q=m+1$, (ii) $q=m+2$, (iii) $q\ge m+3$. Similar explanations hold if we interchange p and q. Clearly, when $p=q\ge m+1$, z_0 is counted 2 times in the left hand side and $p\ge m+1$ times in the right hand side of the above inequality. If $p>q\ge m+1$, in view of *Definition 1.8* we know z_0 is counted $m+\frac{3}{2}$ times in the left hand side of the inequality and $\frac{p+q}{2}\ge m+\frac{3}{2}$ times in the right hand side of the above inequality. If $q>p$, we can explain similarly. Hence the lemma follows. ∎

Lemma 2.4 *Let f and g be two non-constant meromorphic functions such that $E_{m)}(1;f)=E_{m)}(1;g)$, where $1\le m<\infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(\frac{m}{2}-\frac{1}{2}\right)\{\overline{N}_{f\ge m+1}(r,1;f\mid g\ne 1)\\&\quad+\overline{N}_{g\ge m+1}(r,1;g\mid f\ne 1)\}+\left(m-\frac{1}{2}\right)\left\{\overline{N}_L^{m)}(r,1;f)+\overline{N}_L^{m)}(r,1;g)\right\}\\&\quad\le\frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right]\end{aligned}$$

Proof. By the condition that $E_{m)}(1;f)=E_{m)}(1;g)$, we note that every common zero of $f-1$ and $g-1$ up to multiplicity m has the same multiplicities related to f and g. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. If $p=m+1$, then the possible values of q are as follows: (i) $q=m+1$; (ii) $q\ge m+2$; (iii) $q=0$. Similarly, if $p=m+2$, the possible values of q are as follows: (i) $q=m+1$; (ii) $q=m+2$; (iii) $q\ge m+3$; (iv) $q=0$. If $p\ge m+3$, we can find the possible values of q similarly. Now, Lemma 2.4 follows from the above explanation. ∎

Lemma 2.5 *Suppose that F, G share $(1,0)$, $(\infty,0)$ and a is a complex number satisfying $a \neq 0$, 1. If $H \not\equiv 0$ then*

$$\begin{aligned} N(r,H) \;\le\; & \overline{N}(r,0;F \mid\geq 2)+\overline{N}(r,0;G \mid\geq 2)+\overline{N}(r,a;F \mid\geq 2)+\overline{N}(r,a;G \mid\geq 2) \\ & +\overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;F,G)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'}), \end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)(F-a)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined.

Proof. By the definition of H we verify that the possible poles of H result from the following six cases: (i) The multiple zeros of F and G. (ii) The multiple a- points of F and G. (iii) Those common poles of F and G, where each such pole of F and G has different multiplicities related to F and G. (iv) Those common 1-points of F and G, where each such point has different multiplicities related to F and G. (v) The zeros of F' which are not zeros of $F(F-1)(F-a)$. (vi) The zeros of G' which are not zeros of $G(G-1)(G-a)$. Now proceeding as in the proof of *Lemma 2.4*, we can get the result of the lemma. ∎

Lemma 2.6 *Let $E_{m)}(1;F) = E_{m)}(1;G)$, let F, G share $(\infty,0)$ and let a be a complex number satisfying $a \neq 0,1$. If $H \not\equiv 0$, then*

$$\begin{aligned} N(r,H) \le\; & \overline{N}(r,0;F \mid\geq 2)+\overline{N}(r,0;G \mid\geq 2)+\overline{N}(r,a;F \mid\geq 2)+\overline{N}(r,a;G \mid\geq 2) \\ & +\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)+\overline{N}_{F\geq m+1}(r,1;F \mid G\neq 1) \\ & +\overline{N}_{G\geq m+1}(r,1;G \mid F\neq 1)+\overline{N}_*(r,\infty;F,G)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'}), \end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ and $\overline{N}_0(r,0;G^{'})$ has the same meaning as in Lemma 2.5.

Proof. The proof is obvious. ∎

Lemma 2.7 [18] *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f) + O(1)$.*

Next, we set

$$F=\frac{(n-1)(n-2)}{4}f^{n}-\frac{n(n-2)}{2}f^{n-1}+\frac{n(n-1)}{4}f^{n-2}, \tag{2.1}$$

$$G=\frac{(n-1)(n-2)}{4}g^{n}-\frac{n(n-2)}{2}g^{n-1}+\frac{n(n-1)}{4}g^{n-2}, \tag{2.2}$$

where f, g are two non-constant meromorphic functions and $n \geq 3$ is an integer.

Lemma 2.8 *Let F, G be given by* (2.1) *and* (2.2)*, where $n \geq 7$ is an integer and $H \not\equiv 0$. Suppose α_1 and α_2 are the roots of the equation*

$$\frac{(n-1)(n-2)}{4}z^{2}-\frac{n(n-2)}{2}z+\frac{n(n-1)}{4}=0.$$

If F, G share $(1,m)$ and f, g share (∞,k), where $2\le m<\infty$. Then, for a complex number $a(\neq 0,1)$,

$$\begin{aligned}\frac{3n}{2}\left\{T(r,f)+T(r,g)\right\} \;\le\;& 2\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+N_2(r,\alpha_1;f)\\&+N_2(r,\alpha_1;g)+N_2(r,\alpha_2;f)+N_2(r,\alpha_2;g)\\&+N_2(r,a;F)+N_2(r,a;G)+\overline{N}(r,\infty;f)\\&+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\\&\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}&2\{T(r,F)+T(r,G)\}\le \overline{N}(r,1;F)+\overline{N}(r,0;F)+\overline{N}(r,a;F)\\&\quad+\overline{N}(r,\infty;F)+\overline{N}(r,1;G)+\overline{N}(r,0;G)+\overline{N}(r,a;G)+\overline{N}(r,\infty;G)\\&\quad-N_0(r,0;F')-N_0(r,0;G')+S(r,F)+S(r,G).\end{aligned}\tag{2.3}$$

Using Lemmas 2.1, 2.3, 2.5 and 2.7 we see that

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G)\le\;&\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]\\&+N(r,1;F\mid=1)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\\\le\;&\frac{n}{2}\left\{T(r,f)+T(r,g)\right\}+\overline{N}(r,0;f)+\overline{N}(r,0;g)\\&+\overline{N}(r,\alpha_1;f\mid\ge 2)+\overline{N}(r,\alpha_2;f\mid\ge 2)\\&+\overline{N}(r,\alpha_1;g\mid\ge 2)+\overline{N}(r,\alpha_2;g\mid\ge 2)\\&+\overline{N}(r,a;F\mid\ge 2)+\overline{N}(r,a;G\mid\ge 2)\\&+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\&+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g).\end{aligned}\tag{2.4}$$

Using (2.4) in (2.3) the lemma follows in view of Definition 1.6. ∎

Lemma 2.9 *Let F, G be given by* (2.1) *and* (2.2)*, where $n\ge 7$ is an integer and $H\not\equiv 0$. Suppose α_i, $i=1,2$ has the same meaning as given in Lemma 2.8. If $E_{m)}(1;F)=E_{m)}(1;G)$ and f, g share (∞,k), where m, k are integers such that $1\le m<\infty$ and $k\ge 0$. Then for a complex number $a(\neq 0,1)$*

$$\begin{aligned}\frac{3n}{2}\left\{T(r,f)+T(r,g)\right\} \;\le\;& 2\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+N_2(r,\alpha_1;f)\\&+N_2(r,\alpha_1;g)+N_2(r,\alpha_2;f)+N_2(r,\alpha_2;g)\\&+N_2(r,a;F)+N_2(r,a;G)+\overline{N}(r,\infty;f)\\&+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)\end{aligned}$$

$$-\left(\frac{m}{2}-\frac{3}{2}\right)\left\{\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)\right\}$$
$$-\left(m-\frac{3}{2}\right)\left\{\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\right\}+S(r,f)+S(r,g).$$

Proof. Using Lemmas 2.2, 2.4, 2.6 and 2.7 we obtain

$$\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\quad\leq\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+N(r,1;F\mid=1)\\
&\quad-\left(\frac{m}{2}-\frac{1}{2}\right)\left\{\overline{N}_{f\geq m+1}(r,1;f\mid g\neq 1)+\overline{N}_{g\geq m+1}(r,1;g\mid f\neq 1)\right\}\\
&\quad-\left(m-\frac{1}{2}\right)\left\{\overline{N}_L^{m)}(r,1;f)+\overline{N}_L^{m)}(r,1;g)\right\}\\
&\quad\leq\frac{n}{2}\left\{T(r,f)+T(r,g)\right\}+\overline{N}(r,0;f)+\overline{N}(r,0;g)\\
&\quad+\overline{N}(r,\alpha_1;f\mid\geq 2)+\overline{N}(r,\alpha_2;f\mid\geq 2)\\
&\quad+\overline{N}(r,\alpha_1;g\mid\geq 2)+\overline{N}(r,\alpha_2;g\mid\geq 2)+\overline{N}(r,a;F\mid\geq 2)\\
&\quad+\overline{N}(r,a;G\mid\geq 2)+\overline{N}_*(r,\infty;f,g)\\
&\quad-\left(\frac{m}{2}-\frac{3}{2}\right)\left\{\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)\right\}\\
&\quad-\left(m-\frac{3}{2}\right)\left\{\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\right\}\\
&\quad+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g).
\end{aligned}\tag{2.5}$$

Using (2.5) in (2.3), the lemma follows in view of Definition 1.6. ∎

Lemma 2.10 *Let f, g be two non-constant meromorphic functions and suppose α_i, $i=1,2$ has the same meaning as given in Lemma* 2.8. *Then*

$$(n-1)^2(n-2)^2f^{n-2}(f-\alpha_1)(f-\alpha_2)g^{n-2}(g-\alpha_1)(g-\alpha_2)\not\equiv b,$$

where b is a non-zero constant and $n\geq 5$ is an integer.

Proof. On the contrary, suppose that

$$(n-1)^2(n-2)^2f^{n-2}(f-\alpha_1)(f-\alpha_2)g^{n-2}(g-\alpha_1)(g-\alpha_2)\equiv b.\tag{2.6}$$

Let z_0 be a zero of f with multiplicity p. Then z_0 is a pole of g with multiplicity q such that

$$(n-2)p=(n-2)q+2q=nq.\tag{2.7}$$

From (2.7) we see that $2q=(n-2)(p-q)\geq n-2$ and so $p=\dfrac{n}{n-2}\,q\geq\dfrac{n}{2}$.

Let z_0 be a zero of $f-\alpha_i$ $i=1,2$ with multiplicity p. Then z_0 is a pole of g with multiplicity q such that $p=(n-2)q+2q=nq\geq n$.

Since the poles of f are the zeros of g and $g-\alpha_i$ $i=1,2$, we get

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g)+\overline{N}(r,\alpha_1;g)+\overline{N}(r,\alpha_2;g)\\ &\leq \frac{2}{n}N(r,0;g)+\frac{1}{n}\,N(r,\alpha_1;g)+\frac{1}{n}\,N(r,\alpha_2;g)\\ &\leq \frac{4}{n}T(r,g).\end{aligned}$$

By the second fundamental theorem we get

$$\begin{aligned}2T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,\alpha_1;f)+\overline{N}(r,\alpha_2;f)+\overline{N}(r,\infty;f)+S(r,f)\\ &\leq \frac{2}{n}N(r,0;f)+\frac{1}{n}\,N(r,\alpha_1;f)+\frac{1}{n}\,N(r,\alpha_2;f)+\frac{4}{n}T(r,g)+S(r,f)\\ &\leq \frac{4}{n}T(r,f)+\frac{4}{n}T(r,g)+S(r,f).\end{aligned}$$

i.e.,

$$\left(2-\frac{4}{n}\right)\,T(r,f)\leq\frac{4}{n}\,T(r,g)+S(r,f). \tag{2.8}$$

Similarly,

$$\left(2-\frac{4}{n}\right)\,T(r,g)\leq\frac{4}{n}\,T(r,f)+S(r,g) \tag{2.9}$$

Adding (2.8) and (2.9) we get

$$\left(2-\frac{8}{n}\right)\,\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction for $n\geq 5$. This proves the lemma. ∎

Lemma 2.11 [5] *Let f, g be two non-constant meromorphic functions and suppose n (≥ 6) is an integer. If*

$$\begin{aligned}&\frac{(n-1)(n-2)}{2}f^n-n(n-2)f^{n-1}+\frac{n(n-1)}{2}f^{n-2}\\ &\equiv\frac{(n-1)(n-2)}{2}g^n-n(n-2)g^{n-1}+\frac{n(n-1)}{2}g^{n-2},\end{aligned} \tag{2.10}$$

then $f\equiv g$.

Lemma 2.12 [22] *If F, G share $(\infty,0)$ and $V\equiv 0$ then $F\equiv G$.*

Lemma 2.13 *Let F, G be given by* (2.1) *and* (2.2)*, where $n \geq 7$ is an integer and $V \not\equiv 0$. If F, G share $(1,2)$, f, g share (∞, k), where $0 \leq k < \infty$, then the poles of F and G are zeros of V and*

$$\begin{aligned}(nk+n-1)\,\overline{N}(r,\infty;f \mid\geq k+1) &= (nk+n-1)\overline{N}(r,\infty;g \mid\geq k+1)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,\alpha_1;f)+\overline{N}(r,\alpha_2;f)\\ &\quad +\overline{N}(r,0;g)+\overline{N}(r,\alpha_1;g)+\overline{N}(r,\alpha_2;g)\\ &\quad +\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g),\end{aligned}$$

where α_i $i=1,2$ has the same meaning as in Lemma 2.8*.*

Proof. Since f, g share $(\infty;k)$, it follows that F, G share $(\infty;nk)$ and so a pole of F with multiplicity $p(\geq nk+1)$ is a pole of G with multiplicity $r(\geq nk+1)$ and vice versa. We note that F and G have no pole of multiplicity q where $nk < q < nk+n$. Now using the *Milloux theorem* [9, p. 55] and Lemma 2.7 we get from the definition of V

$$m(r,V) = S(r,f)+S(r,g).$$

Hence

$$\begin{aligned}(nk+n-1)\overline{N}(r,\infty;f \mid\geq k+1) &= (nk+n-1)\overline{N}(r,\infty;g \mid\geq k+1)\\ &= (nk+n-1)\overline{N}(r,\infty;F \mid\geq nk+n)\\ &\leq N(r,0;V)\\ &\leq T(r,V)+O(1)\\ &\leq N(r,\infty;V)+m(r,V)+O(1)\\ &\leq N(r,\infty;V)+S(r,f)+S(r,g)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)\\ &\quad +S(r,f)+S(r,g)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,\alpha_1;f)+\overline{N}(r,\alpha_2;f)\\ &\quad +\overline{N}(r,0;g)+\overline{N}(r,\alpha_1;g)+\overline{N}(r,\alpha_2;g)\\ &\quad +\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

This proves the lemma. ■

Lemma 2.14 *Let F, G be given by* (2.1)*, where $n \geq 7$ is an integer. Also let S be given as in Theorem* 1.1*. If $E_f(S,0)=E_g(S,0)$, then $S(r,f)=S(r,g)$.*

Proof. Since $E_f(S,0)=E_g(S,0)$, it follows that F and G share $(1,0)$. We first note that the polynomial

$$p(z) = \frac{(n-1)(n-2)}{4}\,z^n - \frac{n(n-2)}{2}\,z^{n-1} + \frac{n(n-1)}{4}\,z^{n-2} - 1$$

has only simple zeros. In fact,

$$p^{'}(z) = \frac{n(n-1)(n-2)}{4}\,z^{n-3}(z-1)^2.$$

Also we note that $p(0), p(1) \neq 0$. Thus all the zeros of $p(z)$ are simple and we denote them by w_j, $j = 1, 2, ..., n$. Since F, G share $(1, 0)$ from the second fundamental theorem we have

$$\begin{aligned}(n-2)T(r,g) &\leq \sum_{j=1}^{n} \overline{N}\left(r, w_j; g\right) + S(r,g) \\ &= \sum_{j=1}^{n} \overline{N}\left(r, w_j; f\right) + S(r,g) \\ &\leq nT(r,f) + S(r,g).\end{aligned}$$

Similarly, we can deduce

$$(n-2)T(r,f) \leq nT(r,g) + S(r,f).$$

The last inequalities imply $T(r,f) = O\left(T(r,g)\right)$ and $T(r,g) = O\left(T(r,f)\right)$ and so we have $S(r,f) = S(r,g)$. ∎

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1) and (2.2). Since $E_f(S,2) = E_g(S,2)$ and $E_f(\{\infty\},k) = E_g(\{\infty\},k)$ it follows that F, G share $(1,2)$ and $(\infty, nk+n-1)$. So $\overline{N}_*(r,\infty;f,g) = \overline{N}_*(r,\infty;F,G) \leq \overline{N}(r,\infty;F \mid\geq nk+n) = \overline{N}(r,\infty;f \mid\geq k+1)$. By a simple computation it can be easily seen that 1 is a root with multiplicity 3 of $F - \frac{1}{2}$ and hence $F - \frac{1}{2} = (f-1)^3\ Q_{n-3}(f)$, where $Q_{n-3}(f)$ is a polynomial in f of degree $n-3$ and thus

$$\begin{aligned}N_2\left(r, \frac{1}{2}; F\right) &\leq 2\overline{N}(r,1;f) + N\left(r, 0; Q_{n-3}(f)\right) \\ &\leq 2\overline{N}(r,1;f) + (n-3)T(r,f) + S(r,f).\end{aligned}$$

Suppose that $H \not\equiv 0$. Then $F \not\equiv G$. So, it follows from Lemma 2.12 that $V \not\equiv 0$. Hence, from Lemma 2.8 with $a = \frac{1}{2}$, $m = 2$, and Lemma 2.13, we obtain for $\varepsilon (> 0)$

$$\begin{aligned}&\left(\frac{n}{2}+1\right)\ \{T(r,f)+T(r,g)\} \\ &\quad\leq 2\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,1;f)+\overline{N}(r,1;g)\right\}+\overline{N}(r,\infty;f) \\ &\qquad+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)-\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \\ &\quad\leq \left(5-2\Theta(0;f)-2\Theta(1;f)-\Theta(\infty;f)+\varepsilon\right)T(r,f) \\ &\qquad+\left(5-2\Theta(0;g)-2\Theta(1;g)-\Theta(\infty;g)+\varepsilon\right)\ T(r,g) \\ &\qquad+\frac{1}{nk+n-1}\left[3T(r,f)+3T(r,g)\right]+S(r,f)+S(r,g).\end{aligned}$$

That is

$$\begin{aligned}(3.1)\qquad &\left(\frac{n}{2}-4+\Theta_f-\frac{3}{nk+n-1}-\varepsilon\right)\,T(r,f)\\ &+\left(\frac{n}{2}-4+\Theta_g-\frac{3}{nk+n-1}-\varepsilon\right)\,T(r,g)\\ &\le S(r,f)+S(r,g).\end{aligned}$$

Without loss of generality, we may suppose that there exists a set I with infinite linear measure such that

$$T(r,g)\le T(r,f),\qquad r\in I.$$

From (3.1) and Lemma 2.14, we have

$$\left[\Theta_f+\Theta_g-8+n-\frac{6}{nk+n-1}-2\varepsilon\right]\,T(r,g)\ \le S(r,g),\qquad r\in I\backslash E,$$

which leads to a contradiction for sufficiently small $\varepsilon>0$. Hence $H\equiv 0$. Then

$$F\equiv\frac{aG+b}{cG+d},\tag{3.2}$$

where a, b, c, d are constants such that $ad-bc\neq 0$. Also

$$T(r,F)=T(r,G)+O(1).\tag{3.3}$$

We now consider the following cases.

Case I. Let $ac\neq 0$. From (3.2) we get

$$\overline{N}(r,\infty;G)=\overline{N}\left(r,\frac{a}{c};F\right).\tag{3.4}$$

So, in view of (3.3), by the second fundamental theorem we get

$$\begin{aligned}T(r,F)&\le \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}\left(r,\frac{a}{c};F\right)+S(r,F)\\ &= \overline{N}(r,0;f)+2T(r,f)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)\\ &\le 5T(r,f)+S(r,f),\end{aligned}$$

as $r\in I$ and $r\longrightarrow\infty$. This, together with Lemma 2.7, gives $n\le 5$, which contradicts the condition $n\ge 7$.

Case II. Let $a\neq 0$ and $c=0$. Then $F=\alpha G+\beta$, where $\alpha=\dfrac{a}{d}$ and $\beta=\dfrac{b}{d}$.

If F has no 1-point, by the second fundamental theorem we get

$$\begin{aligned}T(r,F)&\le \overline{N}(r,0;F)+\overline{N}(r,\infty;f)+S(r,F)\\ &\le 3T(r,f)+\overline{N}(r,\infty;f)+S(r,f),\end{aligned}$$

as $r \in I$ and $r \longrightarrow \infty$. This, together with Lemma 2.7, gives $n \leq 4$, which contradicts the condition $n \geq 7$.

If F and G have a common 1-point, then $\alpha + \beta = 1$, and so

$$F \equiv \alpha G + 1 - \alpha. \tag{3.5}$$

Suppose $\alpha \neq 1$. If $1-\alpha \neq \frac{1}{2}$, then in view of (3.3) and the second fundamental theorem we get

$$\begin{aligned} 2T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,1-\alpha;F) + \overline{N}\left(r,\frac{1}{2};F\right) + \overline{N}(r,\infty;F) + S(r,F) \\ &\leq 3T(r,f) + \overline{N}(r,0;G) + (n-2)T(r,f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq (n+5)T(r,f) + S(r,f), \end{aligned}$$

as $r \in I$ and $r \longrightarrow \infty$. This, together with Lemma 2.7, gives $n \leq 5$, which contradicts the condition $n \geq 7$. If $\alpha = \frac{1}{2}$, then we have from (3.5)

$$F \equiv \frac{1}{2}(G+1).$$

So, by the second fundamental theorem we can obtain, using (3.3), that

$$\begin{aligned} 2T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r,\frac{1}{2};G\right) + \overline{N}(r,-1;G) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq 3T(r,g) + (n-2)T(r,g) + \overline{N}(r,0;F) + \overline{N}(r,\infty;g) + S(r,g) \\ &\leq (n+5)T(r,g) + S(r,g), \end{aligned}$$

as $r \in I$ and $r \longrightarrow \infty$. This, together with Lemma 2.7, gives $n \leq 5$, which contradicts the condition $n \geq 7$.

So $\alpha = 1$ and hence $F \equiv G$. So, by Lemma 2.11, we get $f \equiv g$.

Case III. Let $a = 0$ and $c \neq 0$. Then $F \equiv \dfrac{1}{\gamma G + \delta}$, where $\gamma = \dfrac{c}{b}$ and $\delta = \dfrac{d}{b}$. If F has no 1-points, then as in Case II we can deduce a contradiction. If F and G have a common 1-point, then $\gamma + \delta = 1$ and so

$$F \equiv \frac{1}{\gamma G + 1 - \gamma}. \tag{3.6}$$

Suppose $\gamma \neq 1$ If $\gamma \neq -1$, then by the second fundamental theorem we get

$$\begin{aligned} 2\,T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,\frac{1}{1-\gamma};F) + \overline{N}\left(r,\frac{1}{2};F\right) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq 3T(r,f) + \overline{N}(r,0;G) + (n-2)T(r,f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq (n+5)T(r,f) + S(r,f), \end{aligned}$$

as $r \in I$ and $r \longrightarrow \infty$. This, together with Lemma 2.7, gives $n \leq 5$, which contradicts the condition $n \geq 7$. If $\gamma = -1$ from (3.6) we have

$$F \equiv \frac{1}{-G+2}.$$

Now, the second fundamental theorem with the help of (3.3) yields

$$\begin{aligned} 2T(r,G) &\le \overline{N}(r,0;G)+\overline{N}\left(r,\frac{1}{2};G\right)+\overline{N}(r,2;G)+\overline{N}(r,\infty;G)+S(r,G)\\ &\le 3T(r,g)+(n-2)T(r,g)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,g)\\ &\le (n+3)T(r,g)+S(r,g), \end{aligned}$$

as $r\in I$ and $r\longrightarrow\infty$. This, together with Lemma 2.7, gives $n\le 3$, which contradicts the condition $n\ge 7$. So, we must have $\gamma=1$ which implies $FG\equiv 1$, which is impossible by Lemma 2.10. This completes the proof of the theorem. ■

Proof of Corollary 1.1. Let F, G be given by (2.1) and (2.2). Since $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$, it follows that f, g share (∞,k) for all large k. Also since $\min\{\Theta_f,\Theta_g\}>4-\frac{n}{2}$, for sufficiently large k we can have $\min\{\Theta_f,\Theta_g\}>4-\frac{n}{2}+\frac{3}{nk+n-1}$ and hence, by Theorem 1.1, we get the conclusion of Corollary 1.1. So, Corollary 1.1 can be treated as a special case of Theorem 1.1. ■

Proof of Theorem 1.2. Let F, G be given by (2.1) and (2.2). Since $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},2)=E_g(\{\infty\},2)$ it follows that F, G share $(1,2)$ and $(\infty,3n-1)$. Suppose that $H\not\equiv 0$. Now proceeding in the same way as done in the proof of Theorem 1.1, using Lemma 2.8 with $a=\dfrac{1}{2}$, $m=2$ and Lemma 2.13 for $k=0$ and $k=2$ we obtain

$$\begin{aligned} &\left(\frac{n}{2}+1\right)\ \{T(r,f)+T(r,g)\}\\ &\quad\le 2\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,1;f)+\overline{N}(r,1;g)\right\}+2\overline{N}(r,\infty;f)\\ &\qquad+\overline{N}(r,\infty;f\mid\ge 3)-\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\quad\le 4\ \{T(r,f)+T(r,g)\}+\frac{6}{n-1}\{T(r,f)+T(r,g)\}\\ &\qquad+\frac{3}{3n-1}\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g), \end{aligned}$$

that is

$$(3.7)\ \left(\frac{n}{2}-3-\frac{6}{n-1}-\frac{3}{3n-1}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g).$$

Clearly (3.7) implies a contradiction for $n\ge 8$ and hence $H\equiv 0$ and the rest of the theorem can be proved in the line of proof of Theorem 1.1. ■

Proof of Theorem 1.3. Let F, G be given by (2.1) and (2.2). $E_{3)}(S,f)=E_{3)}(S,g)$, $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ it follows that $E_{3)}(1,F)=E_{3)}(1;G)$, and F G share (∞,∞). We omit the detail proof since using Lemmas 2.9, 2.10 and 2.11 the proof of the theorem can be carried out along the line of the proof of Theorem 1.1. ■

Proof of Theorem 1.4. Let F, G be given by (2.1) and (2.2). $E_{m)}(S,f) = E_{m)}(S,g)$, $E_f(\{\infty\},k) = E_g(\{\infty\},k)$ it follows that $E_{m)}(1,F) = E_{m)}(1;G)$, and F G share $(\infty, nk+n-1)$. We omit the detail proof since the same can be done in the line of proof of Theorem 1.1. ∎

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