FINITE p-GROUPS DETERMINED BY AN INEQUALITY OF THE ORDER OF ANY TWO-ELEMENTS GENERATED SUBGROUPS

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Abstract. In this note, we study a class of finite p-group determined by an inequality of the order of any two-elements generated subgroups. If $|\langle x, y\rangle| \leq p^i \max\{|x|, |y|\}$ for all $x, y \in G$, then G is called a M_i-group. If $|\langle x, y\rangle| \leq p^i|x|$ for all $x, y \in G$, then G is called a P_i-group. Such groups relate to a problem posed by Berkovich and Janko (Groups of prime order, Walter de Gruyter, Berlin, vol. 1, 2008) (Problem 461 and 237). In this paper, we mainly get the nilpotent class of P_i(or M_i)-groups, the exponent of derived subgroup of P_i(or M_i)-groups and $G^{p^2} \leq Z(G)$.

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1. Introduction

In this paper, only finite p-groups are considered, where p is a prime.

Finite p-groups are often determined by their subgroup structure, especially determined by their two-elements generated subgroups. For example, see [1], [5], and [6]. As in [1], authors have studied the finite p-groups with small subgroups generated by two conjugate elements and define K_1-group. Recall a finite p-group is called a K_1-group if $|\langle x, x^y\rangle : \langle x\rangle| \le p$ for all $x, y \in G$. Generally, a finite p-group is called a K_i-group if $|\langle x, x^y\rangle : \langle x\rangle| \le p^i$ for all $x, y \in G$. We denote this class by K_i.

In this paper, we consider finite p-groups in which any two-elements generated subgroups with large cyclic subgroup, that is, $|\langle x, y\rangle| \le p^i\max\{|x|, |y|\}$ for all $x, y \in G$. We define that a group G is called a M_i-group if $|\langle x, y\rangle| \le p^i\max\{|x|, |y|\}$ for all $x, y \in G$. This problem is posed by Berkovich and Janko in [2] problem 461 and 237. We denote this class by M_i.

In addition, we also consider the stronger case. We define that a group G is called a P_i-group if $|\langle x, y\rangle| \le p^i|x|$ for all $x, y \in G$. We denote this class by P_i.

Clearly, we have $P_i \subseteq M_i \subseteq K_i$. It is obvious that $K_i \nsubseteq M_i$ and $K_i \nsubseteq P_i$. For example, $Q_{2^4} \in K_1$, but $Q_{2^4} \notin M_1$ and $Q_{2^4} \notin P_1$. In addition, we have $D_{2^4} \in M_2$, but $D_{2^4} \notin P_2$

In Section 2, we mainly study P_2-group. We get the following results:

(1) If $p \ge 3$, then $G^p \le Z(G)$ and $\exp(G^{'}) \le p$. Moreover, if $p \ge 5$, then $Cl(G) \le 2$; If $p = 3$, then $Cl(G) \le 3$.

(2) If $p = 2$, then $G^4 \le Z(G)$, $\exp(G^{'}) \le 4$ and $Cl(G) \le 3$.

Moreover, in Section 3, we study the group M_i-groups. Firstly, since $M_1 \subseteq K_1$, we apply another method to get more precise results: $Cl(G) \le 2$ and $G^p \le Z(G)$, if $G \in M_1$. Secondly, we prove that G^{p^i} is cyclic and $G^{p^2} \le Z(G)$, if $G \in M_2$. Furthermore, if $G \in M_2$, we have $Cl(G) \le 4$, where $p \ne 2$ and $p \ne 5$ and the exponent of derived subgroup of G.

We use standard notation. Throughout this paper, G denotes a finite p-group for some prime p. $H \lhd G$ means that H is a normal subgroup of G. $\Omega_n(G) = \langle x \in G \mid x^{p^n} = 1\rangle$, $G^{p^n} = \langle x^{p^n} \| \ x \in G\rangle$. $Cl(G)$ is the nilpotent class of G. $\exp(G)$ is the exponent of G. $\gamma_i(G), Z_i(G)$ is the low central series, the upper central series of G, respectively. $Z(G) = Z_1(G)$. $G' = \gamma_2(G)$.

2. P_2-groups

Lemma 2.1 *Let G be a nonabelian p-group of order p^{n+1} with cyclic subgroup $A = \langle a\rangle$ of index p. Then G is isomorphic to one of the following groups:*

(a) $M_{p^{n+1}} = \langle a, b \mid a^{p^n} = b^p = 1,\ b^{-1}ab = a^{1+p^{n-1}}\rangle$, *where* $n \ge 3$ *if* $p = 2$.

(b) $p = 2$, $D_{2^{n+1}} = \langle a, b \mid a^{2^n} = b^2 = 1, bab = a^{-1}\rangle$.

(c) $p = 2$, $Q_{2^{n+1}} = \langle a, b \mid a^{2^n} = 1, b^2 = a^{2^{n-1}}, b^{-1}ab = a^{-1}\rangle$.

(d) $p = 2$, $SD_{2^{n+1}} = \langle a, b \mid a^{2^n} = b^2 = 1, bab = a^{-1+2^{n-1}}\rangle$, *where* $n > 2$.

Proof. See [2, Theorem 1.2]. ∎

Lemma 2.2 *Let $G \in P_2$ and $x \in G$ of order p. Then, for each $y \in G$, let $H = \langle x, y\rangle$, we have that H is either abelian or $Cl(H) = 2$; and $|H| \leq p^3$ and $\exp(H) = p$, when $p \geq 3$; $H \cong D_8$, when $p = 2$.*

Proof. By $G \in P_2$, we have $|H : \langle x\rangle| \leq p^2$, then $|H| \leq p^3$. If H is nonabelian, when $p \geq 3$, then $|H| = p^3$ and $\exp(H) = p$; when $p = 2$, then $H \cong D_8$. ∎

Theorem 2.3 *Let $G = \langle x, y\rangle$ is a nonabelian p-group. If $G \in P_2$, then*

(1) *when $p > 2$, then $|G| \leq p^4$ and G is a minimal nonabelian p-group;*

(2) *when $p = 2$, then $|G| \leq 2^4$; and when $|G| = 2^4$, we have $G \cong Q_{2^4}$; when $|G| = 2^3$, we have $G \cong D_{2^3}$.*

Proof. Since $G \in P_2$, we have $|\langle x, y\rangle : \langle x\rangle| \leq p^2$. Thus there exists subgroup H such that $|H : \langle x\rangle| \leq p$. Therefore, H has a cyclic subgroup $\langle x\rangle$ of index p. If $p > 2$, by Lemma 2.1, we have $H = \langle x, g\rangle$ and $|g| = p$. Since $H \in P_2$, we get $|H| \leq p^2|g| = p^3$ and $|G| \leq p^4$. If $p = 2$, we have $H = \langle x, g\rangle$ and $|g| \leq p^2$. Therefore, $|H| \leq p^2|g| = p^4$ and $|G| \leq p^5$.

If $p > 2$ and $|G| = p^4$, then G is not a maximal class group. Otherwise, we can choose generators such that $G = \langle x, g\rangle$ and $|g| = p$. We have $|G| \leq p^3$, which is a contradiction to $|G| = p^4$. Therefore, $Cl(G) \leq 2$. If $|G'| = p^2$, then $|\langle h, G'\rangle| = p^3$ for $h \notin G'$ and $\langle h, G'\rangle$ is abelian maximal subgroup of G. We get G is a maximal class group, which is a contradiction. Thus $|G'| = p$, that is, G is a minimal nonabelian p-group.

If $p = 2$ and $|G| = 2^4$, then $|G'| \neq 2$. Otherwise, we have G is a minimal nonabelian 2-group, that is, Q_8, which is a contradiction to $|G| = 2^4$. Therefore, $|G'| = 2^2$ and G is a maximal class 2-group. Since D_{2^4} and SD_{2^4} not in P_2, we have $G \cong Q_{2^4}$.

If $|G| = 2^5$, we have G is a maximal class 2-group when $|G'| = 2^3$. By Lemma 2.1, we get $|G| \leq 2^4$, which is a contradiction to $|G| = 2^5$. When $|G'| = 2$, G is Q_8, which is a contradiction to $|G| = 2^5$. Therefore, $|G'| = 2^2$. It is obvious that $|x| \geq 2^3$ and $|y| \geq 2^3$. Let $H = \langle x^p, y\rangle$, then we have $|H| = 2^4$ and $G' < H < G$. Therefore, $H \cong Q_{2^4}$ and $H = \langle x^p, g\rangle$, where $|g| = 2^2$. Thus, $G = \langle x, g\rangle$ and $|G| \leq 2^4$, which is a contradiction to $|G| = 2^5$.

In addition, it is obvious that D_{2^3} is in P_2, when $|G| = 2^3$. ∎

Theorem 2.4 *Let $G \in P_2$.*

(1) *If $p \geq 3$, then $G^p \leq Z(G)$ and $\exp(G') \leq p$. Moreover, If $p \geq 5$, then $Cl(G) \leq 2$;If $p = 3$, then $Cl(G) \leq 3$.*

(2) *If $p = 2$, then $G^4 \leq Z(G)$, $\exp(G') \leq 4$ and $Cl(G) \leq 3$.*

Proof. Let $x, y \in G$ and $H = \langle x, y\rangle$.

If $p \geq 3$, by Theorem 2.3, we get that H is either a abelian or minimal nonabelian p-group. Therefore $H^p \in Z(H)$ and $|H^{'}| \leq p$, that is, $G^p \leq Z(G)$ and $\exp(G^{'}) \leq 4$. Since $Cl(H) \leq 2$, we get G is a $2-$Engle group. Therefore, the conclusions are obvious.

If $p = 2$, by Theorem 2.3, we have $H \cong Q_{2^4}$ or $H \cong Q_{2^3}$. Therefore, $H^4 \in Z(H)$ and $|H^{'}| \leq 4$, that is, $G^4 \leq Z(G)$ and $\exp(G^{'}) \leq 4$. Let $\bar{G} = G/G^4$, then $\exp(\bar{G}) \leq 4$. Since the quotient group $\bar{G}$ is also G_2-group, by Theorem 2.3, we get $\bar{H} \cong Q_{2^3}$. Thus $\bar{G}$ is a $2-$Engle group. Therefore, $Cl(\bar{G}) \leq 2$. Note that $G^4 \leq Z(G)$, we have $Cl(G) \leq 3$. ∎

3. M_2-groups

Theorem 3.1 *Let $G \in M_i$ $(i \in N)$. Then G^{p^i} is cyclic.*

Proof. If G^{p^i} is noncyclic, let $x \in G$ such that $|x| = \exp(G)$. Then there exits $y \in G$, such that $y^{p^i} \notin \langle x^{p^i}\rangle$. Since $\langle x\rangle\langle y\rangle \subseteq \langle x, y\rangle$ and $\langle x\rangle \cap \langle y\rangle \subseteq \langle y^{p^{i+1}}\rangle$, we have $|\langle x, y\rangle| \geq |\langle x\rangle\langle y\rangle| = \dfrac{|x||y|}{|\langle x\rangle \cap \langle y\rangle|} \geq \dfrac{|x||y|}{|y^{p^{i+1}}|} \geq p^{i+1}|x|$, which contradicts $G \in M_i$. ∎

Remark. Clearly, we have $P_i \subseteq M_i$. Therefore, if $G \in P_i$, then G^{p^i} is cyclic.

Proposition 3.2 *Let $G = \langle a, b\rangle$ nonabelian. Then $G \in M_1$ if and only if G is* (a) *or* (c) *in Lemma* 2.1, *where $G = Q_{2^3}$ if G is* (c).

Proof. Let $|a| = p^n \geq |b|$. Since $G \in M_1$, we have $|\langle a, b\rangle| \leq p|a|$, that is, G be a nonabelian p-group of order p^{n+1} with cyclic subgroup $A = \langle a\rangle$ of index p. Then G is isomorphic to one of groups in Lemma 2.1.

Case 1. G is (b). Since $(ab)^2 = aa^b = 1$ and $G = \langle a, b\rangle = \langle b, ab\rangle$, then $|G| = |\langle b, ab\rangle| \leq 2^2$, which contradicts the fact that G is a nonabelian.

Case 2. G is (c). Since $(ab)^4 = ab^2(b^{-1}ab)ab^2(b^{-1}ab) = b^4 = 1$ and $G = \langle a, b\rangle = \langle b, ab\rangle$, then $|G| = |\langle b, ab\rangle| \leq 2^3$, that is, $G = Q_{2^3}$.

Case 3. G is (d). Since $(ab)^2 = abab = aa^b = a^{2^{n-1}}$, then $|ab| = 4$. By $G \in M_1$ and $G = \langle a, b\rangle = \langle b, ab\rangle$, we have $|G| = |\langle b, ab\rangle| \leq 2^3$, which contradicts $n > 2$. ∎

Theorem 3.3 *Let $G \in M_1$. Then $Cl(G) \leq 2$ and $G^p \leq Z(G)$.*

Proof. Let $H = \langle x, y\rangle$ for $x, y \in G$.

(1) If $p > 3$, then H is (a) by proposition 3.2, that is, $Cl(H) \leq 2$. Thus G is 2-Engle group, by [7, Theorem 3.3.2], we have $Cl(G) \leq 2$.

If $p = 3$, then $Cl(H) \leq 2$ by proposition 3.2 and $|H^{'}| = p$. Thus H is a regular 3-group with cyclic derived subgroup. It follows that G satisfies that every two-elements generate subgroups have cyclic derived subgroups. By [7, Theorem 5.2.11], we have G is a regular 3-group. Let $G = \langle x, y_1, ..., y_k\rangle$, where $|x| = \exp(G)$.

If $[x, y_i] \neq 1$ for every $y_i \in G(i = 1, ..., k)$, by Proposition 3.2, we can choose $y_i \in G(i = 1, ..., k)$ such that $|y_i| = p$. so $G = \langle x\rangle\Omega_1(G)$, where $\exp(\Omega_1(G)) = p$. Since $G \in P_1$, then $\Omega_1(G)$ is elementary abelian. Therefore, $Cl(G) \leq 1 + 1 = 2$.

If there exists $y_i \in G$ such that $[x, y_i] = 1$, then G has $y_j \in G$ such that $[x, y_j] \neq 1$ and $|y_j| = p$(Otherwise G is abelian, conclusion is correct). Since G is a regular 3-group, then we get $|y_iy_j| = |y_i|$ and $[x, y_iy_j] \neq 1$. So, we can substitute y_iy_j for y_i in $G = \langle x, y_1, \ldots, y_k\rangle$ such that $|y_i| = p$. Therefore, $Cl(G) \leq 2$ by proof in above paragraph.

Since $Cl(G) \leq 2$ and $|H^{'}| = p$, we have $[x^p, y] = [x, y]^p = 1$, that is, $G^p \leq Z(G)$.

(2) If $p = 2$, by Proposition 3.2, we also have $Cl(H) \leq 2$. Then, $Cl(G) \leq 2$ and $G^2 \leq Z(G)$. ■

Remark. Clearly, $M_1 \subseteq K_1$. In [1], if $G \in K_1$, authors mainly get the nilpotent class of G and $G^p \leq Z(G)$(or $G^4 \leq Z(G)$). In Theorem 3.3, we get more concise conclusions, if $G \in M_1$.

Lemma 3.4 *Let G is a maximal class 2-group. If $G \in M_2$, Then $|G| \leq 2^4$.*

Proof. Since G is a maximal class 2-group, by Lemma 2.1, then G is one of (b),(c) or (d).

If G is (b), since $G = \langle a, b\rangle = \langle b, ab\rangle$ and $(ab)^2 = 1$, then $|G| = |\langle b, ab\rangle| \leq 2^3$.

If G is (c), since $G = \langle a, b\rangle = \langle b, ab\rangle$ and $(ab)^4 = 1$, then $|G| = |\langle b, ab\rangle| \leq 2^4$.

If G is (d), since $G = \langle a, b\rangle = \langle b, ab\rangle$ and $|ab| = 4$, we have $|G| = |\langle b, ab\rangle| \leq 2^4$. ■

Theorem 3.5 *If $G \in M_2$, then $G^{p^2} \leq Z(G)$.*

Proof. Let $x \in G$ such that $|x| = \exp(G)$. By Theorem 3.1, $G^{p^2} = \langle x^{p^2}\rangle$. If $|x| \leq p^3$, by $|G^{p^2}| \leq p$ and $G^{p^2} \lhd G$, we have $G^{p^2} \leq Z(G)$. Now, suppose that $|x| = p^n \geq p^4$. Let $y \in G$ and let subgroup $H = \langle x, y\rangle$. We will prove that $H^{p^2} = \langle x^{p^2}\rangle \leq Z(H)$.

Since H is also a P_2 group, then $|H| \leq p^{n+2}$. If $|H| = p^{n+1}$, then $|H : \langle x\rangle| = p$. By Lemma 2.1, we get $H = \langle x, y_0\rangle$, where $|y_0| \leq p^2$. Since $|y_0| = p^2$ if and only if $p = 2$ and $H \cong Q_{2^{n+1}}$, it means that $H = \langle xy_0, y_0\rangle$, where $|xy_0| = |y_0| = 2^2$, a contradiction. Hence we have $y_0^p = 1$. By Lemma 2.1 and 3.4, it is easy to get that $Cl(H) \leq 2$ and $H^p \leq Z(H)$.

Now consider the case that $|H| = p^{n+2}$. Let H_1 be the maximal subgroup of H such that $x \in H_1$. Then we have $H_1 \unlhd H$ and $Cl(H_1) \leq 2$. It follows Lemma 2.1 we have $H_1 = \langle x, x_1\rangle$, where $x_1^p = 1, x^{x_1} = x^{1+p^{n-1}}$ or $x^{x_1} = x$. We see that $\Omega_1(H_1) = \langle x^{p^{n-1}}, x_1\rangle$ and $\Omega_1(H_1) \leq \Phi(H)$. Let the quotient group $\bar{H} = H/\Omega_1(H_1)$ and $\bar{x} = x\Omega_1(H_1)$. Then $|\bar{H} : \langle\bar{x}\rangle| = p$. If $p = 2$, by $|\bar{x}| \geq p^3$, we get $\bar{H}$ is a $2-$group of maximal class. Hence there is an element $y_1 \in H$ such that $\bar{H} = \langle\bar{x}, \bar{y_1}\rangle$, where $|\bar{y_1}| \leq 4$. Then we have $\bar{H} = \langle x\bar{y}_1, \bar{y_1}\rangle$, where $|x\bar{y}_1| = |\bar{y_1}| \leq 4$. Since $\Omega_1(H_1) \leq \Phi(H)$, we have $H = \langle xy_1, y_1\rangle$ and $|xy_1| = |y_1| \leq 8$. Since $H \in M_2$, we have $|H| \leq 2^5$. By $2^n \geq 2^4$, which contradicts with $|H| = 2^{n+2} > 2^5$. Hence by Lemma 2.1 we get that $Cl(\bar{H}) \leq 2$. It follows that $\bar{H} = \langle\bar{x}, \bar{y_1}\rangle$, where $|\bar{y_1}| = p$

and $\bar{x}^{\bar{y_1}} = \bar{x}$ or $\bar{x}^{\bar{y_1}} = \bar{x}^{1+p^{n-2}}$. Then we have $H = \langle x, y_1 \rangle$ and $x^{y_1} = xx_1^k$ or $x^{y_1} = x^{1+p^{n-2}}x_1^k$, where $0 \leq k \leq p-1$.

Note that H_1' is a subgroup of order p. If $x^{y_1} = xx_1^k$, by Hall-Petrescu formula, we have

$$(x^{p^2})^{y_1} = (xx_1^k)^{p^2} = x^{p^2}(x_1^k)^{p^2} = x^{p^2},$$

which means that $x^{p^2} \leq Z(H)$. If $x^{y_1} = x^{1+p^{n-2}}x_1^k$, similarly, by Hall-Petrescu formula, we have $(x^{p^2})^{y_1} = (x^{1+p^{n-2}}x_1^k)^{p^2} = x^{p^2}$, and then $x^{p^2} \leq Z(H)$. So, we get that $G^{p^2} \leq Z(G)$. ∎

Lemma 3.6 *Suppose $G = \langle x, y \rangle$ is a p-group. If $G \in M_2$ and $p \geq 3$, then $|\Omega_1(G)| \leq p^3$; If $G \in M_3$ and $p \geq 5$, then $|\Omega_1(G)| \leq p^4$.*

Proof. Let $H = \langle a, b \rangle$ for every $a, b \in G$. If $|a| = |b| = p$ and $G \in M_2$, then $|H| \leq p^3$. Therefore, $|H'| \leq p$. Since $p \geq 3$, then $\Omega_1(H)$ is regular, that is, $\Omega_1(G)$ is. Without loss of generality, we assume $|x| \geq |y|$. Hence, we have $p^2|x| \geq |G| \geq |\langle x, \Omega_1(G) \rangle| = \dfrac{|x||\Omega_1(G)|}{|\langle x \rangle \cap \Omega_1(G)|} = \dfrac{|x||\Omega_1(G)|}{p}$. Thus $|\Omega_1(G)| \leq p^3$.

If $G \in M_3$ and $p \geq 5$, then $|H| \leq p^4$. Therefore, H is regular. Thus $\Omega_1(G)$ is regular. Similarly, we have $p^3|x| \geq |G| \geq |\langle x, \Omega_1(G) \rangle| = \dfrac{|x||\Omega_1(G)|}{|\langle x \rangle \cap \Omega_1(G)|} = \dfrac{|x||\Omega_1(G)|}{p}$. Thus $|\Omega_1(G)| \leq p^4$. ∎

Theorem 3.7 *If $G \in M_2$, then $Cl(G) \leq 4$, where $p \neq 2$ and $p \neq 5$.*

Proof. Let $H = \langle a, b \rangle$ for every $a, b \in G$. By Lemma 3.6, we get $|\Omega_1(H)| \leq p^3$.

If $|\Omega_1(H)| = p^3$, since $p^2|a| \geq |H| \geq |\langle a, \Omega_1(H) \rangle| = \dfrac{|a||\Omega_1(H)|}{p} = p^2|a|$, we have $H = \langle a, \Omega_1(H) \rangle$. Therefore, $H/\Omega_1(H)$ is cyclic. Then, $H' = [H, \Omega_1(H)] < \Omega_1(H)$ and $|H'| \leq p^2$.

If $|\Omega_1(H)| = p^2$, since $\frac{|a|}{p} \leq |H/\Omega_1(H)| \leq |a|$, then $H/\Omega_1(H)$ is cyclic (where $|H/\Omega_1(H)| = \frac{|a|}{p}$) or has a cyclic subgroup of index p (where $|H/\Omega_1(H)| = |a|$).

(1) If $H/\Omega_1(H)$ is cyclic, we get $H' \leq \Omega_1(H)$, that is, $|H'| \leq |\Omega_1(H)| \leq p^2$.

(2) If $H/\Omega_1(H)$ has a cyclic subgroup of index p, when $|H/\Omega_1(H)| = |a| = p^2$, then $|H'| \leq |\Omega_1(H)| \leq p^2$. Without loss of generality, we assume $|a| = p^n \geq p^3$. By Lemma 2.1, there exists $b_1 \in H$ such that $H = \langle a, b_1 \rangle$, where $|b_1| = p^2$.

If $\langle a \rangle \cap \langle b_1 \rangle = 1$, since $p^2|a| \geq |H| = |\langle a, b_1 \rangle| \geq |\langle a \rangle||\langle b_1 \rangle| = p^2|a|$, then $H = \langle a \rangle \langle b_1 \rangle$, that is, H is metacyclic. Therefore $H/\langle b_1 \rangle^H$ is cyclic. Thus, we have $H' \leq \langle b_1 \rangle^H$. By H is metacyclic, we get H is regular. Then, $\exp(\langle b_1 \rangle^H) \leq |b_1| \leq p^2$. We have $|H'| \leq p^2$.

If $\langle a \rangle \cap \langle b_1 \rangle \neq 1$, let $H_1 = \Omega_2(H) = \langle a^{p^{n-2}}, b_1 \rangle$. Since $\langle a^p \rangle \trianglelefteq H$, then $\langle a^{p^{n-2}} \rangle \trianglelefteq H$. Therefore, $H_1 = \langle a^{p^{n-2}} \rangle \langle b_1 \rangle$ and $|H_1| = p^3$. We have $Cl(H_1) \leq 2$. Since we have $(a^{p^{n-2}}b_1^{-1})^p = a^{p^{n-1}}(b_1^{-1})^p = 1 (a^{p^{n-1}} = b_1^p)$, let $b_2 = a^{p^{n-2}}b_1^{-1}$, then we get $H = \langle a, b_1 \rangle = \langle a, b_2 \rangle$, where $|b_2| = p$. Therefore, $H/\Omega_1(H) = \langle a\Omega_1(H), b_2\Omega_1(H) \rangle = \langle a\Omega_1(H) \rangle$, that is , $H/\Omega_1(H)$ is cyclic. Then, we have $|H'| \leq |\Omega_1(H)| \leq p^2$.

Since every two-elements generated subgroup H of G has $|H'| \leq p^2$ and $Cl(H) \leq 3$, then we get G is a 3-Engle group. By [8], we have $Cl(G) \leq 4$, where $p \neq 2$ and $p \neq 5$. ∎

Theorem 3.8 *If* $G \in M_2$, *then:*

(1) *if* $\exp(G) = p$, *then, if and only if* G *is* $Cl(G) \leq 3$, *where* $p = 3$; *or* $Cl(G) = 2$, *where* $p > 3$.

(2) *if* $\exp(G) = p^2$ *and* $p \geq 3$, *then* $\exp(G') \leq p$.

(3) *if* $\exp(G) > p^2$ *and* $p \geq 3$, $\exp(G') \leq p^2$.

Proof. (1) Let $H = \langle a, b\rangle$, for every $a, b \in G$. Since $|H| \leq p^3$, by Theorem 3.3 in [9], (1) holds.

(2) Let $H = \langle a, b\rangle$, for every $a, b \in G$. If $|a| = |b| = p$, then $|H'| \leq p$. Therefore, let $|a| = p^2 \geq |b|$. Since $\langle a^p\rangle = H^p \trianglelefteq G$ and $\exp(G) = p^2$, then $a^p \in Z(G)$. By Theorem 3.7, we have $|H'| \leq p^2$ and $Cl(H) \leq 3$. Therefore, $1 = [a^p, b] = [a, b]^p[a, b, a]^{\frac{p(p-1)}{2}}$. Since $\gamma_3(H) < \gamma_2(H) = H'$, then $[a, b]^p = 1$. We have $G' = \langle [a, b] | a, b \in G\rangle \leq \Omega_1(G)$, by Lemma 3.6, G' is regular and $\exp(G') \leq p$.

(3) Let $H = \langle a, b\rangle$, for every $a, b \in G$. If $|b| \leq |a| \leq p^2$, by (2), then we have $[a, b]^p = 1$. Without loss of generality, we assume $|a| > p^2$ and $|a| \geq |b|$. By Theorem 3.7, we have $|H'| \leq p^2$ and $Cl(H) \leq 3$. Since $\gamma_3(H) < \gamma_2(H) = H'$, we have $[a^p, b] = [a, b]^p[a, b, a]^{\frac{p(p-1)}{2}} = [a, b]^p$. By Theorem 3.5, we get $1 = [a^{p^2}, b] = [a, b]^{p^2}$. Therefore, $G' \leq \Omega_2(G)$.

For every $x, y \in \Omega_2(G)$ and $|y| \leq |x| \leq p^2$, we have $|\langle x, y\rangle| \leq p^4$. If $\langle x, y\rangle$ is a maximal class group, then$|xy| \leq p^2$. If $\langle x, y\rangle$ is not a maximal class group, then $Cl(\langle x, y\rangle) \leq 2$ and $|xy| \leq p^2$. Therefore $\exp(G') \leq p^2$. ∎

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