AN IMPLICIT METHOD FOR SOLVING NONCONVEX VARIATIONAL INEQUALITIES

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Abstract. This paper presents a predictor-corrector algorithm for solving the strongly nonlinear general noncovex variational inequality, which is a class of nonconvex variational inequalities involving three nonlinear operators. We establish the equivalence between the strongly nonlinear general nonconvex variational inequalities and the fixed point problem, and show that the convergence of the predictor-corrector method only requires pseudomonotonicity, which is a weaker condition than monotonicity. Some special cases are also discussed.

Keywords: nonconvex variational inequalities, fixed point problems, predictor-corrector methods.

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1. Introduction

Variational inequality theory, introduced by Stampacchia [1], provides simple and unified framework to study a large number of problems arising in finance, economics, transportation, network and structural analysis, elasticity and optimization. Variational inequalities have been generalized and extended in many different directions using novel and innovative techniques to study wide classes of pure and applied sciences.

The existence and iterative schemes of variational inequalities have been investigated over convex sets, and that is due to the fact that all techniques are mainly based on the properties of the projection operator over convex sets. Recently, the concept of convex sets has been generalized in many different ways, which has a considerable impact in various fields, such as optimization, economics, control theory, etc.. The uniformly prox-regular sets are an immediate consequence of the generalization of convex sets, these sets are nonconvex and include convex sets as a particular case.

Recently, Bounkhel et al. [2], Noor [3], Moudafi [4], and Pang et al. [5] considered the variational inequality problems and equilibrium problems over these nonconvex sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique. In [6]–[10] Noor has shown that the projection technique can be extended to nonconvex variational inequalities and has established the equivalence between the nonconvex variational inequalities and fixed point problems using the projection technique. This equivalent alternative formulation has been used to investigate the existence of a solution of the nonconvex variational inequalities on one hand and to introduce some iterative methods on the other hand.

Motivated by the recent research going on in this area, we have introduced and studied a generalization of nonconvex variational inequalities, which is called the strongly nonlinear general nonconvex variational inequality. We then solved this class using the projection method and the Wiener-Hopf equations technique and showed that the convergece of the two methods requires the strong monotonicity and the Lipschtiz continuity of the operators [14], [15].

In this paper we present a new implicit iterative algorithm using the predictor-corrector technique to solve this class of variational inequalities, and showed that the convergence of the new algorithm requires the pseudomonotonicity, which is a weaker condition than monotonicity, and the Lipschtiz requiring continuity of the operator is not needed.

This paper is divided into four sections, the first is the introduction, the second is the preliminaries and formulation, the third is devoted to the study of the iterative algorithm, and in the fourth section the convergence is proved.

For more information about applications, numerical methods and other aspects of variational inequalities, one may refer to [1]–[18].

2. Preliminaries and formulation

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let K be a nonempty closed subset of H. Recall the following well-known concepts from nonlinear convex analysis and non-smooth analysis [11]–[13].

Definition 2.1. The proximal normal cone of Kat $u \in H$ is given by

$$N_K^P(u) := \{w \in H : \exists\alpha > 0 \mid u \in P_K[u + \alpha w]\},$$

where

$$P_K[u] = \{u^* \in K : d_K(u) \ = \|u - u^*\|\}.$$

and $d_K(\cdot)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v\in K} \|v - u\|$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. *Let K be a nonempty, closed subset in H. Then $w \in N_K^P(u)$, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle w, v-u \rangle \leq \alpha \|v-u\|^2, \forall v \in K.$$

Definition 2.2. ([11]) The Clarke normal cone, denoted by $N_K^C(u)$, is defined by

$$N_K^C(u) = \overline{co}[N_K^P(u)],$$

where $\overline{co}[S]$ denotes the closure of the convex hull of S.

It is true that $N_K^P(u) \subset N_K^C(u)$, while the converse is not true in general. Note that $N_K^C(u)$ is always a closed and convex cone and that $N_K^P(u)$ is always a convex cone but may be non-closed (see [11], [12]). Furthermore, if K is convex all the existing normal cones coincide with the normal cone in the sense of convex analysis $N_K(u)$ given by

$$N_K(u) := \{v \in H : \langle v, u^* - u \rangle \leq 0, \ \forall u^* \in K\}.$$

A new class of nonconvex sets, called *uniformly r-prox-regular sets* (see [13]), has been introduced and studied in [11]. It has been successfully used in many nonconvex applications such as optimization, economic models, dynamical systems, and differential inclusions. This class seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumption on K.

Definition 2.3. ([4]) For a given $r \in (0, \infty]$, a subset K is said to be uniformly r-prox-regular if and only if every nonzero proximal normal to K can be realized by an r-ball, that is, $\forall u \in K$ and $0 \neq w \in N_k^P(u)$, one has

$$\langle w/\|w\|, v-u \rangle \leq (1/2r)\|v-u\|^2, \ \forall v \in K.$$

Recall that for $r = +\infty$ the uniform r-prox-regularity of K is equivalent to the convexity of K. The following lemma summarizes some important consequences of the uniform-prox-regularity and is needed in the sequel.

Lemma 2.2. *Let K be a nonempty closed subset of H, $r \in (0, \infty]$ and set $K_r = \{u \in H : d(u, K) < r\}$. If K is uniformly r-prox-regular, then the following holds:*

(i) $\forall u \in K_r, \ P_K(u) \neq \varnothing$;

(ii) $\forall r \in (0, r)$, *the operator P_K is Lipschitz continuous with constant $\dfrac{r}{r-r'}$ on $K_{r'}$.*

Now we define the strongly nonlinear general nonconvex variational inequality problem (SNGNVIP) as the problem of finding $u \in H$ such that $g(u) \in K$ and

$$\langle Tu, g(v) - g(u) \rangle + \alpha \left\| g(v) - g(u) \right\|^2 \geq \langle A(u), g(v) - g(u) \rangle, \quad \forall v \in H : g(v) \in K, \tag{2.1}$$

where T, A, g are given nonlinear operators from H into itself, and α is a positive parameter. Problem (2.1) was studied by the author in [14], [15].

Special cases

• If $g = I$, the identity operator, then (1) is equivalent to finding $u \in K$, such that

$$\langle Tu, v - u \rangle + \alpha \left\| v - u \right\|^2 \geq \langle A(u), v - u \rangle, \forall v \in K, \tag{2.2}$$

which is known as the strongly nonlinear nonconvex variational inequality, and was studied in [10].

• If $A(u) \equiv 0$, then problem (2.1) is equivalent to finding $u \in K$, such that

$$\langle Tu, g(v) - g(u) \rangle + \alpha \left\| g(v) - g(u) \right\|^2 \geq 0, \forall v \in H : g(v) \in K, \tag{2.3}$$

which is called the general nonconvex variational inequality studied in [7].

• If $g \equiv I$, the identity operator, then problem (2.3) is equivalent to finding $u \in K$, such that

$$\langle Tu, v - u \rangle + \alpha \left\| v - u \right\|^2 \geq 0, \forall v \in K, \tag{2.4}$$

which is called the nonconvex variational inequality.

• if K is a convex set in H, then problem (2.4) is equivalent to finding $u \in K$, such that

$$\langle Tu, v - u \rangle \geq 0, \forall v \in K. \tag{2.5}$$

Problem (2.5) is known as the classical variational inequality, which was introduced and studied by Stampacchia [1].

Now, if K is a nonconvex (uniformly r-prox-regular) set, then problem (2.1) is equivalent to finding $u \in K$, such that

$$0 \in Tu - A(u) + g(u) - g(u) + N_K^P(g(u)), \tag{2.6}$$

where $N_K^P(g(u))$ denotes the normal cone of K at $g(u)$ in the sense of nonconvex analysis. Problem (2.6) is called the nonconvex variational inclusion problem associated with nonconvex variational inequality (2.1). This implies that the variational inequality (2.1) is equivalent to finding a zero of the sum of two monotone operators (2.6).

3. Main results

In this section, we establish the equivalence between the nonconvex variational inequality (SNGNVIP) (2.1) and the fixed point problem using the projection operator.

Lemma 3.1. ([1]) *Let P_K be the projection of H onto the uniformly r-prox-regular set K, then $u \in H : g(u) \in K$ is a solution of the strongly nonlinear nonconvex variational inequality* (2.1) *if and only if $u \in H : g(u) \in K$ satisfies the relation*

$$g(u) = P_K[g(u) - \rho Tu + \rho A(u)], \tag{3.1}$$

Lemma 3.1 implies that the strongly nonlinear general nonconvex variational inequality (2.1) is equivalent to the fixed point problem (3.1). This will be invoked to suggest the following algorithms for solving the (SNGNVIP) (2.1).

Algorithm 3.1. *For a given $u_0 \in K$, find the approximate solution u_{n+1} using the iterative scheme*

$$g(u_{n+1}) = P_K[g(u_n) - \rho Tu_n + \rho A(u_n)], n = 0, 1, 2, ...$$

Algorithm 3.1 is an explicit iterative method for solving the nonconvex variational inequality (2.1).

Now we use the fixed point formulation (3.1) to suggest another iterative method for solving (2.1) and this is the motivation behind this paper.

Algorithm 3.2. *For a given $u_0 \in K$, find the approximate solution u_{n+1} using the iterative scheme*

$$g(u_{n+1}) = P_K[g(u_n) - \rho Tu_{n+1} + \rho A(u_{n+1})], n = 0, 1, 2, ... \tag{3.2}$$

Algorithm 3.2 is an implicit iterative algorithm, which is itself difficult to excute. To overcome this drawback, we use the predictor–corrector technique to suggest a two-step iterative method for solving the nonconvex variational inequalities (2.1). We use Algorithm 3.1 as a predictor and Algorithm 3.2 as a corrector. Consequently, we have the following algorithm.

Algorithm 3.3. *For a given $u_0 \in K$, find the approximate solution u_{n+1} using the iterative scheme*

$$\begin{aligned} g(y_n) &= P_K[g(u_n) - \rho Tu_n + \rho A(u_n)], \quad n = 0, 1, 2, ... \\ g(u_{n+1}) &= P_K[g(u_n) - \rho Ty_n + \rho A(y_n)], \quad n = 0, 1, 2, ... \end{aligned}$$

Algorithm 3.3 is a two-step forward–backward iterative method, and is also known as an extragradient method. The extragradient methods for solving the classical variational inequalities (2.5) were introduced and considered in [7].

Remark 3.1. The predictor-corrector method represented by Algorithm 3.3 and the implicit method represented by Algorithm 3.2 are equivalent. This equivalent formulation is used to prove the convergence of the predictor-corrector method represented by Algorithm 3.3.

4. Convergence

In this section, we consider the convergence analysis of Algorithm 3.2, the following well-known definitions and results are needed for the proof of the convergence.

Definition 4.1. An operator $T : H \to H$ with respect to an arbitrary operator g is called:

(i) *g-monotone* if and only if

$$\langle Tu - Tv, g(u) - g(v)\rangle + \alpha \|g(v) - g(u)\|^2 \geq 0, \forall u, v \in H.$$

(ii) *g-pseudomonotone* if

$$\langle Tu, g(u) - g(v)\rangle + \alpha \|g(v) - g(u)\|^2 \geq 0,$$

implies

$$\langle Tv, g(u) - g(v)\rangle + \alpha \|g(v) - g(u)\|^2 \geq 0, \forall u, v \in H.$$

It is well-known that g-monotonicity implies g-pseudomonotonicity, but the converse is not true.

Definition 4.2. An operator g is called *r-strongly monotone* if and only if for all $u, v \in H$, there exists a constant $r > 0$, such that

$$\langle g(u) - g(v), u - v\rangle \geq r \|u - v\|^2 .$$

Lemma 4.1. ([16]) *Let (a_n) and (β_n) be sequences of nonnegative numbers satisfying:*

$$a_{n+1} \leq (1 + \beta_n)a_n$$

if $\sum_n \beta_n < +\infty$ then (a_n) is convergent.

Now we need the following lemma for the proof of the convergence.

Lemma 4.2. *Let $u \in H : g(u) \in K$ be a solution of* (2.1) *and let u_{n+1} be the approximate solution obtained from Algorithm* 3.2. *If the operator $T - A$ is g-pseudomonotone, then*

$$(1 - 4\alpha\rho_n) \|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2 . \tag{4.1}$$

where (ρ_n) is a sequence of nonnegative numbers.

Proof. Since $u \in H : g(u) \in K$ be a solution of (2.1). Then, using the *g-pseudo-monotonicity* of $T - A$, one has

$$\langle \rho_n Tv - \rho_n A(v), g(v) - g(u)\rangle + \rho\alpha \|g(v) - g(u)\|^2 \geq 0, \forall v \in K \tag{4.2}$$

Taking $v = u_{n+1}$ in (4.2), we have

$$\langle \rho_n Tu_{n+1} - \rho_n A(u_{n+1}), g(u_{n+1}) - g(u)\rangle + \rho\alpha \|g(u_{n+1}) - g(u)\|^2 \geq 0. \tag{4.3}$$

Using Lemma 3.1, equation (3.2) can be written as

$$\begin{aligned} &\langle \rho_n Tu_{n+1} - \rho_n A(u_{n+1}) + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ &\qquad + \alpha\rho_n \|g(v) - g(u_{n+1})\|^2 \geq 0, \ \forall v \in H : g(v) \in K \end{aligned} \tag{4.4}$$

Taking $v = u$ in (4.4), one has

$$\begin{aligned} &\langle \rho_n Tu_{n+1} - \rho_n A(u_{n+1}) + g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ &\qquad + \alpha\rho_n \|g(u) - g(u_{n+1})\|^2 \geq 0. \end{aligned} \tag{4.5}$$

From (4.3) and (4.5), it follows that

$$\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \geq -2\alpha\rho_n \|g(u) - g(u_{n+1})\|^2 .$$

Since

$$\begin{aligned} 2\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle &= \|g(u) - g(u_n)\|^2 \\ &\quad - \|g(u) - g(u_{n+1})\|^2 - \|g(u_{n+1}) - g(u_n)\|^2 \end{aligned}$$

one obtains

$$(1 - 4\alpha\rho_n) \|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2 .$$

Hence the claim. ∎

Theorem 4.1. *Let $u \in H : g(u) \in K$ be a solution of* (2.1) *and let u_{n+1} be the approximate solution obtained from Algorithm* 3.2*. Let the operators T, A be continuous and $T - A$ be g-pseudomonotone. If g is $r-$*strongly monotone, *H is a finite dimensional space and $\sum \rho_n < +\infty$, then the sequence $\{u_n\}$ obtained from Algorithm* 3.2 *converges to a solution u of* (2.1).

Proof. Using Lemma 4.2, then inequality (4.1) holds and we have

$$\|g(u_{n+1}) - g(u)\|^2 \leq \frac{1}{1 - 4\alpha\rho_n} \|g(u) - g(u_n)\|^2 - \frac{1}{1 - 4\alpha\rho_n} \|g(u_n) - g(u_{n+1})\|^2$$

which can be written as

$$\begin{aligned} \|g(u_{n+1}) - g(u)\|^2 &\leq (1 + \frac{4\alpha\rho_n}{1 - 4\alpha\rho_n}) \|g(u) - g(u_n)\|^2 \\ &\quad - \frac{1}{1 - 4\alpha\rho_n} \|g(u_n) - g(u_{n+1})\|^2 \end{aligned} \tag{4.6}$$

Applying Lemma 4.1 to (4.6), we have that the sequence $(\|g(u_n)-g(u)\|)_{n\in N}$ is convergent, which implies that $(g(u_n))_{n\in N}$ is bounded and, by passing to the limit in (4.1), where $\rho_n \to 0$ since $\sum \rho_n < +\infty$, we obtain

$$\lim_{n\to+\infty} \|g(u_n)-g(u_{n+1})\| = 0 \tag{4.7}$$

Since g is r-strongly monotone we have

$$\langle g(u_n)-g(u), u_n-u\rangle \geq r\,\|u_n-u\|^2,$$

i.e.,

$$\begin{aligned}\|u_n-u\|^2 &\leq r^{-1}\langle g(u_n)-g(u), u_n-u\rangle\\ &\leq r^{-1}\|g(u_n)-g(u)\|\,\|u_n-u\|,\end{aligned}$$

that is,

$$\|u_n-u\| \leq r^{-1}\|g(u_n)-g(u)\|,$$

this implies that (u_n) is bounded and thus admits a weak-cluster point $\hat{u}$. This is equivalent to the existence of a subsequence of (u_n) converges to $\hat{u}$.

Passing to the limit on a subsequence in (4.5) and using (4.7) and the continuity of T and A, we have

$$\langle T\hat{u}-A(\hat{u}), g(v)-g(\hat{u})\rangle + \rho\alpha\,\|g(v)-g(\hat{u})\|^2 \geq 0, \forall v\in H: g(v)\in K.$$

Thus, $\hat{u}\in H: g(\hat{u})\in K$ solves the strongly nonlinear general non convex variational inequality (2.1) and we have

$$\|g(u_{n+1})-g(u)\|^2 \leq \left(1+\frac{4\alpha\rho_n}{1-4\alpha\rho_n}\right)\|g(u)-g(u_n)\|^2.$$

Now, since $(\|g(u_n)-g(\hat{u})\|^2)$ converges, set $l(\hat{u}) = \lim_{n\to+\infty}\|g(u_n)-g(\hat{u})\|^2$ and let $\tilde{u}$ be another cluster point. We have

$$\begin{aligned}\|g(u_n)-g(\hat{u})\|^2 &= \|g(u_n)-g(\tilde{u})+g(\tilde{u})-g(\hat{u})\|^2\\ &= \|g(u_n)-g(\tilde{u})\|^2 + 2\langle g(u_n)-g(\tilde{u}), g(\tilde{u})-g(\hat{u})\rangle\\ &+ \|g(\tilde{u})-g(\hat{u})\|^2\end{aligned}$$

by passing to the limit of a subsequence in the last inequality, we obtain

$$l(\hat{u}) = l(\tilde{u}) + \|g(\tilde{u})-g(\hat{u})\|^2 \tag{4.8}$$

because $\lim_{n\to+\infty}\langle g(u_n)-g(\tilde{u}), g(\tilde{u})-g(\hat{u})\rangle = 0$.

Reversing the role of $\hat{u}$ and $\tilde{u}$, we have

$$l(\tilde{u}) = l(\hat{u}) + \|g(\hat{u})-g(\tilde{u})\|^2 \tag{4.9}$$

Adding (4.8) and (4.9), we obtain

$$\|g(\hat{u}) - g(\tilde{u})\|^2 = 0 \Longrightarrow g(\hat{u}) = g(\tilde{u}).$$

The $r-$strong monotonicity of g gives

$$\|\hat{u} - \tilde{u}\| \leq r^{-1} \|g(\hat{u}) - g(\tilde{u})\| = 0 \Longrightarrow \hat{u} = \tilde{u}.$$

and thus the whole sequence converges to a solution of (SNAGGING)(2.1), the required result. ■

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