FINITE GROUPS WITH $6PQ$ ELEMENTS OF THE LARGEST ORDER[1]

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Abstract. It is an interesting topic to determine the structure of a finite group with a given number of elements of the largest order. In this article, it is proved that finite groups with $6pq$ elements of the largest order, where p, q are primes and $13 < p < q$, are solvable.

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1. Introduction

In this paper, all groups considered are finite. Let $\pi_e(G)$ be a set of orders of elements of a group G, k the largest order of elements of G and n the number of cyclic subgroups of order k. If t is a positive integer, then $\pi(t)$ denotes the set of prime divisors of t, especially, $\pi(G) = \pi(|G|)$. $M_t(G)$ denotes the set of elements of order t of G, especially, $M(G) = M_k(G)$. $N_t(G)$ denotes the subgroups generated by elements of order t, e.g., $N_t(G) =< a|a \in M_t(G) >$, especially, $N(G)$ denotes the one generated by elements of the largest order. $\varphi(x)$ the Euler function of x. The other notations and terminology are standard (see [4]).

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Definition. ([9]) Two groups G_1 and G_2 are said to be conformal groups if $\pi_e(G_1) = \pi_e(G_2)$ and $|M_t(G)| = |M_t(S)|$ for all $t \in \pi_e(G_1)$.

Two conformal groups are also called Grassmann equivalent. An open problem was proposed by Thompson (see [9]): *Let G be a conformal group of a finite solvable group. Is G solvable or not?*

It is an interesting topic to determine the structure of finite groups by the number of elements of the largest order. Because it may give useful information to the open problem on the solvability of a group proposed by Thompson. Many authors had investigated the groups with some kinds of restriction to the number of elements of the largest order, they proved that these groups are solvable (see [2], [3], [6]-[8], [12]-[14]). In this paper we will continue this work and obtain the following theorem:

Main Theorem. *Let G be a finite group. If $|M(G)| = 6pq$, where p, q are primes and $13 < p < q$, then G is solvable.*

2. Preliminary results

Lemma 2.1. ([14, Lemma 2.2])] *Suppose that G has m cyclic subgroups of order l. Then $|M_l(G)| = m\varphi(l)$, particularly, $|M(G)| = n\varphi(k)$, where k is the largest order of elements of G.*

Corollary 2.1. *Let G be a group with k and n mentioned as above. If $|M(G)| = 6pq$, where p, q are primes, then all the possibilities for n, k and $\varphi(k)$ are as in Table 1.*

Table 1

n	$\varphi(k)$	k
1	$6pq$	k
3	$2pq$	(1) $r,\ 2r,$ *where* $r = 2pq+1$ *is a prime*; (2) q^2, *where* $q = 2p+1$ *is a prime*
p	$6q$	$r,\ 2r,$ *where* $r = 6q+1$ *is a prime*
$3p$	$2q$	$r,\ 2r,$ *where* $r = 2q+1$ *is a prime*
q	$6p$	$r,\ 2r,$ *where* $r = 6p+1$ *is a prime*
$3q$	$2p$	$r,\ 2r,$ *where* $r = 2p+1$ *is a prime*
pq	6	7, 14, 9, 18
$3pq$	2	3, 4, 6
$6pq$	1	2

Lemma 2.2. ([14, Theorem 1.1]) *Suppose k is the largest order of elements of G. If $|M(G)| = \varphi(k)$, then G is supersolvable.*

Let $n \in \mathbb{N}$. We say that a finite non-abelian simple group G is a simple K_n-group if $|\pi(G)| = n$.

Lemma 2.3. ([5]) *Let G be a simple K_3-group. Then G is isomorphic to one of following simple groups: $A_5(2^2\cdot3\cdot5)$, $A_6(2^3\cdot3^2\cdot5)$, $L_2(7)(2^3\cdot3\cdot7)$, $L_2(8)(2^3\cdot3^2\cdot7)$, $L_2(17)(2^4\cdot3^2\cdot17)$, $L_3(3)(2^4\cdot3^3\cdot13)$, $U_3(3)(2^5\cdot3^3\cdot7)$, $U_4(2)(2^6\cdot3^4\cdot5)$.*

Lemma 2.4. ([10, Theorem 2]) *Let G be a simple K_4-group. Then G is isomorphic to one of following simple groups:*

(1) $A_7(2^3 \cdot 3^2 \cdot 5 \cdot 7)$, $A_8(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $A_9(2^6 \cdot 3^4 \cdot 5 \cdot 7)$, $A_{10}(2^7 \cdot 3^4 \cdot 5^2 \cdot 7)$.

(2) $M_{11}(2^4 \cdot 3^2 \cdot 5 \cdot 11)$, $M_{12}(2^6 \cdot 3^3 \cdot 5 \cdot 11)$, $J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$.

(3) $L_2(16)(2^4 \cdot 3 \cdot 5 \cdot 17)$, $L_2(25)(2^3 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(49)(2^4 \cdot 3 \cdot 5^2 \cdot 7^2)$, $L_2(81)(2^4 \cdot 3^4 \cdot 5 \cdot 41)$, $L_3(4)(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $L_3(5)(2^5 \cdot 3 \cdot 5^3 \cdot 31)$, $L_3(7)(2^5 \cdot 3^2 \cdot 7^3 \cdot 19)$, $L_3(8)(2^9 \cdot 3^2 \cdot 7^2 \cdot 73)$, $L_3(17)(2^9 \cdot 3^2 \cdot 17^3 \cdot 307)$, $L_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 13)$, $S_4(4)(2^8 \cdot 3^2 \cdot 5^2 \cdot 17)$, $S_4(5)(2^6 \cdot 3^2 \cdot 5^4 \cdot 13)$, $S_4(7)(2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4)$, $S_4(9)(2^8 \cdot 3^8 \cdot 5^2 \cdot 41)$, $S_6(2)(2^9 \cdot 3^4 \cdot 5 \cdot 7)$, $O_8^+(2)(2^{12} \cdot 3^5 \cdot 5^2 \cdot 7)$, $G_2(3)(2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $U_3(4)(2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $U_3(5)(2^4 \cdot 3^2 \cdot 5^3 \cdot 7)$, $U_3(7)(2^7 \cdot 3 \cdot 7^3 \cdot 43)$, $U_3(8)(2^9 \cdot 3^4 \cdot 7 \cdot 19)$, $U_3(9)(2^5 \cdot 3^6 \cdot 5^2 \cdot 73)$, $U_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 7)$, $U_5(2)(2^{10} \cdot 3^5 \cdot 5 \cdot 11)$, $Sz(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$, $Sz(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$, ${}^3D_4(2)(2^{12} \cdot 3^4 \cdot 7^2 \cdot 13)$, ${}^2F_4(2)'(2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$.

(4) *$L_2(t)$, where t is a prime satisfying the equation $t^2 - 1 = 2^a \cdot 3^b \cdot u^c$ for some a, b, $c \geq 1$ and a prime $u > 3$.*

(5) *$L_2(2^m)$, satisfies the equations $2^m - 1 = u$ and $2^m + 1 = 3t^b$ for some $t > 3$, $b \geq 1$ and primes u, t.*

(6) *$L_2(3^m)$, satisfies the equations $3^m - 1 = 2u^b$ and $3^m + 1 = 4t$ or $3^m - 1 = 2u$ and $3^m + 1 = 4t^b$, where u and t are odd primes and $b \geq 1$.*

Lemma 2.5. ([10, Theorem 1]) *Let G be a simple K_4-group with $3 \notin \pi(G)$. Then $G \cong Sz(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$ or $G \cong Sz(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$.*

Lemma 2.6. ([14, Theorem 2.2]) *Suppose that $a \in M_r(G)$. If $\pi_e(C_G(a)) \subseteq \pi_e(\langle a \rangle)$, then $C_G(a) \leq N_r(G)$.*

Lemma 2.7. *If $|M(G)| = 2m$, where m is an odd positive integer, then:*

(1) *$k = 4$, q^r or $2q^r$, where q is an odd prime and $r \in \mathbb{N}$.*

(2) *If G is non-solvable, then $k = 2q^r$ for some odd prime q and $2 \| \varphi(k) = (q-1)q^{r-1}$. Moreover, any Sylow 2-subgroup of G contains a maximal subgroup which is elementary abelian.*

(3) *If $k = 14$, $|G| = 2^\alpha \cdot 3 \cdot 7^\beta$ and G is non-solvable, then $G \cong E \times L_2(7)$, where E is an elementary abelian 2-subgroup.*

Proof. By [6, Lemma 2.7], (1)-(2) is true. (3) is hold by the proof of [6, Lemma 3.4(2)]. ∎

Lemma 2.8. ([11, p. 11]) *Suppose that H is a proper subgroup of a group G. Let $H_1, H_2, \cdots, H_n$ be all conjugate classes of H in G. Then $\langle H_1, H_2, \cdots, H_n \rangle = H_1 H_2 \cdots H_n$.*

Lemma 2.9. *Suppose that G has n cyclic subgroups A_i of order k, where $i = 1, 2, ..., n$. Let $\{A_1, A_2, ..., A_s\}$ be a complete representative system of the conjugate classes of n cyclic subgroups of order k, and let n_i be the length of the conjugate classes containing A_i. Then the following statements hold:*

(1) $n_i = |G : N_G(A_i)|$, $n = \sum_{i=1}^{n} n_i$, $\pi(n_i) \bigcup \pi(A_i) = \pi(n_1) \bigcup \pi(A_1)$, *where* $i = 1, 2, ..., s$.

(2) $\pi(C_G(A_i)) = \pi(A_i)$, $|N_G(A_i) : C_G(A_i)| \mid \varphi(k)$ *and* $|G| = |G : N_G(A_i)||N_G(A_i) : C_G(A_i)||C_G(A_i)|$, *where* $i = 1, 2, ..., s$.

(3) *Let* $A = \langle a \rangle$, $|a| = k$. *If* $i = 1$, $|M(G)| = 2m$, *where* m *is an odd positive integer, then* G *is solvable.*

Proof. By [3, Lemma 2.6], (1), (2) are true. Now, we prove (3). Since $|a| = k$ is an element of the largest order, we have $\pi_e(C_G(a)) \subseteq \pi_e(\langle a \rangle)$. By Lemma 2.6, $C_G(A) \leq N(G)$. If $i = 1$, then the all cyclic subgroups A_i of order k are conjugate in G, so $N(G) = A^G = AA^{g_1} \cdots A^{g_{n-1}}$ by Lemma 2.8. Since $|M(G)| = \varphi(k) = 2m$, then by Lemma 2.7, $k = 4$, q^r or $2q^r$, where q is an odd positive integer, thus $\pi(N(G)) \leq 2$, so $N(G)$ is solvable. Since $N(G) \lhd G$, we get $|G/N(G)| \mid |G : C_G(A)| = n \cdot |N_G(A) : C_G(A)| \mid |M(G)|$, so $G/N(G)$ is solvable, hence G is solvable. ∎

Lemma 2.10. *([11]) Let $G = L_2(q)$. Then G has a subgroup which is isomorphic to the dihedral group containing the cyclic subgroup of order $\frac{q \pm 1}{2}$.*

3. Proof of Main Theorem

Proof of Main Theorem. We will prove the theorem step by step according to the possible values of $\varphi(k)$.

Case 1. If $n = 1$ and $\varphi(k) = 6pq$, it can be shown that G is supersolvable from Lemma 2.1 and Lemma 2.2.

Case 2. If $n = 3$ and $\varphi(k) = 2pq$, then by Corollary 2.1, (1) $k = r$ or $2r$, where $r = 2pq + 1$ is a prime; (2) $k = q$ or q^2 where $q = 2p + 1$ is a prime. Let a be an element with largest order k of G and let $A = \langle a \rangle$. Then it is a well-known fact that

$$|G| = |G : N_G(A)| \cdot |N_G(A) : C_G(A)| \cdot |C_G(A)|. \qquad (*)$$

Suppose that G is non-solvable, by Lemma 2.7 (2), we get $k = 2r$. Let $|C_G(A)| = 2^\alpha \cdot r^\beta$, where $\alpha, \beta \in \mathbb{N}$. Clearly, $C_G(A)$ is solvoble. The fact $N_G(A)/C_G(A) \leq Aut(A)$ implies that $|N_G(A)/C_G(A)| \mid \varphi(k)$, thus $N_G(A)/C_G(A)$ is solvable, hence $N_G(A)$ is solvable. Let η be the permutation representation of G on right cosets of $N_G(A)$ in G. Then $G/ker\eta \lesssim S_3$. Since $ker\eta \leq N_G(A)$ is solvable, we get G is solvable, a contradiction.

Case 3. If $n = p$ and $\varphi(k) = 6q$, then by Corollary 2.1, $k = r$ or $2r$, where $r = 6q + 1$ is a prime. Suppose that G is non-solvable, by Lemma 2.7 (2), we get $k = 2r$. Using the same argument as *Case 2*, we get $N_G(A)$ is solvable. By Lemma 2.9 (2) and (3), we have $i \geq 2$, so there exists a i such that $p \nmid n_i$, thus G is a $\{2, 3, q, r\}$-group from $(*)$. Let $|C_G(A)| = 2^\alpha \cdot r^\beta$, where $\alpha, \beta \in \mathbb{N}$. Then $C_G(A)$ has no elements of order 2^2 and r^2. If $\beta \geq 2$, then $C_G(A)$ has at least $r^2 - 1$ elements of order $2r$. By $q > p$, we get $r^2 - 1 = 6q(6q + 2) > 6pq$, a contradiction. Thus $\beta = 1$. Let R be a Sylow r-subgroup of $C_G(A)$. If $R \ntriangleleft C_G(A)$, then $C_G(A)$ has at least $r + 1$ Sylow r-subgroups, so $C_G(A)$ has at least $(r + 1) \cdot (r - 1) = r^2 - 1$ elements of order $2r$, a contradiction. So $R \lhd C_G(A)$, thus $R\, Char\, C_G(A) \lhd N_G(A)$, hence $N_G(A) \leq N_G(R)$. Since $|G : N_G(A)| = |G : N_G(R)| \cdot |N_G(R) : N_G(A)|$ and $|G : N_G(A)| < p$. By Sylow Theorem, $|G : N_G(R)| = 1$, so $R \lhd G$. Thus G/R is a $\{2, 3, q\}$-group. By Lemma 2.3 and $q > p > 13$, G/R is solvable, so is G, a contradiction.

Case 4. If $n = 3p$ and $\varphi(k) = 2q$, then by Corollary 2.1, $k = r$ or $2r$, where $r = 2q + 1$ is a prime. Suppose that G is non-solvable, by Lemma 2.7 (2), we get $k = 2r$. Using the same argument as *Case 2*, we get $N_G(A)$ is solvable and $\pi(G) \subseteq \{2, 3, p, q, r\}$. By Lemma 2.9 (2) and (3), we have $i \geq 2$. If $3 \nmid (n_1, n_2, ..., n_s)$, then there exists a i such that $3 \nmid n_i$. By $|G| = n_i \cdot |N_G(A) : C_G(A)| \cdot |C_G(A)|$, we get G is a $\{2, p, q, r\}$-group. Since p, q are primes and $q > p > 13$, we have G is solvable by Lemma 2.3 and Lemma 2.5, a contradiction. If $3 \mid (n_1, n_2, ..., n_s)$, then $3 \parallel (n_1, n_2, ..., n_s)$ by $n_1 + n_2 + \cdots + n_s = 3p$. Let $n_i = 2^{\alpha_i} \cdot 3^{\beta_i} \cdot p^{\gamma_i} \cdot q^{\theta_i} \cdot r^{\lambda_i}$, where $\alpha_i, \beta_i, \gamma_i, \theta_i, \lambda_i \in \mathbb{N}$. By $|G| = n_i \cdot |N_G(A_i) : C_G(A_i)| \cdot |C_G(A_i)|$ and $n_i < 3p$, there exists a i such that $\beta_i = 1, \gamma_i = \theta_i = \lambda_i = 0$, so G is a $\{2, 3, q, r\}$-group and $3 \parallel |G|$. Since G is non-solvable and $q > p > 13$, we get no section of G which is isomorphic to the one of simple K_3-groups, so G has a section W which is isomorphic to the one of simple K_4-groups. By Lemma 2.4, W is isomorphic to one of following simple K_4-groups:

(Ⅰ): Assume that $W \cong L_2(t)$, where t is a prime satisfying the equation $t^2 - 1 = 2^a \cdot 3^b \cdot u^c$ for some a, b, $c \geq 1$ and a prime $u > 3$.

Now, $t = r$, $u = q$. Then $r^2 - 1 = 2^a \cdot 3^b \cdot q^c$. Clearly, $b = 1$, so $2q(r + 1) = 2^a \cdot 3 \cdot q^c$, thus $2q + 2 = r + 1 = 2^{a-1} \cdot 3 \cdot q^{c-1}$. If $c \geq 2$, then $q \mid 2$, a contradiction. Hence $c = 1$. By Lemma 2.10, $L_2(r)$ has a subgroup which is isomorphic to the dihedral group containing the cyclic subgroup of order $\frac{r \pm 1}{2}$, then it has a cyclic subgroup of order $\frac{r+1}{2} = 2^{a-2} \cdot 3$, so it has the element of order 2^{a-2}. If $a \geq 5$, then G has an element of order 8, contrary to the Lemma 2.7 (2), thus $a \leq 4$, hence $q + 1 = 2^{a-2} \cdot 3 \leq 12$, a contradiction. Therefore, $W \ncong L_2(r)$.

(Ⅲ): Assume that $W \cong L_2(2^m)$, satisfies the equations $2^m - 1 = u$ and $2^m + 1 = 3t^b$ for some $t > 3$, $b \geq 1$ and primes u, t.

Now, if $u = q$, $t = r$, then $2 = 3r^b - q = 3(2q + 1)^b - q \geq 5q + 3 > 3$, a contradiction. If $u = r$, $t = q$, then $2 = 3q^b - r = 3q^b - (2q + 1)$, so $3q^b - 3 = 2q$, thus $3 \mid 2q$, hence $3 \mid q$, a contradiction. Therefore, $W \ncong L_2(2^m)$.

(ⅢⅠ): Assume that $W \cong L_2(3^m)$, satisfies the equations $3^m - 1 = 2u^b$ and $3^m + 1 = 4t$ or $3^m - 1 = 2u$ and $3^m + 1 = 4t^b$, where u and t are odd primes and $b \geq 1$.

Since $|L_2(3^m)| = \frac{3^m \cdot (3^m+1) \cdot (3^m-1)}{2}$ and $|L_2(3^m)| \mid |G|$, we have $m = 1$, it is impossible. Therefore, $W \ncong L_2(3^m)$.

Case 5. If $n = q$ and $\varphi(k) = 6p$, then by Corollary 2.1, $k = r$ or $2r$, where $r = 6p + 1$ is a prime. Suppose that G is non-solvable, by Lemma 2.7 (2), we get $k = 2r$. Using the same argument as *Case 2*, we get $N_G(A)$ is solvable and $\pi(G) \subseteq \{2, 3, p, q, r\}$. By Lemma 2.9 (2) and (3), we have $i \geq 2$. The same discussion as *Case 4*, we get G is a $\{2, 3, p, r\}$-group and $3 \parallel |G|$. Since G is non-solvable and $q > p > 13$, by Lemma 2.3, if there is a section Y of G which is isomorphic to the simple K_3-groups, then $Y \cong L_2(17)$, so $3^2 \mid |G|$, a contradiction. Thus G has a section W which is isomorphic to the one of simple K_4-groups. By Lemma 2.4, W is isomorphic to one of following simple K_4-groups:

(Ⅰ): Assume that $W \cong L_2(t)$, where t is a prime satisfying the equation $t^2 - 1 = 2^a \cdot 3^b \cdot u^c$ for some a, b, $c \geq 1$ and a prime $u > 3$.

Now, $t = r$, $u = p$. Then $r^2 - 1 = 2^a \cdot 3^b \cdot p^c$. Clearly, $b = 1$, so $6p(r + 1) = 2^a \cdot 3 \cdot p^c$, $6p + 2 = r + 1 = 2^{a-1} \cdot p^{c-1}$. If $c \geq 2$, then $p \mid 2$, a contradiction. Hence $c = 1$. By Lemma 2.10, $L_2(r)$ has a subgroup which is isomorphic to the dihedral group containing the cyclic subgroup of order $\frac{r \pm 1}{2}$, then it has a cyclic subgroup of order $\frac{r+1}{2} = 2^{a-2}$, so it has the element of order 2^{a-2}. If $a \geq 5$, then G has an element of order 8, contrary to the Lemma 2.7 (2), thus $a \leq 4$, hence $3p + 1 = 2^{a-2} \leq 4$, a contradiction. Therefore, $W \ncong L_2(r)$.

(Ⅲ): Assume that $W \cong L_2(2^m)$, satisfies the equations $2^m - 1 = u$ and $2^m + 1 = 3t^b$ for some $t > 3$, $b \geq 1$ and primes u, t.

Now, if $u = p$, $t = r$, then $2 = 3r^b - q = 3(6p + 1)^b - p \geq 17p + 3 > 3$, a contradiction. If $u = r$, $t = p$, then $2 = 3p^b - r = 3p^b - (6p + 1)$, so $p^b - 2p = 1$, thus $p \mid 1$, a contradiction. Therefore, $W \ncong L_2(2^m)$.

(ⅢⅠ): Assume that $W \cong L_2(3^m)$, satisfies the equations $3^m - 1 = 2u^b$ and $3^m + 1 = 4t$ or $3^m - 1 = 2u$ and $3^m + 1 = 4t^b$, where u and t are odd primes and $b \geq 1$.

Since $|L_2(3^m)| = \frac{3^m \cdot (3^m+1) \cdot (3^m-1)}{2}$ and $|L_2(3^m)| \mid |G|$, we have $m = 1$, it is impossible. Therefore, $W \ncong L_2(3^m)$.

Case 6. If $n = 3q$ and $\varphi(k) = 2p$, then by Corollary 2.1, $k = r$ or $2r$, where $r = 2p + 1$ is a prime. Suppose that G is non-solvable. Using the same argument as *Case 5*, we get a contradiction.

Case 7. If $n = pq$ and $\varphi(k) = 6$, then by Corollary 2.1, $k = 7$, 14, 9, 18. Suppose that G is non-solvable, by Lemma 2.7 (2), we get $k = 14$, 18. Assume that $k = 14$. By $n_1 + n_2 + \cdots + n_s = pq$, there exists a $i \in \mathbb{N}$ such that $3 \nmid n_i$, so $3^2 \nmid |G|$ by $(*)$. Since p, q are primes and $q > p > 13$, we get G is a $\{2, 3, 7\}$-group. Then by Lemma 2.7 (3), $G \cong A \times L_2(7)$, where A is an elementary abelian 2-subgroup, so G has $(2^a - 1) \times 48$ elements of order 14. Clearly, $(2^a - 1) \times 48 \neq 6pq$, a contradiction. Assume that $k = 18$. By Lemma 2.9 (2) and (3), we have $i \geq 2$. The same discussion as *Case 4*, we get G is a $\{2, 3, p\}$-group or $\{2, 3, q\}$-group. Since G is non-solvable and $q > p > 13$, by Lemma 2.3, we get G is a $\{2, 3, p\}$-group and $p = 17$. Now, G has a section W which is isomorphic to $L_2(17)$, then $8 \in \pi(L_2(17))$, contrary to the Lemma 2.7 (2).

Case 8. If $n = 3pq$ and $\varphi(k) = 2$, then by Corollary 2.1, $k = 3$, 4, 6. Let a be an element with largest order k of G. Then $C_G(\langle a \rangle)$ is a $\{2\}$-group or $\{2, 3\}$-group. Since $|N_G(\langle a \rangle) : C_G(\langle a \rangle)| \mid \varphi(k) = 2$, $k \leq 6$ and $q > p > 13$. The same discussion as *Case 4*, we get G is a $\{2\}$-group or $\{2, 3\}$-group. Hence G is solvable.

Case 9. If $n = 6pq$ and $\varphi(k) = 1$, then G is an elementary abelian 2-group for $k = 2$, the number of maximal order elements in G is $2^t - 1$ for some positive integer t, a contradiction. ∎

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