FIXED POINT THEOREMS ON INTUITIONISTIC FUZZY QUASI-METRIC SPACES WITH APPLICATION TO THE DOMAIN OF WORDS

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Abstract. In this paper, we prove intuitionistic fuzzy quasi-metric version of the Banach contraction principle which extend the famous Grabiec fixed point theorem. By using this result we show the existence of fixed point for contraction mappings on the domain of words and apply this approach to deduce the existence of solution for some recurrence equations associated to the analysis of Quicksort algorithms and divide and Conquer algorithms, respectively.

Key words and phrases: Intuitionistic fuzzy quasi-metric space, Common fixed point, G-complete intuitionistic fuzzy quasi-metric space, B-contraction, Domain of words.
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1. Introduction

Fuzzy set theory was first introduced by Zadeh [23] in 1965 to describe the situations where data are uncertain. Thenafter the concept of fuzzy sets was generalized as intuitionistic fuzzy set by Atanassov [2, 3] in 1984, has a wide range of applications in various fields.

With the help of continuous t-norm the concept of fuzzy metric space was modified by Kramosil and Michalek [13] and George and Veeramani [7]. The concept of fuzzy quasi-metric space was introduced by Gregori and Romaguera [10] by generalizing the concept of fuzzy metric space given by Kramosil and Michalek [13].

In [16], Park generalized the notion of fuzzy metric space given by George and Veeramani [7] and introduced the notion of intuitionistic fuzzy metric space. Alaca et al. [1] defined the notion of intuitionistic fuzzy metric space as a generalization of fuzzy metric space by Kramosil and Michalek [13]. On the other hand Alaca, Turkoglu and Yildiz [1] proved intuitionistic fuzzy versions of the celebrated Banach fixed point theorem and Edelstein fixed point theorem by using the notion of intuitionistic fuzzy metric space.

The concept of intuitionistic fuzzy quasi-metric space was introduced by Tirado [21] by generalizing the notion of intuitionistic fuzzy metric space given by Alaca, Turkoglu and Yildiz [1] to the quasi-metric setting and gave intuitionistic fuzzy quasi-metric version of the Banach contraction principle. Our basic references are [5], [7], [8], [9], [10], [17], [21].

In [8], Grabiec proved fuzzy versions of celebrated Banach fixed point theorem and Edelstein fixed point theorem. Romaguera, Sapena and Tirado [17] proved the Banach fixed point theorem in fuzzy quasi-metric spaces and applied the result to the domain of words.

In this paper, we prove Banach fixed point theorem in intuitionistic fuzzy quasi-metric space. The existence of a solution for a recurrence equation which appears in the average case analysis of Quicksort algorithms is obtained as an application. We generalize the results of Romaguera, Sapena and Tirado [17] and also generalize several known results.

2. Priliminaries

Definition 2.1. ([19]) A binary operation $* : [0, 1]\times[0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is commutative and associative.

(2) $*$ is continuous.

(3) $a * 1 = a$ for all $a \in [0, 1]$.

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0, 1]$.

Definition 2.2. ([19]) A binary operation $\Diamond : [0, 1]\times[0, 1] \to [0, 1]$ is a continuous t-conorm if it satisfies the following conditions:

(1) $\Diamond$ is commutative and associative.

(2) $\Diamond$ is continuous.

(3) $a \Diamond 0 = a$ for all $a \in [0, 1]$.

(4) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0, 1]$.

Remark 2.1. The concept of triangular norms (t-norm) and triangular conorms (t-conorm) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions respectively. These concepts were originally introduced by Menger [15] in his study of statistical metric spaces. Several examples for these concepts were purposed by many authors [5, 11, 12, 22].

Definition 2.3. ([21]) A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy quasi-metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M and N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(1) $M(x, y, t) + N(x, y, t) \leq 1 \; \forall x, y \in X$ and $t > 0$.

(2) $M(x, y, 0) = 0 \; \forall x, y \in X$.

(3) $M(x, y, t) = M(y, x, t) = 1$ iff $x = y \; \forall t > 0$.

(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \; \forall x, y, z \in X$ and $s, t > 0$.

(5) for all $x, y \in X$, $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

(6) $N(x, y, 0) = 1 \; \forall x, y \in X$.

(7) $N(x, y, t) = N(y, x, t) = 0$ iff $x = y \; \forall t > 0$.

(8) $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s) \; \forall x, y, z \in X$ and $s, t > 0$.

(9) for all $x, y \in X$, $N(x, y, \cdot) : [0, \infty) \to [0, 1]$ is right continuous.

In this case, we say that $(M, N, *, \Diamond)$ is an intuitionistic fuzzy quasi-metric (an ifqm) on X. If in addition M and N satisfy $M(x, y, t) = M(y, x, t)$ and $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$ then $(M, N, *, \Diamond)$ is called intuitionistic fuzzy metric on X and $(X, M, N, *, \Diamond)$ is called an intuitionistic fuzzy metric space.

Example 2.1. Let (X, d) be a quasi-metric space. Define t-norm $a * b = \min\{a, b\}$ and t-conorm $a \Diamond b = \max\{a, b\}$ and for all $x, y \in X$ and $t > 0$,

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{and} \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy quasi-metric space. We call this intuitionistic fuzzy quasi-metric (M, N) induced by the metric d the standard intuitionistic fuzzy-quasi metric. Furthermore it is easy to check that $(M_d)^{-1} = M_{d^{-1}}$, $(M_d)^i = M_{d^i}$, $(N_d)^{-1} = N_{d^{-1}}$, $(N_d)^i = N_{d^i}$. The topology $\Im_d$ generated by d coincides with the topology $\Im_{MN_d}$ generated by the induced intuitionistic fuzzy quasi-metric $(M, N, *, \Diamond)$.

Remark 2.2. It is clear that if $(X, M, N, *, \Diamond)$ is an ifqm-space then $(X, M, *)$ is a fuzzy quasi-metric space. Conversely if $(X, M, *)$ is a fuzzy quasi-metric space on X, then $(X, M, 1-M, *, \Diamond)$ is an ifqm-space where $a \Diamond b = 1-[(1-a)*(1-b)]$ for all $a, b \in [0, 1]$. If $(M, N, *, \Diamond)$ is an ifqm on X, then $(M^{-1}, N^{-1}, *, \Diamond)$ is also an ifqm on X where M^{-1} and N^{-1} are the fuzzy sets in $X \times X \times (0, \infty)$ defined by

$M^{-1}(x,\, y,\, t) = M(y,\, x,\, t)$ and $N^{-1}(x,\, y,\, t) = N(y,\, x,\, t)$. Moreover if we denote M^i and N^s, the fuzzy sets on $X^2 \times [0,\, \infty)$ given by $M^i(x,\, y,\, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$ and $N^s(x, y, t) = \max\{N(x, y, t), N^{-1}(x, y, t)\}$. Then $(M^i, N^s, *, \Diamond)$ is an intuitionistic fuzzy metric on X. In order to construct a suitable topology on an ifqm-space $(X,\, M,\, N,\, *,\, \Diamond)$ it seems natural to consider balls $B(x,\, r,\, t)$ defined similarly to Park [16] and Alaca, Turkoglu and Yildiz [1] by $B(x,\, r,\, t) = \{y \in X : M(x,\, y,\, t) > 1 - r,\, N(x,\, y,\, t) < r \text{ for all } x \in X\}$ $r \in (0,\, 1)$ and $t > 0$. Then one can prove as in Park [16] that the family of sets of the form $\{B(x,\, r,\, t) : x \in X,\, 0 < r < 1,\, t > 0\}$ is a base for the topology $\Im_{M,\, N}$ on X.

Definition 2.4. ([16]) Let $(X,\, M,\, N,\, *,\, \Diamond)$ be an intuitionistic fuzzy metric space. A sequence $\{x_n\}_n$ in X is called a Cauchy if for each $\varepsilon \in (0,\, 1)$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n,\, x_m,\, t) > 1 - \varepsilon$, $N(x_n,\, x_m,\, t) < \varepsilon$ whenever $n,\, m \geq n_0$. We say that $(X,\, M,\, N,\, *,\, \Diamond)$ is complete if every Cauchy sequence is convergent.

Definition 2.5. ([16]) A sequence $\{x_n\}_n$ in an intuitionistic fuzzy metric space $(X,\, M,\, N,\, *,\, \Diamond)$ is said to be converges to a point $x \in X$ if and only if

$$\lim_{n\to\infty} M(x, x_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x, x_n, t) = 0 \text{ for all } t > 0.$$

Definition 2.6. ([1]) Let $(X,\, M,\, N,\, *,\, \Diamond)$ be an intuitionistic fuzzy metric space. A sequence $\{x_n\}_n$ in X is called G-Cauchy if for each $p \in N$ and each $t > 0$, $\lim\limits_{n\to\infty} M(x_{n,}\, x_{n+p},\, t) = 1$ and $\lim\limits_{n\to\infty} N(x_{n,}\, x_{n+p},\, t) = 0$. We say that $(X, M, N, *, \Diamond)$ is G-complete if every G-Cauchy sequence is convergent.

3. The Banach fixed point theorem in intuitionistic fuzzy quasi-metric space

Definition 3.1. ([1]) A B-contraction on an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is a self mapping f on X such that there is a constant $k \in (0,\, 1)$ satisfying

$$M(f(x),\; f(y),\; kt) \geq M(x,\; y,\; t) \text{ and } N(f(x),\; f(y),\; kt) \leq N(x,\; y,\; t)$$

for all $x,\, y \in X$, $t > 0$.

Theorem A. ([1]) *Let $(X,\, M,\, N,\, *,\, \Diamond)$ be a G-complete intuitionistic fuzzy metric space and $f : X \to X$ be a self-map such that*

$$M(f(x),\; f(y),\; kt) \geq M(x,\; y,\; t) \text{ and } N(f(x),\; f(y),\; kt) \leq N(x,\; y,\; t),$$

for all $x,\, y \in X$, $t > 0$ with $k \in (0,\, 1)$. Then f has a unique fixed point.

Generalizing in a natural way the notations of G-completeness and B-contraction to intuitionistic fuzzy quasi-metric spaces we introduce the following concepts.

Definition 3.2. ([21]) A sequence $\{x_n\}_n$ in an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \Diamond)$ is said to be G-Cauchy if it is a G-Cauchy sequence in the intuitionistic fuzzy metric space $(X, M^i, N^s, *, \Diamond)$.

Definition 3.3. ([21]) An intuitionistic fuzzy quasi-metric space $(X, M, N, *, \Diamond)$ is called G-bicomplete if the intuitionistic fuzzy metric space $(X, M^i, N^s, *, \Diamond)$ is G-complete. In this case we say that $(M, N, *, \Diamond)$ is a fuzzy quasi-metric on X.

Definition 3.4. A B-contraction on an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \Diamond)$ is a self mapping f on X such that there is a constant $k \in (0, 1)$ satisfying

$$M(f(x),\ f(y),\ kt) \geq M(x,\ y,\ t) \text{ and } N(f(x),\ f(y),\ kt) \leq N(x,\ y,\ t),$$

for all $x, y \in X$, $t > 0$. The number k is called a contraction constant of f .

Theorem 3.1. *Let $(X, M, N, *, \Diamond)$ be a G-bicomplete intuitionistic fuzzy quasi-metric space such that*

$$\lim_{t\to\infty} M(x,\ y,\ t) = 1 \textit{ and } \lim_{t\to\infty} N(x,\ y,\ t) = 0 \textit{ for all } x,\ y \in X.$$

Then every B-contraction on X has a unique fixed point.

Proof. Let $f : X \to X$ be a B-contraction on X with contraction constant $k \in (0,1)$. Then

$$M(f(x),\ f(y),\ kt) \geq M(x,\ y,\ t) \text{ and } N(f(x),\ f(y),\ kt) \leq N(x,\ y,\ t),$$

for all $x, y \in X$, $t > 0$. It immediately follows that

$$M^i(f(x),\ f(y),\ kt) \geq M^i(x,\ y,\ t) \text{ and } N^s(f(x),\ f(y),\ kt) \leq N^s(x,\ y,\ t),$$

for all $x, y \in X$, $t > 0$. Hence f is a B-contraction on the G-complete fuzzy metric space $(X, M^i, N^s, *, \Diamond)$ and by Theorem A, f has a unique fixed point.

4. G-bicompleteness in non-Archimedean intuitionistic fuzzy quasi-metric space

Definition 4.1. ([21]) An intuitionistic fuzzy quasi-metric space $(X, M, N, *, \Diamond)$ is called a non-Archimedean intuitionistic fuzzy quasi-metric space if $(M, N, *, \Diamond)$ is a non-Archimedean intuitionistic fuzzy quasi-metric on X, that is,

$$\begin{aligned} M(x,\ y,\ t) &\geq \min\{M(x,\ z,\ t),\ M(z,\ y,\ t)\} \\ \text{and } N(x,\ y,\ t) &\leq \max\{N(x,\ z,\ t),\ N(z,\ y,\ t)\}, \end{aligned}$$

for all $x, y, z \in X$ and $t > 0$.

Lemma 4.1. *Each G-Cauchy sequence in a non-Archimedean intuitionistic fuzzy quasi-metric space is a Cauchy sequence.*

Proof. Let (x_n) be a G-Cauchy sequence in the non-Archimedean intuitionistic fuzzy quasi-metric space $(X,\ M,\ N,\ *,\ \diamondsuit)$, then for each $t > 0$, we have

$$\lim_{n\to\infty} M^i(x_n,\ x_{n+1},\ t) = 1 \text{ and } \lim_{n\to\infty} N^s(x_n,\ x_{n+1},\ t) = 0,$$

which implies that, for each $\varepsilon \in (0,\ 1)$, there is $n_0 \in N$ such that

$$M^i(x_n,\ x_{n+1},\ t) > 1-\varepsilon \text{ and } N^s(x_n,\ x_{n+1},\ t) < \varepsilon, \text{ for each } n \geq n_0,$$

Now let $m > n \geq n_0$. Then $m = n + j$, for some $j \in N$. So

$$\begin{aligned} M^i(x_n,\ x_m,\ t) &\geq \min\{M^i(x_n,\ x_{n+1},\ t),\ M^i(x_{n+1},\ x_{n+2},\ t),\\ &\quad ...,\ M^i(x_{n+j-1},\ x_{n+j},\ t)\}\\ &> 1-\varepsilon \end{aligned}$$

and

$$\begin{aligned} N^s(x_n,\ x_m,\ t) &\leq \max\{N^s(x_n,\ x_{n+1},\ t),\ N^s(x_{n+1},\ x_{n+2},\ t),\\ &\quad ...,\ N^s(x_{n+j-1},\ x_{n+j},\ t)\}\\ &< \varepsilon. \end{aligned}$$

We conclude that (x_n) is a Cauchy sequence in $(X,\ M,\ N,\ *,\ \diamondsuit)$.

Theorem 4.1. *Each bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space is G-bicomplete.*

Proof. Let (x_n) be a G-Cauchy sequence in the bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space $(X,\ M,\ N,\ *,\ \diamondsuit)$. By Lemma 4.1, (x_n) is a Cauchy sequence in $(X,\ M,\ N,\ *,\ \diamondsuit)$. Hence there is $x \in X$ such that $\lim_{n\to\infty} M^i(x, x_n, t) = 1$ and $\lim_{n\to\infty} N^s(x, x_n, t) = 0$, for all $t > 0$. We conclude that $(X, M^i, N^s, *, \diamondsuit)$ is G-complete, that is, $(X, M, N, *, \diamondsuit)$ is G-bicomplete.

Corollary 4.2. *Each complete non-Archimedean intuitionistic fuzzy metric space is G-complete.*

5. Application to the domain of words

Let Σ be a non-empty alphabet. Let Σ^∞ be the set of all finite and infinite sequences ("words") over Σ, where we adopt the convention that the empty sequence ϕ is an element of Σ^∞. The symbol $\sqsubseteq$ denote the prefix order on Σ^∞, that is, $x \sqsubseteq y \iff x$ is a prefix of y. Now, for each $x \in \Sigma^\infty$ denote by $l(x)$ the length of x. Then $l(x) \in [1, \infty)$ whenever $x \neq \phi$ and $l(\phi) = 0$. For each $x, y \in \Sigma^\infty$ let $x \sqcap y$ be the common prefix of x and y. Thus the function $d_\sqsubseteq$ defined on $\Sigma^\infty \times \Sigma^\infty$ by

$$d_{\sqsubseteq}(x,\ y) = \begin{cases} 0, & \text{if } x \sqsubseteq y \\ 2^{-l(x \sqcap y)}, & \text{otherwise} \end{cases}$$

is a quasi-metric on Σ^∞ (We adopt the convention that $2^{-\infty} = 0$). Actually, $d_{\sqsubseteq}$ is a non-Archimedean quasi-metric on Σ^∞ and the non-Archimedean quasi-metric $(d_{\sqsubseteq}\ )^s$ is the Baire metric on Σ^∞, that is, $(d_{\sqsubseteq}\ )^s(x,\ x) = 0$ and $(d_{\sqsubseteq}\ )^s(x,\ y) = 2^{-l(x \sqcap y)}$ for all $x,\ y \in \Sigma^\infty$ such that $x \neq y$. It is well known that $(d_{\sqsubseteq}\ )^s$ is complete. From this fact it is clear that $d_{\sqsubseteq}$ is bicomplete. The quasi-metric $d_{\sqsubseteq}$, which was introduced by Smyth [20], will be called the Baire quasi-metric. Observe that condition $d_{\sqsubseteq}\ (x,\ y) = 0$ can be used to distinguish between the case that x is a prefix of y and the remaining cases.

Example 5.1. Let $d_{\sqsubseteq}$ be a (non-Archimedean) quasi-metric on a set X and let $M_{d_{\sqsubseteq}}$ and $N_{d_{\sqsubseteq}}$ are fuzzy sets in $X \times X \times [0,\ \infty)$ given by

$$M_{d_{\sqsubseteq}}(x,\ y,\ t) = \frac{t}{t + d_{\sqsubseteq}(x,\ y)} \quad \text{and} \quad N_{d_{\sqsubseteq}}(x,\ y,\ t) = \frac{d_{\sqsubseteq}(x,\ y)}{t + d_{\sqsubseteq}(x,\ y)},$$

for all $x,\ y \in X$ and $t > 0$. Then $(M_{d_{\sqsubseteq}},\ N_{d_{\sqsubseteq}},\ \wedge,\ \vee)$ is a (non-Archimedean) intuitionistic fuzzy quasi-metric on X, where $\wedge$ denotes the continuous t-norm and $\vee$ denotes the continuous t-conorm given by $a \wedge b = \min\{a,\ b\}$ and $a \vee b = \max\{a,\ b\}$. It is clear that $\Im_{MN_{d_{\sqsubseteq}}} = \Im_{d_{\sqsubseteq}}$ and that $(X,\ M_{d_{\sqsubseteq}},\ N_{d_{\sqsubseteq}},\ \wedge,\ \vee)$ is bicomplete if and only if $(X,\ d_{\sqsubseteq})$ is bicomplete.

Proposition 5.1. $(\Sigma^\infty,\ M_{d_{\sqsubseteq}},\ N_{d_{\sqsubseteq}},\ \wedge,\ \vee)$ *is a G-bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space.*

Consequently, Theorem 3.1 can be applied to this useful space.

Proposition 5.2. $(\Sigma^\infty,\ M_{d_{\sqsubseteq 1}},\ N_{d_{\sqsubseteq 0}},\ \wedge,\ \vee)$ *is a G-bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space.*

The intuitionistic fuzzy non-Archimedean quasi-metric $(M_{d_{\sqsubseteq 1}},\ N_{d_{\sqsubseteq 0}},\ \wedge,\ \vee)$ is given by

$M_{d_{\sqsubseteq 1}}(x,\ y,\ 0) = 0$ and $N_{d_{\sqsubseteq 0}}(x,\ y,\ 0) = 1$ for all $x,\ y \in \Sigma^\infty$.

$M_{d_{\sqsubseteq 1}}(x,\ y,\ t) = 1$ and $N_{d_{\sqsubseteq 0}}(x,\ y,\ t) = 0$ if x is a prefix of y and $t > 0$,

$M_{d_{\sqsubseteq 1}}(x,\ y,\ t) = 1 - 2^{-l(x \sqcap y)}$
and $N_{d_{\sqsubseteq 0}}(x,\ y,\ t) = 2^{-l(x \sqcap y)}$ if x is not a prefix of y and $t \in (0,\ 1)$,

$M_{d_{\sqsubseteq 1}}(x,\ y,\ t) = 1$ and $N_{d_{\sqsubseteq 0}}(x,\ y,\ t) = 0$ if x is not a prefix of y and $t > 1$.

Proposition 5.2 allows us to apply any of the Proposition 5.1 and Theorem 3.1 to the complexity analysis of quicksort algorithm, to show, in direct way, the existence and uniqueness of solution for the following recurrence equation:

$$T(1) = 0 \text{ and } T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} T(n-1),\ n \geq 2.$$

The average case analysis of Quicksort is discussed in [14] (see also [6]), where the above recurrence equation is obtained

Consider as an alphabet Σ the set of non-negative real numbers, that is, $\Sigma = [0, \infty)$. We associate to T the functional $\Phi : \Sigma^\infty \to \Sigma^\infty$ given by

$$(\Phi(x))_1 = T(1) \quad \text{and} \quad (\Phi(x))_n = \frac{2(n-1)}{n} + \frac{n+1}{n} x_{n-1}, \text{ for all } n \geq 2.$$

If $x \in \Sigma^\infty$ has length $n < \infty$, we write $x = x_1 x_2 x_3 ... x_n$, and if x is an infinite word we write $x = x_1 x_2 x_3 ...$ Next we show that Φ is a B-contraction on the G-bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space $(\Sigma^\infty, M_{d_\sqsubseteq}, N_{d_\sqsubseteq}, \wedge, \vee)$ with contraction constant $\frac{1}{2}$.

To this end, we first note that, by construction, we have $l(\Phi(x)) = l(x) + 1$ for all $x \in \Sigma^\infty$ (in particular $l(\Phi(x)) = \infty$ whenever $l(x) = \infty$).

Furthermore, it is clear that

$$x \sqsubseteq y \Longleftrightarrow \Phi(x) \sqsubseteq \Phi(y),$$

and consequently

$$\Phi(x \sqcap y) \sqsubseteq \Phi(x) \sqcap \Phi(y), \text{ for all } x, y \in \Sigma^\infty.$$

Hence

$$l(\Phi(x \sqcap y)) \leq l(\Phi(x) \sqcap \Phi(y)), \text{ for all } x,\ y \in \Sigma^\infty.$$

From the preceding observations we deduce that for all $x,\ y \in X$, if x is a prefix of y, then

$$M_{d_\sqsubseteq}\left(\Phi(x),\ \Phi(y),\ \frac{t}{2}\right) = M_{d_\sqsubseteq}(x,\ y,\ t) = 1$$

and

$$N_{d_\sqsubseteq}\left(\Phi(x),\ \Phi(y),\ \frac{t}{2}\right) = N_{d_\sqsubseteq}(x,\ y,\ t) = 0.$$

and if x is not a prefix of y, then for all $t > 0$,

$$\begin{aligned} M_{d_\sqsubseteq}\left(\Phi(x),\ \Phi(y),\ \frac{t}{2}\right) &= \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x)\sqcap\Phi(y))}} \\ &\geq \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x\sqcap y))}} \\ &\geq \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-(l(x\sqcap y)+1)}} \\ &\geq \frac{t}{t + 2^{-l(x\sqcap y)}} \\ &\geq M_{d_\sqsubseteq}(x,\ y,\ t) \end{aligned}$$

and

$$\begin{aligned}
N_{d_{\sqsubseteq}}\left(\Phi(x),\ \Phi(y),\ \frac{t}{2}\right) &= \frac{2^{-l(\Phi(x)\sqcap\Phi(y))}}{\frac{t}{2}+2^{-l(\Phi(x)\sqcap\Phi(y))}}\\
&\leq \frac{2^{-l(\Phi(x\sqcap y))}}{\frac{t}{2}+2^{-l(\Phi(x\sqcap y))}}\\
&\leq \frac{2^{-(l(x\sqcap y)+1)}}{\frac{t}{2}+2^{-(l(x\sqcap y)+1)}}\\
&\leq \frac{2^{-l(x\sqcap y)}}{t+2^{-l(x\sqcap y)}}\\
&\leq N_{d_{\sqsubseteq}}(x,\ y,\ t).
\end{aligned}$$

Therefore, Φ is a B-contraction on $(\Sigma^\infty,\ M_{d_{\sqsubseteq}},\ N_{d_{\sqsubseteq}},\ \wedge,\ \vee)$ with contraction constant $\frac{1}{2}$. So, by Theorem 3.1, Φ has a unique fixed point $z = z_1z_2z_3...$, which is obviously the unique solution to the recurrence equation T, that is, $z_1 = 0$ and $z_n = \frac{2(n-1)}{n} + \frac{n+1}{n}z_{n-1}$ for all $n \geq 2$.

Conclusion

We conclude the paper by applying our results to the complexity analysis of Divide and Conquer algorithm. Recall [4], [18] that Divide and Conquer algorithms solve a problem by recursively splitting it into subproblems each of which is solved separately by the same algorithm, after which the results are combined into a solution of the original problem. Thus, the complexity of a Divide and Conquer algorithm typically is the solution to the recurrence equation given by

$$T(1) = c \text{ and } T(n) = aT\left(\frac{n}{b}\right) + h(n).$$

where $a,\ b,\ c \in N$ with $a,\ b \geq 2$, n range over the set $\{b^p : p = 0, 1, 2, ...\}$, and $h(n) \geq 0$ for all $n \in N$. As in the case of Quicksort algorithm, take $\Sigma = [0, \infty)$ and put $\Sigma^N = \{x \in \Sigma^\infty : l(x) = \infty\}$. Clearly Σ^N is a closed subset of $(\Sigma^\infty, (M_{d_{\sqsubseteq}})^i, (N_{d_{\sqsubseteq}})^s, \wedge, \vee)$, $(\Sigma^N, M_{d_{\sqsubseteq}}, N_{d_{\sqsubseteq}}, \wedge, \vee)$ is a non-Archimedean intuitionistic G-bicomplete fuzzy quasi-metric space by Proposition 5.1.

Now, we associate to T the functional $\Phi : \Sigma^N \to \Sigma^N$ given by

$$\begin{aligned}
(\Phi(x))_1 &= T(1) \text{ and } (\Phi(x))_n = \frac{ax_n}{b} + h(n) \text{ if } n \in \{b^p : p = 1, 2, ...\},\\
\text{and } (\Phi(x))_n &= 0 \text{ otherwise for all } x \in \Sigma^N.
\end{aligned}$$

For our purposes here it suffices to observe that for each $x,\ y \in \Sigma^N$, the following inequality holds $l(\Phi(x) \sqcap \Phi(y)) \geq 1 + l(x \sqcap y)$. In fact, If $l(x \sqcap y) = 0$, then $l(\Phi(x)\sqcap\Phi(y)) \geq 1$ and if $b^p > l(x\sqcap y)\geq b^{p-1}$, $p \geq 1$, then $b^{p+1} > l(\Phi(x)\sqcap\Phi(y))\geq b^p$.

Hence, for each $x,\ y \in \Sigma^N$ and $t > 0$, we obtain

$$\begin{aligned}
M_{d_\sqsubseteq}\left(\Phi(x),\ \Phi(y),\ \frac{t}{2}\right) &= \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x)\sqcap\Phi(y))}} \\
&\geq \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x\sqcap y))}} \\
&\geq \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-(l(x\sqcap y)+1)}} \\
&\geq \frac{t}{t + 2^{-l(x\sqcap y)}} \\
&\geq M_{d_\sqsubseteq}(x,\ y,\ t)
\end{aligned}$$

and

$$\begin{aligned}
N_{d_\sqsubseteq}\left(\Phi(x),\ \Phi(y),\ \frac{t}{2}\right) &= \frac{2^{-l(\Phi(x)\sqcap\Phi(y))}}{\frac{t}{2} + 2^{-l(\Phi(x)\sqcap\Phi(y))}} \\
&\leq \frac{2^{-l(\Phi(x\sqcap y))}}{\frac{t}{2} + 2^{-l(\Phi(x\sqcap y))}} \\
&\leq \frac{2^{-(l(x\sqcap y)+1)}}{\frac{t}{2} + 2^{-(l(x\sqcap y)+1)}} \\
&\leq \frac{2^{-l(x\sqcap y)}}{t + 2^{-l(x\sqcap y)}} \\
&\leq N_{d_\sqsubseteq}(x,\ y,\ t).
\end{aligned}$$

Therefore Φ is a B-contraction on $(\Sigma^N, M_{d_\sqsubseteq}, N_{d_\sqsubseteq}, \wedge, \vee)$ with contraction constant $\frac{1}{2}$. So, by Theorem 3.1, Φ has a unique fixed point $z = z_1 z_2 z_3 ...$ Consequently, the function F defined on $\{b^p : p = 0,\ 1,\ 2, ...\}$ by $F(b^p) = z_{b^p}$ for all $p \geq 0$, is the unique solution to the recurrence equation of the given Divide and Conquer algorithm.

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