FINITE GROUPS WITH THEIR AUTOMORPHISM GROUPS HAVING ORDERS $4pq^2$ $(2 < p < q)$

Yu Zeng

Guiyun Chen[1]

School of Mathmatics and Satistics
Southwest University
Chongqing 400715
China
e-mails: zenyu@swu.edu.cn
gychen1963@163.com

Abstract. The authors find all finite nilpotent groups with automorphism groups having orders $4pq^2$ and prove that there exists no finite non-nilpotent group such that $|\mathrm{Aut}(G)| = 4pq^2(2 < p < q,\, p \nmid q^2 - 1)$. So, the authors classify finite groups with their automorphism groups having orders $4pq^2(2 < p < q,\, p \nmid q^2 - 1)$.

Keywords: a finite group; automorphism group; Sylow subgroup; order; classification.

1. Introduction

Given a positive integer n, to find out all finite groups in equation $|\mathrm{Aut}(G)| = n$ is a very difficult and interesting topic. Iyer proved in [8] that there are at most finite numbers of groups satisfy $|\mathrm{Aut}(G)| = n$. Flannery and Machale solved equations $|\mathrm{Aut}(G)| = p^n(1 \le n \le 4)$ or pq and proved that $|\mathrm{Aut}(G)| = p^n(5 \le n \le 7)$ has no solution in [5, 14]. Curran proved that $|\mathrm{Aut}(G)| = p^n(1 \le n \le 5)$ has no solution in [3]. Flym and Machale found all finite groups with $|\mathrm{Aut}(G)| = 2^n(1 \le n \le 7)$ in [6]. Chen solved the equation $|\mathrm{Aut}(G)| = p_1p_2\cdots p_n$ or pq^2 in [2]. Li found all finite groups such that $|\mathrm{Aut}(G)| = 2^3p$, p^3q or p^2q^2 in [10, 11, 13]. Huang solved the equation $|\mathrm{Aut}(G)| = prq^2$ in [7]. Xia and Chen classified finite groups with their automorphism groups having orders $4p_1p_2\cdots p_n$ in [17].

In this paper, we find all finite nilpotent groups with $|\mathrm{Aut}(G)| = 4pq^2$ and prove that there exists no finite non-nilpotent group such that $|\mathrm{Aut}(G)| = 4pq^2$ $(2 < p < q,\, p \nmid q^2 - 1)$. That is to say, we get the classification of finite groups G with $|\mathrm{Aut}(G)| = 4pq^2(2 < p < q,\, p \nmid q^2 - 1)$.

[1]Corresponding author: gychen1963@163.com

All groups considered are finite. We use C_n to denote cyclic group of order n, Cen(G) the group of central automorphisms of G, and $[A]B$ the semi-direct product of groups A and B with A normal in $[A]B$. All other notations are standard.

2. Preliminaries

In this section, we list some known results in this topic and establish some lemmas needed.

Lemma 2.1. [4] *Let P be a non-cyclic p-group such that $|P| \geq p^3$. Suppose that $|P/Z(P)| \leq p^4$. Then $|P| \mid |Aut(P)|$.*

Lemma 2.2. [9] *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, p_2, ..., p_n$ are distinct primes. Set $W(n) = \sum_{i=1}^{r} \alpha_i$. Then there exists no finite group G such that $|Aut(G)|$ is odd and $W(|Aut(G)|) \leq 4$.*

Lemma 2.3. [17] *Let G be a nilpotent group of order m, where $m = 2^{\alpha} p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ and $p_1, p_2, ..., p_n$ are distinct primes such that $p_1 < p_2 < \cdots < p_n$. If Sylow p_i-subgroup of G is cyclic, Sylow 2-subgroup of G is cyclic or is isomorphic to $C_2 \times C_2$, then $2^n \mid |Aut(G)|$ and $2^{\alpha+n-1} \mid |Aut(G)|$ when $\alpha \geq 1$.*

Lemma 2.4. [16] *Let G be a group without nontrivial abelian direct factor, and p a prime. If $p \mid (|G/G'|, |Z(G)|)$, then G has a central automorphism of order p.*

Lemma 2.5. [16] *Let $\pi(G) = \{p_1, p_2, \ldots, p_r\}$. Suppose that the Sylow decomposition of G/G' and $Z(G)$ respectively are*

$$\begin{aligned} G/G' &= \overline{G_{p_1}} \times \overline{G_{p_2}} \times \cdots \times \overline{G_{p_r}}, \quad \text{and} \\ Z(G) &= Z_{p_1} \times \cdots \times Z_{p_r}, \end{aligned}$$

where the groups on the right side may be 1. If G does not contain nontrivial abelian direct factor, then $|Cen(G)| = \prod_{i=1}^{r} \prod_{j=1}^{k_i} |Z_{p_i,j}|^{r_{i,j}}$, where $Z_{p_i,j} = \{z \in Z_{p_i} \mid z^{p_i^j} = 1\}$, $p_i^{k_i}$ is the order of $\overline{G_{p_i}}$ and $r_{i,j}$ is the number of direct factors of order p_i^j in the direct composition of $\overline{G_{p_i}}$.

Lemma 2.6. [20] *If all Sylow subgroups of a finite non-abelian group G are cyclic. Then G is a metacyclic group such that $G = \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r, ((r-1)n, m) = 1, r^n \equiv 1 (mod\ m) \rangle$.*

Lemma 2.7. [12] *For every prime p, suppose that $|Aut(G)|$ can not be divided by p^3. Then G has a Sylow Tower.*

Lemma 2.8. [11] *Suppose $G = [A]B$ and A is abelian, k is an interger such that $(k, |A|) = 1$, then the map*

$$\sigma : ab \mapsto a^k b,\ a \in A,\ b \in B,$$

is an automorphism of G.

Lemma 2.9. [17] *Suppose*

$$G = \left\langle a, b \mid a^m = b^n = 1,\ b^{-1}ab = a^r,\ ((r-1)n, m) = 1,\ r^n \equiv 1(mod\ m) \right\rangle.$$

Then one of the following holds:

(1) *If $Z(G) = 1$, then $|Aut(G)| = m\varphi(m)$.*

(2) *If $|Z(G)| = 2$, then $|Aut(G)| = 2m\varphi(m)$.*

Lemma 2.10. *Let G be a group of order $2q^2$, where q is an odd prime. Then G is supersolvable.*

Proof. It follows from the definition relations of finite groups with order $2q^2$ in [21]. ∎

Lemma 2.11. [20] *Suppose G/N and G/K are both nilpotent, then $G/(N \cap K)$ is nilpotent.*

Lemma 2.12. [13] *Let G be a group of even order such that $G = [Q]R$, where $R \in Syl_2(G)$, $Q \in Syl_q(G)$. Suppose that $Q \cong C_q \times C_q$ and $C_R(Q) = 1 = Z(G)$, where q is an odd prime. Then $2^3 \mid |Aut(G)|$.*

Lemma 2.13. *Let G be a group of even order such that $G = [P \times Q]R$, where $R \in Syl_2(G)$, $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Suppose that $P \cong C_p$, $Q \cong C_q \times C_q$ and $C_R(PQ) = 1 = Z(G)$, where p and q are distinct odd primes. Then $2^3 \mid |Aut(G)|$.*

Proof. If $|R| \geq 2^3$, consider the inner automorphisms induced by elements of R, since $Z(G) = 1$, we get $2^3 \mid |\mathrm{Aut}(G)|$. If $|R| \leq 4$, since $PQ \trianglelefteq G$, we investigate the automorphism I_a of PQ induced by any $a \in R$. Let $\widehat{R} = \{I_a \mid a \in R\}$. Since $C_R(PQ) = 1$, we get $\widehat{R} \cong R$. ∎

If $|R| = 4$, we consider $\mathrm{Aut}(PQ)$. Since $PQ = P \times Q$, then $GL(2, q) \cong Aut(Q) \leq Aut(PQ)$. Because $2^3 \mid |GL(2, q)|$, we can find a 2-element $\sigma \in Aut(PQ)\backslash\widehat{R}$ which normalizes $\widehat{R}$. Then $\sigma^{-1}I_a\sigma = I_{a'}$ holds for every $a \in R$. Because of $C_R(PQ) = 1$, a' mentioned above is unique. Since $(I_{ab})^\sigma = (I_aI_b)^\sigma = I_{a'b'}$ holds for arbitrary $a, b \in R$, we have

$$(ab)' = a'b',\ a,\ b \in R.$$

Define a map ρ as

$$(ax)^{\rho} = a'x^{\sigma},\ a \in R,\ x \in PQ.$$

Then, for every $a, b \in R$ and $x, y \in PQ$, we have

$$(axby)^{\rho}{=}(ab)'(b^{-1}xby)^{\sigma}{=}a'b'x^{I_b\sigma}y^{\sigma}{=}a'b'x^{\sigma I_{b'}}y^{\sigma}{=}a'x^{\sigma}b'y^{\sigma}{=}(ax)^{\rho}(by)^{\rho}$$

This implies that ρ is an automorphism of G. By the definition of $a^{'}$, we see that there exists some $a \in R$ such that $a \neq a^{'}$, hence $\rho \neq 1$. Obviously, $|\rho| \bigm| |\sigma|$. If $\rho \in$Inn(G), then there exists a 2-element of G, for example $y^{-1}by$ ($y \in PQ$, $b \in R$), such that $x^{y^{-1}by} = x^{\sigma}$ for any $x \in PQ$. Consequently, $x^b = x^{\sigma}$, which implies that $\sigma = I_b \in \widehat{R}$, a contradiction. Therefore $2^3 \bigm| |\mathrm{Aut}(G)|$.

If $|R| = 2$, then $|G| = 2pq^2$. But $P, Q \trianglelefteq G$, it follows from Lemma 2.10 and Lemma 2.11 that G is supersolvable. Let $R = \langle a \rangle$ and $P = \langle z \rangle$. Because G is supersolvable and $C_Q(R) = 1$, we can take x and y such that $Q = \langle x,\ y \rangle$ and $x^a = x^{-1}$, $y^a = y^{-1}$. Then $I_a \in Z(\mathrm{Aut}(Q))$ and thus $\alpha^{-1}I_a\alpha = I_a$ holds for every $\alpha \in$ Aut(Q). Every element σ of Aut(Q) can be extended to an element ρ of Aut(G), through the following definition of ρ:

$$(a'z'w')^{\rho} = a'z'(w')^{\sigma},\ a' \in R,\ z' \in R,\ w' \in Q.$$

Thus Aut(Q) is a subgroup of Aut(G) and then $2^3 \bigm| |\mathrm{Aut}(G)|$. ∎

Lemma 2.14. [13] *Let p be an odd prime. Suppose H is a p'-group acting on a p-group V and A is an H-invariant maximal abelian subgroup of V. Then H acts trivially on V only if H acts trivially on A.*

Lemma 2.15. [20] *Suppose H is a π'-group acting on an abelian π-group G, A is an H-invariant subgroup and a direct factor of G. Then there exists an H-invariant subgroup K of G such that $G = A \times K$.*

3. Main results

In this section, G is a finite group such that $|\mathrm{Aut}(G)| = 4pq^2$, where p and q are odd primes such that $2 < p < q$.

Theorem 3.1. *Suppose G is nilpotent. Then G is isomorphic to one of following groups:*

$G_1 = C_{p_1^2} \times C_{p_2}$, *where* $p_1 = q,\ \ p_1 = 2p + 1$ *and* $p_2 = 2q + 1$.
$G_2 = C_2 \times C_{p_1^2} \times C_{p_2}$, *where* $p_1 = q,\ \ p_1 = 2p + 1$ *and* $p_2 = 2q + 1$.
$G_3 = C_{p_1} \times C_{p_2}$, *where* $p_1 = 2q + 1$ *and* $p_2 = 2pq + 1$.
$G_4 = C_2 \times C_{p_1} \times C_{p_2}$, *where* $p_1 = 2q + 1$ *and* $p_2 = 2pq + 1$.
$G_5 = C_{p_1} \times C_{p_2^3}$, *where* $p_1 = 3$ *and* $p_2 = 2p + 1 = q$.
$G_6 = C_2 \times C_{p_1} \times C_{p_2^3}$, *where* $p_1 = 3$ *and* $p_2 = 2p + 1 = q$.
$G_7 = C_4 \times C_{p_1}$, *where* $p_1 = 2pq + 1$.
$G_8 = C_4 \times C_{p_1^3}$, *where* $p_1 = 2p + 1 = q$.

Proof. Since G is nilpotent, we may assume that $G = R \times P_1 \times P_2 \times \cdots \times P_m$, where $R \in Syl_2(G)$, $P_1 \in Syl_{p_1}(G)$, ... , $P_m \in Syl_{p_m}(G)$, $p_i(1 \leq i \leq m)$ are odd primes such that $p_1 < p_2 < \cdots < p_m$. Then

$$4pq^2 = |\mathrm{Aut}(G)| = (\textstyle\prod_{i=1}^{m} |\mathrm{Aut}(P_i)|) \times |\mathrm{Aut}(\mathrm{R})|.$$

We divide the proof into 5 steps.

(1) P_i is cyclic and $|P_i| \bigm| p_i^2$, where $1 \leq i \leq m$.

If $|P_i| \geq p_i^3$. By $|P_i/Z(P_i)| \bigm| |P_i/P_i \cap Z(G)|$, we have

$$|P_i/P_i \cap Z(G)| \bigm| |P_iZ(G)/Z(G)| \bigm| |G/Z(G)| \bigm| |\mathrm{Aut}(G)|.$$

This implies $|P_i/Z(P_i)| \leq p_i^2$. Then $|P_i| \bigm| |\mathrm{Aut}(P_i)|$ by Lemma 2.1. But $|\mathrm{Aut}(P_i)| \bigm| |\mathrm{Aut}(G)|$, it follows that $p_i^3 \bigm| 4pq^2$, a contradiction. Therefore $|P_i| \bigm| p_i^2$. If P_i is not cyclic, then $P_i \cong C_{p_i} \times C_{p_i}$ and thus $8 \bigm| |\mathrm{Aut}(P_i)| = 4pq^2$, a contradiction too.

(2) R is cyclic of order ≤ 8 or $R \cong C_2 \times C_2$.

Otherwise R is non-cyclic and $|R| \geq 2^3$. Similar to step (1), we can prove that $|R/Z(R)| \leq 4$. Then $|R| \bigm| |\mathrm{Aut}(R)|$ by Lemma 2.1. But $|\mathrm{Aut}(R)| \bigm| |\mathrm{Aut}(G)|$, so $8 \bigm| 4pq^2$, a contradiction. Hence R is cyclic or $|R| \leq 4$.

Now if R is cyclic, let $|R| = 2^s(s > 1)$, then $2^{s-1} \bigm| |\mathrm{Aut}(R)| \bigm| |\mathrm{Aut}(G)|$, so $2^{s-1} \leq 4$, which yields $s \leq 3$. Hence (2) follows.

(3) While $|R| = 8$ one has that $G = R \cong C_8$ and $|\mathrm{Aut}(G)| = 4$, a contradiction.

In the following steps, we need to notice the fact that $2q^2 + 1$ is not a prime when q is an odd prime such that $q > 3$.

(4) While $|R| = 1$ or 2, by Lemma 2.3, we may assume that $G = RP_1P_2$ $(P_1 \neq 1 \neq P_2)$.

If $R = 1$. Then one of the following occurs:

If $G \cong C_{p_1^2} \times C_{p_2}$, then $|\mathrm{Aut}(G)| = p_1(p_1 - 1)(p_2 - 1)$. Hence $|\mathrm{Aut}(G)| = 4pq^2(2 < p < q)$ if only if $p_1 = q$, $p_1 = 2p + 1$ and $p_2 = 2q + 1$.

If $G \cong C_{p_1^2} \times C_{p_2^2}$, then $|\mathrm{Aut}(G)| = p_1p_2(p_1 - 1)(p_2 - 1)$, it is impossible.

If $G \cong C_{p_1^2} \times C_{p_2^3}$, then $|\mathrm{Aut}(G)| = p_1p_2(p_1 - 1)(p_2 - 1)$, which concludes $p_1 - 1 = p_2 - 1 = 2$, a contradiction.

If $G \cong C_{p_1} \times C_{p_2}$, then $|\mathrm{Aut}(G)| = (p_1 - 1)(p_2 - 1)$. Hence $|\mathrm{Aut}(G)| = 4pq^2(2 < p < q)$ if only if $p_1 = 2q + 1$ and $p_2 = 2pq + 1$.

If $G \cong C_{p_1} \times C_{p_2^2}$, then $|\mathrm{Aut}(G)| = p_2(p_1 - 1)(p_2 - 1)$, which is impossible.

If $G \cong C_{p_1} \times C_{p_2^3}$, then $|\mathrm{Aut}(G)| = p_2^2(p_1 - 1)(p_2 - 1)$. Hence $|\mathrm{Aut}(G)| = 4pq^2(2 < p < q)$ if only if $p_1 = 3$ and $p_2 = 2p + 1 = q$.

Therefore G is isomorphic to G_1, G_3 or G_5.

If $|R| = 2$, then G is isomorphic to G_2, G_4 or G_6 according to the conclusion above.

(5) While $|R| = 4$, we can only have that $G = RP_1$ by Lemma 2.3.

If $G \cong C_4 \times C_{p_1}$, then $|\mathrm{Aut}(G)| = 2(p_1 - 1)$. So $|\mathrm{Aut}(G)| = 4pq^2$ $(2 < p < q)$ if only if $p_1 = 2pq + 1$.

If $G \cong C_4 \times C_{p_1^2}$, then $|\mathrm{Aut}(G)| = 2p_1(p_1 - 1)$, which is impossible.

If $G \cong C_4 \times C_{p_1^3}$, then $|\mathrm{Aut}(G)| = 2p_1^2(p_1 - 1)$. Hence $|\mathrm{Aut}(G)| = 4pq^2$ $(2 < p < q)$ if only if $p_1 = 2p + 1 = q$.

If $G \cong C_2 \times C_2 \times C_{p_1}$, then $|\mathrm{Aut}(G)| = 6(p_1 - 1)$. So $|\mathrm{Aut}(G)| = 4pq^2$ $(2 < p < q)$ if only if $p_1 = 2q^2 + 1$, $p = 3$, a contradiction.

If $G \cong C_2 \times C_2 \times C_{p_1^2}$, then $|\mathrm{Aut}(G)| = 6p_1(p_1 - 1)$, which is impossible.

If $G \cong C_2 \times C_2 \times C_{p_1^3}$, then $|\mathrm{Aut}(G)| = 6p_1^2(p_1 - 1)$, which is impossible. Therefore G is isomorphic to G_7 or G_8. The proof is completed. ■

Now we begin to study the non-nilpotent groups. We will prove that no non-nilpotent group G exists if $p \nmid q^2 - 1$. To make the proof clear, we give the following lemma.

Lemma 3.2. *Suppose that G is a non-nilpotent group such that $|Aut(G)| = 4pq^2$, where $p \nmid q^2 - 1$. Then G has a nontrivial abelian direct factor.*

Proof. Suppose that G has no nontrivial abelian direct factor.

Obviously, we may assume that $G = RPQ$, where $R \in Syl_2(G)$, $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Because $|\mathrm{Aut}(G)| = 4pq^2$, G obtains a Sylow tower by Lemma 2.7, which implies that G has a normal series $Q \trianglelefteq PQ \trianglelefteq G$ and $P \trianglelefteq RP$.

(a) $|R \cap Z(G)| \leq 2$, meanwhile $|R| \mid 4$ if $R \cap Z(G) = 1$, and $R \cong C_4$ if $|R \cap Z(G)| = 2$.

If $R \cap Z(G) = 1$, then $|R| \mid 4$ since $|R| = |RZ(G)/Z(G)|$.

If $R \cap Z(G) \neq 1$, let $H = PQ$. We prove $|R \cap Z(G)| = 2$. Since $|G/HZ(G)| \mid |G/Z(G)|_2$, $G/HZ(G)$ is abelian and thus $G' \leq HZ(G)$. We assert that $G \neq HZ(G)$. Otherwise R is a nontrivial abelian direct factor of G, which contradicts to our assumption. Hence $2 \mid (|G/G'|, |Z(G)|)$, then $2 \mid |\mathrm{Cen}(G)|$ and thus $|\mathrm{Cen}(G)|_2 = 2$ or 4 by Lemma 2.4.

If $|\mathrm{Cen}(G)|_2 = 4$, then every Sylow 2-subgroup of $G/Z(G)$ centralizes $G/Z(G)$, which means

$$G/Z(G) = RZ(G)/Z(G) \times HZ(G)/Z(G).$$

This implies that R acts trivially on $H/H \cap Z(G)$ and $H \cap Z(G)$, and then acts trivially on H. Thus $G = R \times H$. Hence by our assumption, R is non-abelian and then $|R| \geq 8$. However, $|R/Z(R)| \bigm| |R/R \cap Z(G)| \bigm| 4$, then $8 \bigm| |\mathrm{Aut}(R)|$ by Lemma 2.1. It follows that $8 \bigm| |\mathrm{Aut}(G)|$, a contradiction.

If $|\mathrm{Cen}(G)|_2 = 2$. By Lemma 2.5, we have that $|R \cap Z(G)| = 2$ and every Sylow 2-subgroup of G/G' is cyclic. This implies $R/R \cap G' \cong RG'/G'$ is cyclic. Since $G/H(R \cap Z(G)) \cong R/R \cap Z(G)$ is abelian, $G' \leq H(R \cap Z(G))$. Then $R \cap G' \leq R \cap H(R \cap Z(G)) = R \cap Z(G)$ and thus $R/R \cap Z(G)$ is cyclic, which implies R is abelian. Hence $G/H \cong R$ is abelian and $G' \leq H$, therefore $R \cap G' = 1$ and R is cyclic. Since $|R \cap Z(G)| = 2$, one has that $|R| \bigm| 8$. If $|R| = 2$, then $R \leq Z(G)$ and hence R is a nontrivial abelian direct factor of G, a contradiction. If $|R| = 8$, then $|RZ(G)/Z(G)| = |R/R \cap Z(G)| = 4$ and thus $|\mathrm{Inn}(G)|_2 = 4$. Hence $|Z(\mathrm{Inn}(G))|_2 \neq 1$ since $|\mathrm{Cen}(G)|_2 = 2$ and $4 \,||\, |\mathrm{Aut}(G)|$. Let I_u be an element of order 2 in $Z(\mathrm{Inn}(G))$, where I_u is an inner automorphism induced by $u \in R$. Hence $\langle u \rangle$ acts trivially on $H/H \cap Z(G)$ and $H \cap Z(G)$, which forces $[\langle u \rangle,\, H] = 1$. Because R is abelian, $u \in Z(G)$, a contradiction. Thus $|R \cap Z(G)| = 2$ and $R \cong C_4$.

(b) $P \cap Z(G) = 1$ and $|P| \bigm| p$.

We may assume $P \neq 1$. Because G is solvable and has a normal series $Q \trianglelefteq PQ \trianglelefteq G$, G must have a $\{2, p\}$-Hall subgroup L. Moreover $L \cap PQ \in Syl_p(G)$, which implies that L has a normal p-Sylow subgroup. Without loss of generality, we assume that $L = RP$ and $P \trianglelefteq L$. Since $|P/Z(P)| \bigm| p$, we have that P is abelian and thus $P = C_P(R) \times [R,\, P]$. Because $P \cap Z(G) \leq C_P(R)$ and $|P/P \cap Z(G)| \bigm| p$, $|P/C_P(R)| \bigm| p$ and $|[R,\, P]| = 1$ or p.

Assume $|[R, P]| = 1$. Let $B = RQ$, then B is normal in G, and $G' \leq B$ for G/B is abelian. This implies $p \bigm| |G/G'|$. Moreover if $P \cap Z(G) \neq 1$, then $p \bigm| |\mathrm{Cen}(G)|$. We have $p \bigm| |\mathrm{Inn}(G)|_p$ for there exists no nontrivial abelian direct factor of G. However $|\mathrm{Aut}(G)|_p = p$, then $p \bigm| |Z(\mathrm{Inn}(G))|$. Take $I_v \in Z(\mathrm{Inn}(G))$ such that $|I_v| = p$, where I_v is an inner automorphism induced by $v \in P$. Then $\langle v \rangle$ acts trivially on $B/B \cap Z(G)$ and $B \cap Z(G)$. Because P is abelian, $[\langle v \rangle,\, P] = 1$ and thus $v \in Z(G)$ which contradicts $|I_v| = p$. Hence $P \cap Z(G) = 1$, $|P| \mid p$.

If $|[R,\, P]| = p$, then it follows by $|P/C_P(R)| \bigm| |P/P \cap Z(G)| = p$, that $P \cap Z(G) = C_P(R)$. This implies $G = (P \cap Z(G)) \times [R,\, P]B$. Since there exists no nontrivial abelian direct factor of G, we get that $P \cap Z(G) = 1$ and $|P| \bigm| p$.

(c) $G = [P \times Q]R$.

Since $|P| \bigm| p$ and $p \nmid q^2 - 1$, P acts trivially on $Q/Q \cap Z(G)$ and $Q \cap Z(G)$ by conjugation. This gives $[P, Q] = 1$ and then $G = [P \times Q]R$.

In the following, we divide our arguments into several parts by considering the order of $G/Z(G)$.

(1) There exists no finite group G such that $|\mathrm{Aut}(G)| = 4pq^2$ and $|G/Z(G)|_q \bigm| q$.

At first, we assert that $Q\cap Z(G)=1$ and $|Q| \bigm| q$. Since $|Q/Z(Q)| \bigm| |G/Z(G)|_q \bigm| q$, Q is abelian and thus $Q = C_Q(RP)\times[RP, Q]$. Clearly, $G = RP[RP, Q]\times C_Q(RP)$. Because there is no nontrivial abelian direct factor of G, $C_Q(RP) = 1$. Therefore $Q \cap Z(G) \le C_Q(RP) = 1$, which gives $|Q| = |Q/Q \cap Z(G)| \bigm| q$.

Now, we continue the proof by investigating R.

If R is cyclic, then every Sylow subgroup of G is cyclic. It follows from Lemma 2.6 that $G = \langle a,\ b \mid a^m = b^t = 1,\ b^{-1}ab = a^r,\ ((r-1)t, m) = 1,\ r^t \equiv 1 (mod\ m)\rangle$. If $Z(G) = 1$, then $m\varphi(m) = |\mathrm{Aut}(G)| = 4pq^2$ by Lemma 2.9. If $q \bigm| m$, then $q^3 \bigm| m\varphi(m)$ or $q \,\|\, m\varphi(m)$, a contradiction. If $q^2 \bigm| \varphi(m)$, then $m \bigm| 4p$, again a contradiction. If $|Z(G)| = 2$, then $2m\varphi(m) = |\mathrm{Aut}(G)| = 4pq^2$ by Lemma 2.9, which leads to a contradiction similarly.

If $R \cong C_2 \times C_2$. When P and Q are both trivial, we have that $|\mathrm{Aut}(G)| = 6$, a contradiction. If only one of P and Q is trivial, we may assume $Q \ne 1$ and $P = 1$. In this case $Z(G) = 1$ by (a), (b) and $Q \cap Z(G) = 1$. But $C_R(Q) = 1$, so $\mathrm{Aut}(Q)$ contains a subgroup which is isomorphic to $C_2 \times C_2$, a contradiction. If $P \ne 1$ and $Q \ne 1$, then

$$G=\langle a, b, x, y \mid a^2=b^2=x^p=y^q=1=[a, b]=[x, y], x^a=x^{-1}, y^a=y, x^b=x, y^b=y^{-1}\rangle.$$

It is easy to check that $|\mathrm{Aut}(G)| = pq(p-1)(q-1)$, which contradicts $|\mathrm{Aut}(G)| = 4pq^2$.

(2) There exists no finite group G such that $|\mathrm{Aut}(G)| = 4pq^2$ and $|G/Z(G)| = q^2$ or pq^2.

Since $G = [P \times Q]R$, it follows that G is nilpotent by Lemma 2.10 and 2.11, a contradiction.

(3) There exists no finite group G such that $|\mathrm{Aut}(G)|=4pq^2$ and $|G/Z(G)|=2q^2$.

In this case, R is cyclic, $|R| \bigm| 4$ and $P=1$ by (a) and (b).

Now, we show that Q is non-abelian. Suppose that Q is abelian, then $Q = C_Q(R) \times [R,\ Q]$ and $G = R[R,\ Q] \times C_Q(R)$. Because G contains no nontrivial abelian direct factor, $C_Q(R) = 1$ holds. This gives $Q \cap Z(G) \le C_Q(R) = 1$ and then $|Q| = q^2$. If Q is cyclic, then any automorphism of Q can be extended to an automorphism of G, which forces that $\mathrm{Aut}(G)$ contains an outer automorphism of order q, a contradiction to $|G/Z(G)| = 2q^2$. Then $Q \cong C_q\times C_q$. If we can prove $Z(G) = 1$, and then by Lemma 2.12 we come to a contradiction: $2^3 \bigm| |\mathrm{Aut}(G)|$.

To prove $Z(G) = 1$, it suffices to prove that $R \cong C_2$. Otherwise, $R \cong C_4$. According to Lemma 2.10 , G is supersolvabe since $|G/Z(G)| = 2q^2$. By Lemma 2.15 we can choose x and y such that $Q = \langle x\rangle \times \langle y\rangle$, $\langle x\rangle \trianglelefteq G$ and $\langle y\rangle \trianglelefteq G$. Let $G = [\langle x\rangle]K$, where $K = R\langle y\rangle$. Define

$$\alpha\colon\ x^t b \mapsto x^{-t} b,\ b \in K,\ t \in \mathbb{Z},$$

then α is an automorphism of G of order 2. Let $R = \langle a \rangle$ and I_a be an inner automorphism of G induced by a, then $|a| = 4$ and $|I_a| = 2$. Since $C_Q(R) = 1$, it is easy to show that

$$x^{I_a} = x^{-1},\ y^{I_a} = y^{-1} \ \text{ and } \ \alpha I_a = I_a \alpha.$$

Therefore, $\langle I_a, \alpha\rangle = \langle I_a\rangle \times \langle \alpha\rangle$ makes itself a Sylow 2-subgroup of $\mathrm{Aut}(G)$. On the other hand, by Lemma 2.4, there exists a central automorphism σ ($\in \mathrm{Aut}(G)$) of order 2 since $|R \cap Z(G)| = 2$. Replacing σ by its suitable conjugate, we may assume $\sigma \in \langle I_a, \alpha\rangle$. But neither I_a nor α is a central automorphism of G, which forces $\sigma = \alpha I_a$. Then

$$yZ(G) = (yZ(G))^{\sigma} = (yZ(G))^{\alpha I_a} = y^{-1}Z(G).$$

This implies $y^2 \in Z(G)$, which contradicts $Q \cap Z(G) = 1$. Thus Q is non-abelian.

Now, we show that G does not exist.

Since Q is non-abelian and $|Q/Z(Q)| \mid |Q/Q{\cap}Z(G)|{=}q^2$, $Z(Q){=}Q \cap Z(G){>}1$ and $Q/Z(Q) \cong C_q \times C_q$. And since $|G/Z(G)| = 2q^2$, $G/Z(G)$ is supersolvable, which means that there exists $w \in Q$ such that $\langle wZ(G)\rangle \trianglelefteq G/Z(G)$. For $a \in R$, there exists an integer k such that $w^k w^a \in Q \cap Z(G) = Z(Q)$, which implies that $\langle wZ(Q)\rangle$ is R-invariant. Let $A/Z(Q) = \langle wZ(Q)\rangle$, then by Lemma 2.15, it follows that

$$Q/Z(Q) = A/Z(Q) \times B/Z(Q),$$

where both of $A/Z(Q)$ and $B/Z(Q)$ are R-invariant subgroups of $Q/Z(Q)$ with order q. This implies that A and B are R-invariant abelian subgroups of Q. By Lemma 2.14, it follows that $[R, A] \neq 1 \neq [R, B]$ (Otherwise $[R, Q] = 1$, which contradicts the fact that there exists no nontrivial abelian direct factor of G). Let $[R, A] = \langle x\rangle$ and $[R, B] = \langle y\rangle$, where x and y are elements of order q. And let $G = [A]K$, where $K = R\langle y\rangle$. Notice that A is an abelian q-group. Then for every m such that $(|A|, m) = 1$, there is an automorphism α_m of G by Lemma 2.8, defined as $(au)^{\alpha_m} = a^m u$, $a \in A$, $u \in K$. If A contains an element of order q^2, then we can take m such that $|\alpha_m| = q$. Obviously α_m is an outer automorphism of G. But $|\alpha_m \mathrm{Inn}(G)| \mid (q, 2p)$, by $|\mathrm{Aut}(G)| = 4pq^2$, one has that $\alpha_m \in TMInn(G)$, a contradiction. Therefore A an elementary abelian q-group. At this moment we can take an automorphism α_m of G , such that $|\alpha_m| = q - 1$. Clearly α_m acts untrivially on $Q \cap Z(G)$, which means that α_m is an outer automorphism of G too. But $|\alpha_m \mathrm{Inn}(G)| \mid (q - 1, 2p)$, by $|\mathrm{Aut}(G)| = 4pq^2$, it follows that $(q - 1, 2p) = 2$ by $p \nmid q^2 - 1$. Therefore $|\alpha_m \mathrm{Inn}(G)| = 2$ and $|\alpha_m^2| = \frac{q-1}{2}$ since α_m is an outer automorphism. Thus $\frac{q-1}{2} \mid 2q^2$ and so $\frac{q-1}{2} \mid 2$, which contradicts $q > p > 2$ and $p \nmid q^2 - 1$.

(4) There exists no finite group G such that $|\mathrm{Aut}(G)|{=}4pq^2$ and $|G/Z(G)|{=}4q^2$.

In this case $R \cap Z(G) = 1$, $|R| = 4$ and $P = 1$ by (a) and (b). This implies $C_R(Q) = 1$.

By Lemma 2.3 of [12] , one has that $Q = [A]B$, where A is abelian, B is cyclic, and $A = Q$ if Q is abelian. It is easy to take an automorphism α of Q having order 2, defined as

$$\alpha : ab \mapsto a^{-1}b,\ a \in A,\ b \in B.$$

If $|Q| \geq q^3$, then $Z = Q \cap Z(G) \cap A \neq 1$ and $\alpha\big|_Z \neq 1$. Let X=$\{\sigma \in Aut(Q) \mid Z^\sigma$=$Z\}$ and $\mathrm{Aut}_Z(Q) = \{\sigma \in \mathrm{Aut}(Q) \mid \sigma\big|_Z = 1\}$, then both of X and $\mathrm{Aut}_Z(Q)$ are groups such that $\mathrm{Aut}_Z(Q) \trianglelefteq X$. Since $C_R(Q) = 1$, one has that $R \leq \mathrm{Aut}_Z(Q)$. Obviously, $2 \bigm| |X/\mathrm{Aut}_Z(Q)|$ for $\alpha \in X\backslash \mathrm{Aut}_Z(Q)$ and $|\alpha| = 2$. Thus $8 \bigm| |X|$ and R is not a Sylow 2-subgroup of X. Then there exists $\beta \in X\backslash R$ such that $R^\beta = R$ and β is a 2-element. Clearly β can be extended to an outer automorphism of G, which contradicts $|G/Z(G)| = 4q^2$ for $|\beta|$ is a power of 2. This forces $|Q| = q^2$. If Q is cyclic, then every automorphism of Q can be extended to an automorphism of G, which means that there is an outer automorphism of G with order q, a contradiction. Therefore $Q = C_q \times C_q$ and $Q \cap Z(G) = 1$, which implies $Z(G) = 1$. And then by Lemma 2.12, we come to a contradiction: $2^3 \bigm| |\mathrm{Aut}(G)|$.

(5) There exists no finite group G such that $|Aut(G)|$=$4pq^2$ and $|G/Z(G)|$=$2pq^2$.

In this case R is cyclic, $|R| \bigm| 4$ and $P \cong C_p$. Since $G = [P \times Q]R$, we may define $\sigma_i \in\mathrm{Aut}(G)$ as:

$$\sigma_i : xyz \mapsto xy^iz,\ x \in R,\ y \in P,\ z \in Q,\ (i, p) = 1.$$

It is easy to show that $\{\sigma_i \mid (i,\, p) = 1,\ i \in \mathbb{Z}\}$ forms a cyclic group of $p - 1$. For any $z \in Q$ and any $i \in \mathbb{Z}$ satisfying $(i,\, p) = 1$, we have $I_z\sigma_i = \sigma_i I_z$. Thus $|\langle\sigma_i,\, I_z \mid (i,\, p) = 1,\ i \in \mathbb{Z},\ z \in Q\rangle| = (p - 1)q^2$, which implies that $(p - 1)q^2 \bigm| 4pq^2$. Noticing $p \nmid q^2 - 1$, we get that $p = 5$.

Now, we show that Q is non-abelian. Otherwise, if Q is abelian, then $Q = C_Q(RP) \times [RP,\, Q]$ and $G = RP[RP,\, Q] \times C_Q(RP)$. Because G contains no nontrivial abelian direct factor, $C_Q(RP) = 1$. This gives that $Q \cap Z(G) \leq C_Q(RP) = 1$ and then $|Q| = q^2$. If Q is cyclic, then any automorphism of Q can be extended to an automorphism of G, which forces that $\mathrm{Aut}(G)$ contains an outer automorphism of order q, a contradiction. Hence $Q \cong C_q \times C_q$. If we can prove that $Z(G) = 1$, and then by Lemma 2.13 we come to a contradiction: $2^3 \bigm| |\mathrm{Aut}(G)|$.

To prove $Z(G) = 1$, it suffices to prove that $R \cong C_2$. Otherwise, $R \cong C_4$. By Lemma 2.10 and Lemma 2.11, G is supersolvable since $G = [P \times Q]R$, $|G/Z(G)\big/PZ(G)/Z(G)| = 2q^2$ and $|G/Z(G)\big/QZ(G)/Z(G)|$=$2p$. By Lemma 2.15, we can take x and y such that $Q = \langle x\rangle \times \langle y\rangle$, $\langle x\rangle \trianglelefteq G$ and $\langle y\rangle \trianglelefteq G$. Let $R = \langle a\rangle$ and $P = \langle z\rangle$. Define α by

$$\alpha : a^iz^jx^ky^l \mapsto a^iz^jx^{-k}y^l,\ i, j, k, l \in \mathbb{Z},$$

then α is an automorphism of G having order 2. Let I_a be an inner automorphism of G induced by a, then $|I_a| = 2$ and $C_Q(R) = 1$ for $|R \cap Z(G)| = 2$ and $[P, Q] = 1 = C_Q(RP)$. Therefore,

$$x^{I_a} = x^{-1},\ y^{I_a} = y^{-1},\ z^{I_a} = z^{-1} \text{ or } z.$$

Clearly $\alpha I_a = I_a\alpha$, thus $\langle I_a, \alpha\rangle = \langle I_a\rangle \times \langle\alpha\rangle$ is a Sylow 2-subgroup of $\mathrm{Aut}(G)$. By Lemma 2.4, there exists a central automorphism $\sigma(\in\mathrm{Aut}(G))$ of order 2 since $|R \cap Z(G)| = 2$. Replacing σ by its suitable conjugate, we may assume $\sigma \in \langle I_a, \alpha\rangle$. But neither I_a nor α is a central automorphism of G, which forces $\sigma = \alpha I_a$. Then

$$yZ(G) = (yZ(G))^{\sigma} = (yZ(G))^{I_a\alpha} = y^{-1}Z(G),$$

which implies $y^2 \in Z(G)$, a contradiction to $Q\cap Z(G) = 1$. Thus Q is non-abelian.

Now, we show that G doesn't exist.

Because Q is non-abelian and $|Q/Z(Q)| \bigm| |Q/Q \cap Z(G)| = q^2$, one has that $Z(Q) = Q \cap Z(G) > 1$ and $Q/Z(Q) \cong C_q \times C_q$. Since

$$G/Z(G) = [PZ(G)/Z(G) \times QZ(G)/Z(G)]RZ(G)/Z(G),$$

it follows by Lemma 2.10 and Lemma 2.11 that $G/Z(G)$ is supersolvable, which implies that there exists $w \in Q$ such that $\langle wZ(G)\rangle \leq QZ(G)/Z(G)$ and $\langle wZ(G)\rangle \trianglelefteq G/Z(G)$. For $a \in RP$, there exists an integer k such that $w^k w^a \in Q \cap Z(G) = Z(Q)$, consequently $\langle wZ(Q)\rangle$ is RP-invariant. Let $A/Z(Q) = \langle wZ(Q)\rangle$, then by Lemma 2.15, one has that

$$Q/Z(Q) = A/Z(Q) \times B/Z(Q),$$

where both of $A/Z(Q)$ and $B/Z(Q)$ are RP-invariant subgroups of $Q/Z(Q)$ of order q. This implies A and B are RP-invariant abelian subgroups of Q.

If $[RP, Q] = 1$, then $G = RP \times Q$ and $[R, P] \neq 1$ for G has no nontrivial abelian direct factor. Because $|G/Z(G)| = 2pq^2$, we have $RP \cong \langle a, b \mid a^2 = b^5 = 1, b^a = b^{-1}\rangle$ or $\langle a, b \mid a^4 = b^5 = 1, b^a = b^{-1}\rangle$. Clearly, $|\mathrm{Aut}(RP)| = 20$ or 40. But $|\mathrm{Aut}(RP)||\mathrm{Aut}(Q)| \bigm| 20q^2$, we have that $|\mathrm{Aut}(Q)| \bigm| q^2$. By Lemma 2.2, we come to a contradiction $|Q| \leq 2$.

If $[RP, Q] \neq 1$, then $[RP, A] \neq 1 \neq [RP, B]$ by Lemma 2.14. Let $[RP, A] = \langle x\rangle$, $[RP, B] = \langle y\rangle$, where x and y are two elements of order q. And let $G = [A]K$, where $K = RP\langle y\rangle$. Notice that A is an abelian q-group. Then for every m such that $(|A|, m) = 1$, there is an automorphism α_m of G by Lemma 2.8, defined by $(au)^{\alpha_m} = a^m u$, $a \in A$, $u \in K$. If A contains an element of order q^2, then there exists an integer m such that $|\alpha_m| = q$. Obviously α_m is an outer automorphism of G. On the other hand, $|\alpha_m \mathrm{Inn}(G)| \bigm| (q, 2)$, it forces that $\alpha_m \in \mathrm{Inn}(G)$, a contradiction. Hence A is an elementary abelian q-group. By the same way we can take an automorphism α_m of G such that $|\alpha_m| = q - 1$. Clearly α_m acts untrivially on $Q \cap Z(G)$, which means that α_m is an outer automorphism of G. But $|\alpha_m \mathrm{Inn}(G)| \bigm| (q - 1, 2)$, it follows that $(q - 1, 2) = 2$. Consequently $|\alpha_m \mathrm{Inn}(G)| = 2$ and $|\alpha_m^2| = \frac{q-1}{2}$. Therefore $\frac{q-1}{2} \bigm| 2$. But $\frac{q-1}{2} \bigm| 2pq^2$ and $p \nmid q^2 - 1$, which contradicts the assumptions $q > p > 2$ and $p \nmid q^2 - 1$.

(6) There exists no finite group G such that $|\mathrm{Aut}(G)|=4pq^2$ and $|G/Z(G)|=4pq^2$.

In this case, $R \cap Z(G) = 1$, $|R| = 4$ and $P \cong C_p$. This implies $C_R(PQ) = 1$. We show that G doesn't exist.

Since $G = [P \times Q]R$, we can define $\sigma_i \in$Aut(G) as:

$$\sigma_i : xyz \mapsto xy^i z,\ x \in R,\ y \in P,\ z \in Q,\ (i, p) = 1.$$

It is easy to show that $\{\sigma_i \mid (i,\, p) = 1,\, i \in \mathbb{Z}\}$ forms a cyclic group of $p-1$. For any $z \in Q$ and any $i \in \mathbb{Z}$ satisfying $(i, p) = 1$, we have $I_z\sigma_i = \sigma_i I_z$. Thus $|\langle \sigma_i, I_z \mid (i,p) = 1, i \in \mathbb{Z}, z \in Q\rangle| = (p-1)q^2$. This implies that $(p-1)q^2 \bigm| 4pq^2$. But $p \nmid q^2 - 1$, we get that $p = 5$. Because $4 \bigm|\bigm| |\mathrm{Aut}(G)|$, $R \cap Z(G) = 1$ and $|R| = 4$, we have $\langle \sigma_i \mid (i,\, 5) = 1\rangle$ consists of inner automorphisms of G. That is to say, replacing R by its suitable conjugate, we can get the subgroup of inner automorphism group $RZ(G)/Z(G) = \langle \sigma_i \mid (i,\, 5) = 1\rangle$, then $[R, Q] = 1$, $[RP, Q] = 1$ and $[R, P] \neq 1$. Hence $G = RP \times Q$ ($R \cong C_4$, $P \cong C_5$). Therefore, RP is isomorphic to $\langle a,\, b \mid a^4 = b^5 = 1,\, b^a = b^2\rangle$ or $\langle a,\, b \mid a^4 = b^5 = 1,\, b^a = b^{-2}\rangle$. We can easily see that $|\mathrm{Aut}(RP)| = 40$, which contradicts to $|\mathrm{Aut}(RP)||\mathrm{Aut}(Q)| \bigm| 20q^2$.

This is the end of the proof. ∎

Theorem 3.3. *Suppose that G is a non-nilpotent group such that $|Aut(G)| = 4pq^2$ and $p \nmid q^2 - 1$. Then G does not exist.*

Proof. Assume that the theorem is not true, then there exists a non-nilpotent group G such that $|\mathrm{Aut}(G)| = 4pq^2$.

By Lemma 3.2, we have $G = H \times A$, where H is a non-nilpotent group without nontrivial abelian direct factor and A is a nontrivial abelian group. Hence

$$|\mathrm{Aut}(H)||\mathrm{Aut}(A)| \bigm| 4pq^2.$$

We get that $|\mathrm{Aut}(H)|$ cannot be $4pq^2$ or $4q^2$ by Lemma 3.2 and [13, Theorem 3.2]. Since q is an odd prime, by Lemma 2.2, neither $|\mathrm{Aut}(H)|$ nor $|\mathrm{Aut}(A)|$ is equal to any one of q, q^2, p, pq and pq^2. Hence,

$$|\mathrm{Aut}(H)| \in \{2p,\, 2q,\, 2pq,\, 2q^2,\, 2pq^2\}.$$

(1) If $|\mathrm{Aut}(H)| = 2p$ or $2q$, then H is nilpotent by [2, Theorem 1], a contradiction.

(2) If $|\mathrm{Aut}(H)| = 2pq$, then it follows by [2, Theorem 1] that $H \cong \langle a,\, b \mid a^m = b^q = 1,\, a^{-1}ba = b^t,\, t^m \equiv 1(\mathrm{mod} q),\, m \bigm| 2p\rangle$, $q = 2p+1$, a contradiction to $p \nmid q^2 - 1$.

(3) If $|\mathrm{Aut}(H)| = 2pq^2$, then H doesn't exist by [7, Lemma 2.2], so G does not exist.

(4) If $|\mathrm{Aut}(H)| = 2q^2$, then H doesn't exist by [2, Theorem 1], and G does not exist either.

Thus there doesn't exist a finite non-nilpotent group G such that $|\mathrm{Aut}(G)| = 4pq^2$, where $p \nmid q^2 - 1$. ■

Acknowledgements. This work is supported by National Natural Foundation of China (Grant Nos. 11171364, 11001226, 11271301), NSF of Chongqing (Grant No. CStC 2011jj A14965).

References

[1] Chen, G.Y., *Finite groups with automorphism group having all Sylow subgroups cyclic,* Proc. Royal Irish Acad., 92A (1) (1992), 37-40.

[2] Chen, G.Y., *Finite groups whose automorphism groups have order* $p_1p_2\cdots p_n$ *or* pq^2 (Chinese), Journal of Southwest Normal University (Natural Science Edition), 15 (1) (1992), 21-27.

[3] Curran, M. J., *Automorphism of certain p-groups(p odd)*, Bulletin Australian Mathematical Soc., 38 (1988), 299-305.

[4] Davitt, R. M., *On the automorphism group of a finite p-group with a small central quotient*, Canadian J. of Math., 32 (5) (1980), 1168-1176.

[5] Flannery, D., MacHale, D., *Some finite groups which are rarely automorphism groups* (I), Proc. Royal Irish Acad., 91A (1981), 209-215.

[6] Flym, J., MacHale, D., O'Brein, E.A., Sheehy, R., *Finite groups whose automorphism groups are 2-groups*, Proc. Royal Irish Acad., 94A (1994), 137-145.

[7] Huang, P.A., Qian, G.H., *Finite groups whose automorphism groups have order* prq^2 (Chinese), Acta Math. Sinica (Chin. Ser.), 22A (2) (2001), 199-204.

[8] Iyer, H.K., *On solving the equation* $Aut(X) = G$, Rocky Mountain J. Math., 9 (1979), 653-670.

[9] Li, S.R., *Automorphism groups of some finite groups*, Sci. China, Ser. A, 1993, 1276-1282.

[10] Li, S.R., *Finite Groups with automorphism Group of order* 2^3p, Proc. Royal Irish Acad., 94A (2) (1994), 193-205.

[11] Li, S.R., *Finite Groups with automorphism Group of order* p^3q, Proc. Royal Irish Acad., 94A (2) (1994), 207-218.

[12] Li, S.R., *Finite Groups whose automorphism Group has cube-free*, Chin. Ann. Math., Ser. B, 18B (1997), 301-308.

[13] Li, S.R., *On the solution of equation* $|Aut(G)| = p^2q^2$, Chin. Ann. Math., Ser. A, 22A (2) (2001), 199-204.

[14] Machale, D., *Some finite groups which are rarely automorphism groups*, Proc. Royal Irish Acad., 83A (1983), 189-196.

[15] Miller, G.A., *Groups with the same group of isomorphisms,* Trans. Amer. Math. Soc., 1 (1900), 395-401.

[16] Sanders, P.R., *The central automorphism of a finite group*, J. of the London Math. Soc., s1-44 (1) (1969), 225-228.

[17] Xia, Q.Z., Chen, G.Y., Cao, H.P., *Finite groups with their automorphism groups having orders* $4p_1p_2\cdots p_n$, Science in China (Mathematics) (Chinese Ed.), 42 (6) (2012), 611-617.

[18] Kurzweil H., Stellmacher B., *The Theory of Finite Groups*, Springer-Verlag, New York 2004.

[19] Rose J., *A Course on Group Theory*, Cambrige University Press, London, 1978.

[20] Xu, M.Y., *Finite groups* (Chinese), Science Press, Beijing, 2001.

[21] Zhang, Y.D., *The stucture of finite groups* (Chinese), Science Press, Beijing, 1982.

Accepted: 23.10.2013