OPERATION VIA-REGULAR OPEN SETS

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Abstract. The aim of this paper is to introduce and study the concepts of γ-regular open sets and their related notions in topological spaces.

Keywords: γ-open set, γ-regular open set, γ-δ-open set.

2000 Mathematics Subject Classification: 54D10.

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Kasahara [3] defined the concept of an operation on topological spaces and introduce the concept of γ-closed graphs of a function. Ogata [6] introduced the notion γ-open sets in a topological space (X, τ). In this paper, we have introduce and study the notion of γ-regular open sets by using the operation γ on a topological space (X, τ). We also introduce the almost (γ, β)-continuous functions and investigate some of its important properties.

2. Preliminaries

Definition 2.1 Let (X, τ) be a topological space. An operation γ [3] on the topology τ is a function from τ onto a power set $\mathcal{P}(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. It is denoted by $\gamma : \tau \to \mathcal{P}(X)$.

Definition 2.2 A subset A of a topological space (X, τ) is called γ-open [6] set if, for each $x \in A$, there exists an open set U such that $x \in U$ and $U^\gamma \subset A$. τ_γ denotes the set of all γ-open sets in (X, τ). The complement of a γ-open set is called γ-closed.

Definition 2.3 Let A be subset of a topological space (X, τ) and γ be an operation on τ. Then

(i) the τ_γ-closure of A is defined as the intersection of all γ-closed sets containing A. That is, $\tau_\gamma\text{-cl}(A) = \bigcap\{F : F \text{ is } \gamma\text{-closed and } A \subset F\}$.

(ii) the τ_γ-interior of A is defined as the union of all γ-open sets contained in A. That is, $\tau_\gamma\text{-Int}(A) = \bigcup\{U : U \text{ is } \gamma\text{-open and } U \subset A\}$.

Definition 2.4 A subset A of a topological space (X, τ) is said to be

(i) γ-semiopen [5] if $A \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))$.

(ii) γ-preopen [4] if $A \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$.

(iii) γ-α-open [2] if $A \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)))$.

(iv) γ-β-open [1] if $A \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)))$.

The complement of a γ-semiopen (resp. γ-preopen, γ-α-open, γ-β-open) set is called a γ-semiclosed (resp. γ-preclosed, γ-α-closed, γ-β-closed) set. The family of all γ-semiopen (resp. γ-preopen, γ-α-open, γ-β-open) sets of (X, τ) is denoted by $\gamma SO(X)$ (resp. $\gamma PO(X)$, $\gamma\alpha(X)$, $\gamma\beta(X)$)

Definition 2.5 A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(i) (γ, β)-continuous [6] at a point $x \in X$ if, for each β-open subset V in Y containing $f(x)$, there exists a γ-open subset U of X containing x such that $f(U) \subset V$;

(ii) (γ, β)-continuous [6] if it has this property at each point of X.

3. γ-Regular open sets

Throughout this paper, the operator γ is defined on (X,τ) and the operator β is defined on (Y,σ).

Definition 3.1 A subset A of a topological space (X,τ) is said to be γ-regular open set, if $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A)) = A$. We call a subset A of X is γ-regular closed, if its complement is γ-regular open.

The family of all γ-regular open (resp. γ-regular closed) sets of (X,τ) is denoted by $\gamma RO(X)$ (resp. $\gamma RC(X)$).

Lemma 3.2 *For a topological space* (X,τ)*, we have* $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A)) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A)$.

Proof. It is obvious that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))))\subseteq\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))$. Conversely, $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))=\tau_\gamma\text{-Int}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A)))\subseteq\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))))$. ■

Definition 3.3 An operation γ on τ is said to be regular, if for any open neighborhoods U, V of $x \in X$, there exists an open neighborhood W of x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$.

Lemma 3.4 *If A and B are two γ-open subsets and γ is a regular operator. then $A \cap B$ is a γ-open set.*

Remark 3.5 The condition in the above Lemma that γ is a regular operator can not be omitted as we see in the following example

Example 3.6 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Let $\gamma : \tau \to \mathcal{P}(X)$ be an operation defined as follows: for every $A \in \tau$,

$$A^\gamma = \begin{cases} A & \text{if b} \in \text{A}, \\ \text{cl}\,(A) & \text{if b} \notin \text{A}, \end{cases}$$

Then the sets $\{a, b\}$ and $\{a, c\}$ are γ-open sets but their intersection $\{a\}$ is not a γ-open set.

Lemma 3.7 *Let A and B be subsets of a topological space (X,τ). Then the following properties hold:*

(i) $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))$ is γ-regular open.

(ii) If A and B are γ-regular open, then $A \cap B$ is also γ-regular open.

Proof. (i). Follows from Lemma 3.2. (ii). Let A and B be γ-regular open sets of X. Then using Lemma 3.4, we have $A \cap B = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A)) \cap \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(B)) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A) \cap \tau_\gamma\text{-cl}\,(B)) \supset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A \cap B)) \supset \tau_\gamma\text{-Int}(A \cap B) = A \cap B$. Therefore, we obtain $A \cap B = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A \cap B))$. This shows that $A \cap B$ is γ-regular open. ■

Remark 3.8 The following example shows that the union of any two γ-regular open sets need not be γ-regular open.

Example 3.9 [6] Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $\gamma : \tau \to \mathcal{P}(X)$ be an operation defined as follows: for every $A \in \tau$,

$$A^\gamma = \begin{cases} A & \text{if c} \notin \text{A}, \\ \text{cl}\,(A) & \text{if c} \in \text{A}, \end{cases}$$

Then the sets $\{a\}$ and $\{b\}$ are γ-regular open sets but their union $\{a, b\}$ is not a γ-regular open set.

Theorem 3.10 *The following statements are true:*

(i) A γ-open set A is γ-regular open if and only if τ_γ-Int(τ_γ-cl$\,(A)) \subseteq A$.

(ii) For every γ-closed set A, the set τ_γ-Int(A) is γ-regular open.

Proof. (i). It suffices to prove that every γ-open set A satisfying τ_γ-Int(τ_γ-cl$\,(A)) \subset A$ is γ-regular open. Since $A \subset \tau_\gamma$-cl$\,(A)$ holds, then $A = \tau_\gamma$-Int$(A) \subset \tau_\gamma$-Int(τ_γ-cl$\,(A))$ is true, so we have $A = \tau_\gamma$-Int(τ_γ-cl$\,(A))$. (ii). If A is γ-closed, then the following holds: τ_γ-Int$(A) \subset \tau_\gamma$-cl$\,(\tau_\gamma$-Int$(A)) \subset \tau_\gamma$-cl$\,(A) = A$. Hence τ_γ-Int$(A) = \tau_\gamma$-Int(τ_γ-cl$\,(\tau_\gamma$-Int$(A))) \subset \tau_\gamma$-Int$(A)$. So, τ_γ-Int$(A) = \tau_\gamma$-Int(τ_γ-cl$\,(\tau_\gamma$-Int$(A))$ holds. That is, τ_γ-Int(A) is γ-regular open. ■

Theorem 3.11 *For a subset A of X, the following properties are equivalent:*

(i) *A is γ-preopen.*

(ii) *there exists a γ-regular open subset $G \subset X$ such that $A \subset G$ and τ_γ-cl$\,(A) = \tau_\gamma$-cl$\,(G)$.*

(iii) *$A = G \cap D$, where G is γ-regular open and D is γ-dense.*

(iv) *$A = G \cap D$, where G is γ-open and D is γ-dense.*

Proof. (i) ⇒ (ii): Let A be γ-preopen. We have $A \subset \tau_\gamma$-Int(τ_γ-cl$\,(A)) \subset \tau_\gamma$-cl$\,(A)$ which implies that τ_γ-cl$\,(A) \subset \tau_\gamma$-cl$\,(\tau_\gamma$-Int($\tau_\gamma$-cl$\,(A))) \subset \tau_\gamma$-cl$\,(A)$ and so τ_γ-cl$\,(A) = \tau_\gamma$-cl$\,(\tau_\gamma$-Int(τ_γ-cl$\,(A)))$. Let $G = \tau_\gamma$-Int(τ_γ-cl$\,(A))$. Then $A \subset G$ and τ_γ-Int(τ_γ-cl$\,(G)) = \tau_\gamma$-Int(τ_γ-cl$\,(\tau_\gamma$-Int(τ_γ-cl$\,(A)))) = \tau_\gamma$-Int(τ_γ-cl$\,(A)) = G$ which implies that G is γ-regular open. Also τ_γ-cl$\,(G) = \tau_\gamma$-cl$\,(\tau_\gamma$-Int(τ_γ-cl$\,(G))) = \tau_\gamma$-cl$\,(A)$. (ii) ⇒ (iii): Let G be a γ-regular open set such that $A \subset G$ and τ_γ-cl$\,(A) = \tau_\gamma$-cl$\,(G)$. Let $D = A \cup (X \backslash G)$. Then $A = G \cap D$ where G is γ-regular open. Now, τ_γ-cl$\,(D) = \tau_\gamma$-cl$\,(A \cup (X \backslash G)) = \tau_\gamma$-cl$\,((A) \cup \tau_\gamma$-cl$\,(X \backslash G) = \tau_\gamma$-cl$\,(G) \cup \tau_\gamma$-cl$\,(X \backslash G) = \tau_\gamma$-cl$\,((G \cup (X \backslash G)) = \tau_\gamma$-cl$\,(X) = X$. Hence D is γ-dense. (iii) ⇒ (iv): The proof follows from the fact that every γ-regular open set is γ-open. (iv) ⇒ (i): Suppose $A = G \cap D$ where G is γ-open and D is γ-dense. Now $G = G \cap X = G \cap \tau_\gamma$-cl$\,(D) \subset \tau_\gamma$-cl$\,(G \cap D)$ and so $G = \tau_\gamma$-Int$(G) \subset \tau_\gamma$-Int(τ_γ-cl$\,(G \cap D) = \tau_\gamma$-Int($\tau_\gamma$-cl$\,(A)$ which implies that $A \subset \tau_\gamma$-Int(τ_γ-cl$\,(A))$. Hence A is γ-preopen. ■

Theorem 3.12 *For a subset A of X, the following are equivalent:*

(i) *A is γ-regular closed.*

(ii) *A is γ-preclosed and γ-semiopen.*

(iii) *A is γ-α-closed and γ-β-open.*

Proof. (i) $\Rightarrow$ (ii): Let A be γ-regular closed. Then $A = \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-}Int(A)$ and A is γ-preclosed and γ-semiopen. (ii) $\Rightarrow$ (iii): Let A be γ-preclosed and γ-semiopen. Then $A \subset \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A)$ and $\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A) \subset A$. Therefore, we have $\tau_\gamma\text{-cl}\,(A) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A)$ and hence $\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A)))) = \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A)) \subset A$. This shows that A is γ-α-closed. Since every γ-semiopen set is γ-β-open set, it is obvious that A is γ-β-open. (iii) $\Rightarrow$ (i): Let A be γ-α-closed and γ-β-open. Then $A = \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A)))$ and hence $\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A)) = \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A)))) = A$. Therefore, A is γ-regular closed.**Proof.**

Definition 3.13 A topological space (X, τ) is said to be τ_γ-extremally disconnected, if $\tau_\gamma\text{-cl}\,(A) \in \tau_\gamma$, for every $A \in \tau_\gamma$.

Theorem 3.14 *For a topological space (X, τ), the following properties are equivalent:*

(i) *X is τ_γ-extremally disconnected.*

(ii) *Every γ-regular open subset of X is γ-closed.*

(iii) *Every γ-regular closed subset of X is γ-open.*

Proof. (i) $\rightarrow$ (ii): Let X be τ_γ-extremally disconnected. Let A be a γ-regular open subset of X. Then $A = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))$. Since A is a γ-open set, then $\tau_\gamma\text{-cl}\,(A) \in \tau_\gamma$. Thus, $A = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A)) = \tau_\gamma\text{-cl}\,(A)$; hence A is γ-closed. (ii) $\rightarrow$ (iii): Suppose that every γ-regular open subset of X is γ-closed in X. Let $A \in \tau_\gamma$. Since $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))$ is γ-regular open, then it is γ-closed in X. This implies that $\tau_\gamma\text{-cl}\,(A) \subset \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))$ since $A \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))$. Thus, $\tau_\gamma\text{-cl}\,(A) \in \tau_\gamma$; hence X is τ_γ-extremally disconnected. (ii) $\Leftrightarrow$ (iii): Obvious. ∎

Theorem 3.15 *For a topological space (X, τ), the following properties are equivalent:*

(i) *X is τ_γ-extremally disconnected.*

(ii) *γ-regular open sets coincide with γ-regular closed sets.*

Proof. (i) $\Rightarrow$ (ii): Suppose A is a γ-regular open subset of X. Since γ-regular open sets are γ-open, by (i), $A = \tau_\gamma\text{-cl}\,(A) = \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A))$ and so A is γ-regular closed. If A is γ-regular closed, then $A = \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A)) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A))) = \tau_\gamma\text{-Int}(A)$ so A is γ-open. Also, $A = \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A)) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(A))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))$. Hence A is γ-regular open. (ii) $\Rightarrow$ (i):

Let A be a γ-open subset of X. Then τ_γ-Int(τ_γ-cl(A)) is γ-regular open and so it is γ-regular closed, by (ii). Hence τ_γ-Int(τ_γ-cl(τ_γ-Int(τ_γ-cl$((A)))))$ = τ_γ-Int(τ_γ-cl(A)) which implies that τ_γ-cl(τ_γ-Int(τ_γ-cl(A))) = τ_γ-Int(τ_γ-cl(A)). Therefore, τ_γ-cl(A) = τ_γ-cl(τ_γ-Int(A)) $\subset$ τ_γ-cl(τ_γ-Int(τ_γ-cl(A))) = τ_γ-Int(τ_γ-cl(A)) and so τ_γ-cl(A) = τ_γ-cl(τ_γ-Int(A)). Hence τ_γ-cl(A) is γ-open. This shows that X is τ_γ-extremally disconnected. ∎

Theorem 3.16 *For a topological space (X, τ), the following properties are equivalent:*

(i) *X is τ_γ-extremally disconnected.*

(ii) *Every γ-regular closed set is γ-preopen.*

Proof. (i) $\Rightarrow$ (ii): The proof follows from the fact that every γ-regular open sets is γ-preopen set. (ii) $\Rightarrow$ (i): If A is γ-open, then τ_γ-cl(τ_γ-Int(A)) is γ-regular closed and so it is γ-preopen. Therefore, τ_γ-cl(A) = τ_γ-cl(τ_γ-Int(A)) $\subset$ τ_γ-Int(τ_γ-cl(τ_γ-cl(τ_γ-Int(A)))) = τ_γ-Int(τ_γ-cl(τ_γ-Int(A))) = τ_γ-Int(τ_γ-cl(A)). Thus, τ_γ-cl(A) = τ_γ-Int(τ_γ-cl(A)) which implies that τ_γ-cl(A) is γ-open. Hence X is τ_γ-extremally disconnected. ∎

Theorem 3.17 *If (X, τ) is a τ_γ-extremally disconnected, then the following properties hold:*

(i) *$A \cap B$ is γ-regular closed for all γ-regular closed subsets of A and B of X.*

(ii) *If γ is regular open. Then $A \cap B$ is γ-regular open for all γ-regular open subsets A and B of X.*

Proof. (i). Let X be τ_γ-extremally disconnected. Let A and B be γ-regular closed subsets of X. Since A and B are γ-closed, τ_γ-Int(A) and τ_γ-Int(B) are γ-closed. This implies that $A \cap B$ = τ_γ-cl(τ_γ-Int(A)) $\cap$ τ_γ-cl(τ_γ-Int(B)) = τ_γ-Int(A) $\cap$ τ_γ-Int(B) = τ_γ-Int$(A \cap B)$ $\subset$ τ_γ-cl(τ_γ-Int$(A \cap B)$). On the other hand, we have τ_γ-cl(τ_γ-Int$(A \cap B)$) = τ_γ-cl(τ_γ-Int(A) $\cap$ τ_γ-Int(B)) $\subset$ τ_γ-cl(τ_γ-Int(A)) $\cap$ τ_γ-cl(τ_γ-Int(B))=$A \cap B$. Thus, $A \cap B$ is γ-regular closed. (ii). It follows from (i). ∎

Theorem 3.18 *For a topological space (X, τ), the following properties are equivalent:*

(i) *X is τ_γ-extremally disconnected.*

(ii) τ_γ-cl$(A) \in \tau_\gamma$ *for every* $A \in \gamma SO(X)$.

(iii) τ_γ-cl$(A) \in \tau_\gamma$ *for every* $A \in \gamma PO(X)$.

(iv) τ_γ-cl$(A) \in \tau_\gamma$ *for every* $A \in \gamma RO(X)$.

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii): Let A be a γ-semiopen (γ-preopen) set. Then A is γ-β-open; hence $\tau_\gamma\text{-cl}\,(A) \in \tau_\gamma$. (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv): $A \in \gamma RO(X)$. Then $A \in \gamma SO(X)$ and $A \in \gamma PO(X)$ and hence $\tau_\gamma\text{-cl}\,(A) \in \tau_\gamma$. (iv) $\Rightarrow$ (i). Suppose that the γ-closure of every γ-semi open subset of X is γ-open. Let $A \subset X$ be a γ-open set. This implies that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))$ is a γ-open set. Then $\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A)))$ is γ-open. We have $\tau_\gamma\text{-cl}\,(A) \subset (\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(A))$. Thus, $\tau_\gamma\text{-cl} \in \tau_\gamma$; hence X is γ-extremally disconnected. ■

Definition 3.19 Let (X, τ) be a topological space, $S \subset X$ and $x \in X$. Then

(i) x is called γ-δ-cluster point of S, if $S \cap \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(U)) \neq \emptyset$, for each τ_γ-open set U containing x.

(ii) The family of all γ-δ-cluster point of S is called the γ-δ-closure of S and is denoted by $\tau_\gamma\text{-cl}_\delta(S)$.

(iii) A subset S is said to be γ-δ-closed, if $\tau_\gamma\text{-cl}_\delta(S) = S$. The complement of an γ-δ-closed set is said to be an γ-δ-open set.

Lemma 3.20 *Let A and B be subsets of a topological space (X, τ). Then the following properties hold:*

(i) $A \subset \tau_\gamma\text{-cl}_\delta(A)$.

(ii) *If $A \subset B$, then $\tau_\gamma\text{-cl}_\delta(A) \subset \tau_\gamma\text{-cl}_\delta(B)$.*

(iii) $\tau_\gamma\text{-cl}_\delta(A) = \cap\{F \subset X : A \subset F$ *and F is γ-δ-closed*$\}$.

(iv) *If A is an γ-δ-closed set of X for each $\alpha \in \Delta$, then $\cap\{A_\alpha : \alpha \in \Delta\}$ is γ-δ-closed.*

(v) $\tau_\gamma\text{-cl}_\delta(A)$ *is γ-δ-closed.*

Proof. (i). For any $x \in A$ and any γ-open set V containing x, we have $\emptyset \neq A \cap V \subset A \cap \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(V))$ and hence $x \in \tau_\gamma\text{-cl}_\delta(A)$. This shows that $A \subset \tau_\gamma\text{-cl}_\delta(A)$. (ii). Suppose that $x \notin \tau_\gamma\text{-cl}_\delta(B)$. Then there exists a γ-open set V containing x such that $\emptyset = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(V)) \cap B$. Hence $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(V)) \cap A = \emptyset$. Therefore, we have $x \notin \tau_\gamma\text{-cl}_\delta(A)$. (iii). Suppose that $x \in \tau_\gamma\text{-cl}_\delta(A)$. For any γ-open set V containing x and any γ-δ- closed set F containing A, we have $\emptyset \neq A \cap \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(V)) \subset F \cap \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(V))$ and hence $x \in \tau_\gamma\text{-cl}_\delta(F) = F$. This shows that $x \in \cap\{F \subset X : A \subset F$ and F is γ-δ-closed$\}$. Conversely, suppose that $x \notin \tau_\gamma\text{-cl}_\delta(A)$. Then there exists a γ-open set V containing x such that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(V)) \cap A = \emptyset$. Then $X \backslash \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}\,(V))$ is a γ-δ-closed set which contains A and does not contain x. Therefore, we have $x \notin \cap\{F \subset X : A \subset F$ and F is γ-δ-closed$\}$. (iv). For each $\alpha \in \Delta$, $\tau_\gamma\text{-cl}_\delta(\bigcap_{\alpha\in\Delta} A_\alpha) \subset \tau_\gamma\text{-cl}_\delta(A_\alpha) = A_\alpha$ and hence $\tau_\gamma\text{-cl}_\delta(\bigcap_{\alpha\in\Delta} A_\alpha) \subset (\bigcap_{\alpha\in\Delta} A_\alpha)$. By (i) we obtain $\tau_\gamma\text{-cl}_\delta(\bigcap_{\alpha\in\Delta} A_\alpha) = (\bigcap_{\alpha\in\Delta} A_\alpha)$. This shows that $\bigcap_{\alpha\in\Delta} A_\alpha$ is γ-δ-closed. (v). This follows immediately from (iii) and (iv). ■

Proposition 3.21 *Let A and B be subsets of a topological space (X,τ). Then the following properties hold:*

(i) *If A is γ-regular open, then it is γ-δ-open.*

(ii) *Every γ-δ-open set is the union of a family of γ-regular open sets.*

Proof. (i). Let A be any γ-regular open set. For each $x \in A$, $(X\backslash A) \cap A = \emptyset$ and A is γ-regular open. Hence $x \notin \tau_\gamma\text{-cl}_\delta(X\backslash A)$ for each $x \in A$. This shows that $x \notin (X\backslash A) \Rightarrow x \notin \tau_\gamma\text{-cl}_\delta(X\backslash A)$. Therefore we have $\tau_\gamma\text{-cl}_\delta(X\backslash A) \subset (X\backslash A)$. Since in general, for any subset S of X, $S \subseteq \tau_\gamma\text{-cl}_\delta(S)$, $\tau_\gamma\text{-cl}_\delta(X\backslash A) = (X\backslash A)$; A is γ-δ-open. (ii). Let A be a γ-δ-open set. Then $(X\backslash A)$ is γ-δ-closed and hence $(X\backslash A) = \tau_\gamma\text{-cl}_\delta(X\backslash A)$. For each $x \in A$, $x \notin \tau_\gamma\text{-cl}_\delta(X\backslash A)$ and there exists a γ-open set V_x such that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V_x)) \cap (X\backslash A) = \emptyset$. Therefore, $x \in V_x \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V_x)) \subset A$ and hence $A = \cup\{\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V_x)) : x \in A\}$. By (i) $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V_x))$ is γ-regular open for each $x \in A$. ∎

Theorem 3.22 *Let (X,τ) be topological space and $\tau_{\gamma\delta} = \{A \subset X : A$ is a γ-δ-open set of $(X,\tau)\}$.*

Proof. (i). It is obvious that $\emptyset, X \in \tau_{\gamma\delta}$. (ii). Let $V_\alpha \in \tau_{\gamma\delta}$ for each $\alpha \in \Delta$. Then $X\backslash V_\alpha$ is γ-δ-closed, for each $\alpha \in \Delta$. By Proposition 3.21, $\underset{\alpha\in\Delta}{\cap}(X\backslash V_\alpha)$ is γ-δ-closed and $\underset{\alpha\in\Delta}{\cap}(X\backslash V_\alpha) = X\backslash \underset{\alpha\in\Delta}{\cup} V_\alpha$. Hence $\underset{\alpha\in\Delta}{\cup} V_\alpha$ is γ-δ-open. (iii) Let $A, B \in \tau_{\gamma\delta}$. Then $A = \underset{\kappa\omega\in\Delta_1}{\cup} A_\kappa$ and $B = \underset{\omega\in\Delta_2}{\cup} B_\omega$, where A_κ and B_ω are γ-regular open sets for each $\kappa \in \Delta_1$ and $\omega \in \Delta_2$. Thus, $A \cap B = \cup\{A_\kappa \cap B_\omega : \kappa \in \Delta_1, \omega \in \Delta_2\}$. Since $A_\kappa \cap B_\omega$ is γ-regular open, $A \cap B$ is a γ-δ-open set. ∎

Remark 3.23 It is clear that γ-δ-open sets in (X,τ) form a topology $\tau_{\gamma\delta}$ on X weaker than τ_γ for which the γ-regular open sets of X form a base.

Lemma 3.24 *If A is a γ-β-open set in a topological space (X,τ), then $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}_\delta(A)$.*

Proof. Let A be a γ-β-open set. Suppose that $x \notin \tau_\gamma\text{-cl}(A)$. Then there exists an γ-open set U containing x such that $U \cap A = \varnothing$. We have $U \cap \tau_\gamma\text{-cl}(A) = \emptyset$. This implies that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(U)) \cap \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))) = \emptyset$. Since A is a γ-β-open set, then $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(U)) \cap A = \emptyset$. Thus, $x \notin \tau_\gamma\text{-cl}_\delta(A)$ and $\tau_\gamma\text{-cl}(A) \supset \tau_\gamma\text{-cl}_\delta(A)$. On the other hand, we have $\tau_\gamma\text{-cl}(A) \subset \tau_\gamma\text{-cl}_\delta(A)$. Hence, we obtain $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}_\delta(A)$. ∎

Lemma 3.25 *If A is a γ-semiopen set in a topological space (X,τ), then $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}_\delta(A)$.*

Proof. Let A be a γ-semiopen set. We have $\tau_\gamma\text{-cl}(A) \subset \tau_\gamma\text{-cl}_\delta(A)$. Suppose that $x \notin \tau_\gamma\text{-cl}(A)$. Then there exists a γ-open set U containing x such that $U \cap A = \emptyset$. We have $U \cap \tau_\gamma\text{-Int}(A) = \emptyset$. This implies that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(U)) \cap \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) = \emptyset$. Since A is a γ-semiopen set, then $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(U)) \cap A = \emptyset$. Thus, $x \notin \tau_\gamma\text{-cl}_\delta(A)$ and $\tau_\gamma\text{-cl}(A) \supset \tau_\gamma\text{-cl}_\delta(A)$. Hence $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}_\delta(A)$. ∎

Theorem 3.26 *For a topological space (X, τ), the following properties are equivalent:*

(i) *X is τ_γ-extremally disconnected.*

(ii) τ_γ-cl$(A) \in \tau_\gamma$ *for every* $A \in \gamma SO(X)$.

(iii) τ_γ-cl$(A) \in \tau_\gamma$ *for every* $A \in \gamma PO(X)$.

(iv) τ_γ-cl$(A) \in \tau_\gamma$ *for every* $A \in \gamma RO(X)$.

Theorem 3.27 *For a topological space (X, τ), the following properties are equivalent:*

(i) *X is τ_γ-extremally disconnected.*

(ii) τ_γ-cl$_\delta(A) \in \tau_\gamma$ *for every* $A \in \gamma SO(X)$.

(iii) τ_γ-cl$_\delta(A) \in \tau_\gamma$ *for every* $A \in \gamma PO(X)$.

(iv) τ_γ-cl$_\delta(A) \in \tau_\gamma$ *for every* $A \in \gamma RO(X)$.

Proof. The proof follows from Theorems 3.22, 3.26 and Lemmas 3.24, 3.25. ∎

4. Almost (γ, β)-continuous functions

Definition 4.1 A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(i) almost (γ, β)-continuous at a point $x \in X$ if, for each β-open subset V in Y containing $f(x)$, there exists a γ-open set U of X containing x such that $f(U) \subset \tau_\beta$-Int$(\tau_\beta$-cl$(V))$;

(ii) almost (γ, β)-continuous, if it has this property at each point of X.

Remark 4.2 Almost (γ, β)-continuity implies (γ, β)-continuity. But the converse is not true in general as the following examples shows.

Example 4.3 Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\varnothing, \{a\}, \{a, b\}, X\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then f is almost (id, id)-continuous but not (id, id)-continuous, where "*id*" denotes the identity operator.

Theorem 4.4 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) *f is almost (γ, β)-continuous at $x \in X$;*

(ii) $x \in \tau_\gamma$-Int$(f^{-1}(\tau_\gamma$-Int$(\tau_\gamma$-cl$(V))))$ *for every β-open set V of Y containing $f(x)$;*

(iii) $x \in \tau_\gamma$-Int$(f^{-1}(V))$ *for every* β*-regular open set* V *of* Y *containing* $f(x)$*;*

(iv) *For any* β*-regular open set* V *containing* $f(x)$*, there exists a* γ*-open set* U *containing* x *such that* $f(U) \subset V$*.*

Proof. (i)⇒(ii): Let V be any β-open set V of Y containing $f(x)$. By (i), there exists a γ-open set U of X containing x such that $f(U) \subset \tau_\beta$-Int$(\tau_\beta$-cl$\,(V))$. Since $x \in U \subset f^{-1}(\tau_\beta$-Int$(\tau_\beta$-cl$\,(V)))$, we have $x \in \tau_\gamma$-Int$(f^{-1}(\tau_\beta$-Int$(\tau_\beta$-cl$\,(V))))$. (ii)⇒(iii): Let V be any β-regular open set V of Y containing $f(x)$. Then since $V = \tau_\beta$-Int$(\tau_\beta$-cl$\,(V))$, by (ii), we have $x \in \tau_\gamma$-Int$(f^{-1}(V))$. (iii)⇒(iv): Let V be any β-regular open set of Y containing $f(x)$. From (iii), there exists a γ-open set U containing x such that $U \subset f^{-1}(V)$. Hence we have (iv). (iv)⇒(i): Let V be any β-open set V of Y containing $f(x)$. Then $f(x) \in V \subset \tau_\beta$-Int$(\tau_\beta$-cl$\,(V))$. Since τ_β-Int$(\tau_\beta$-cl$\,(V))$ is β-regular open, by (iv), there exists a γ-open set U containing x such that $f(U) \subset \tau_\beta$-Int$(\tau_\beta$-cl$\,(V))$. Hence f is almost (γ, β)-continuous at $x \in X$. ∎

Theorem 4.5 *For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following statements are equivalent:*

(i) f *is almost* (γ, β)*-continuous;*

(ii) $f^{-1}(V)$ *is* γ*-open for every* β*-regular open set* V *of* Y*;*

(iii) $f^{-1}(V)$ *is* γ*-closed for every* β*-regular closed set* V *of* Y*;*

(iv) $f(\tau_\gamma$-cl$\,(A)) \subset \tau_\gamma$-cl$_\delta(f(A))$ *for every subset* A *of* X*;*

(v) σ_β-cl$\,(f^{-1}(B)) \subset f^{-1}(\tau_\gamma$-cl$_\delta(B))$ *for every subset* B *of* Y*;*

(vi) $f^{-1}(F)$ *is* γ*-open for every* β*-*δ*-open set* F *of* Y*;*

(vii) $f^{-1}(V)$ *is* γ*-closed for every* β*-*δ*-closed set* V *of* Y*.*

Proof. (i)⇒(ii): Clear. (ii)⇒(iii): Let $F \in \beta RC(Y)$. Then $Y \backslash F \in \beta RO(Y)$. Take $x \in f^{-1}(Y \backslash F)$, then $f(x) \in Y \backslash F$ and since f is almost (γ, β)-continuous, there exists a β open set W_x of X such that $x \in W_x$ and $f(W_x) \subset Y \backslash F$. Then $x \in W_x \subset f^{-1}(Y \backslash F)$ so that $f^{-1}(Y \backslash F) = \bigcup_{x \in f^{-1}(Y \backslash F)} W_x$. Since any union of γ-open sets is γ-open, $f^{-1}(Y \backslash F)$ is γ-open in X and hence $f^{-1}(F) \in \beta RC(X)$. (iii)⇒(iv): Let A be a subset of X. Since σ_β-cl$_\delta(f(A))$ is β-δ-closed in Y, it is equal to $\bigcap\{F_\alpha : F_\alpha$ is β-regular closed in Y, $\alpha \in \Lambda\}$, where Λ is an index set. From (iii), we have $A \subset f^{-1}(\tau_\gamma$-cl$_\delta(f(A))) = \bigcap\{f^{-1}(F_\alpha) : \alpha \in \Lambda\} \in \gamma RC(X)$ and hence τ_γ-cl$\,(A) \subset f^{-1}(\sigma_\beta$-cl$_\delta(f(A)))$. Therefore, we obtain $f(\tau_\gamma$-cl$\,(A)) \subset \sigma_\beta$-cl$_\delta(f(A))$. (iv)⇒(v): Set $A = f^{-1}(B)$ in (iv), then $f(\tau_\gamma$-cl$\,(f^{-1}(B))) \subset \sigma_\beta$-cl$_\delta(f(f^{-1}(B))) \subset \sigma_\beta$-cl$_\delta(B)$ and hence τ_γ-cl$\,(f^{-1}(B)) \subset f^{-1}(\sigma_\beta$-cl$_\delta(B))$. (v)⇒(vi): Let F be β-δ-closed set of Y, then τ_γ-cl$\,f^{-1}(F) \subset f^{-1}(F)$ so $f^{-1}(F) \in \gamma RC(X)$. (vi)⇒(vii): Let V be β-δ-open set of Y, then $Y \backslash V$ is β-δ-closed set in Y. This gives $f^{-1}(Y \backslash V) \in \gamma RC(X)$ and hence $f^{-1}(V) \in \beta O(X)$. (viii)⇒(i): Let V be any β-regular open set of Y. Since V is β-δ-open in Y, then $f^{-1}(V) \in \tau_\gamma$ and hence from $f(f^{-1}(V)) \subset V$ $= \sigma_\beta$-Int$(\sigma_\beta$-cl$\,(V))$. Therefore, f is almost (γ, β)-continuous. ∎

Theorem 4.6 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) *f is almost (γ, β)-continuous;*

(ii) *$f^{-1}(V) \subset \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V))))$ for every β-open set V in Y;*

(iii) *$\tau_\gamma\text{-cl}\,(f^{-1}(\tau_\beta\text{-cl}\,(\tau_\beta\text{-Int}(F)))) \subset f^{-1}(F)$ for every β-closed set F in Y;*

(iv) *$\tau_\gamma\text{-cl}\,(f^{-1}(\tau_\beta\text{-cl}\,(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(B))))) \subset f^{-1}(\tau_\beta\text{-cl}\,(B))$ for every subset B in Y;*

(v) *$f^{-1}(\tau_\beta\text{-Int}(B)) \subset \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(\tau_\beta\text{-Int}(B)))))$ for every subset B in Y.*

Proof. (i)⇒(ii): Let V be any β-open set in Y and $x \in f^{-1}(V)$. Then there exists a γ-open set U containing x such that $f(U) \subset \tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V))$. This implies $x \in \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V))))$. Hence $f^{-1}(V) \subset \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V))))$. (ii)⇒(iii): Let F be any β-closed set in Y. Then $f^{-1}(Y \backslash F) \subset \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(Y \backslash F)))) = X \backslash \tau_\gamma\text{-cl}\,(f^{-1}(\tau_\beta\text{-cl}\,(\tau_\beta\text{-Int}(F))))$. Hence $\tau_\gamma\text{-cl}\,(f^{-1}(\tau_\beta\text{-cl}\,(\tau_\beta\text{-Int}(F)))) \subset f^{-1}(F)$. (iii)⇒(iv) and (iv)⇒(v): It is obvious. (v)⇒(i): Let V be any β-regular open set in Y. Since $\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(\tau_\beta\text{-Int}(V))) = V$, from (v), it follows $f^{-1}(V) \subset \tau_\beta\text{-Int}(f^{-1}(V))$ and so $f^{-1}(V) = \tau_\beta\text{-Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is γ-open in X. By Theorem 4.5 (ii), f is almost (γ, β)-continuous. ∎

Theorem 4.7 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) *f is almost (γ, β)-continuous;*

(ii) *$\tau_\gamma\text{-cl}\,(f^{-1}(G)) \subset f^{-1}(\tau_\beta\text{-cl}\,(G))$ for every β-β-open set G of Y;*

(iii) *$\tau_\gamma\text{-cl}\,(f^{-1}(G)) \subset f^{-1}(\tau_\beta\text{-cl}\,(G))$ for every β-semiopen set G of Y;*

(iv) *$\tau_\gamma\text{-cl}\,(f^{-1}(G)) \subset f^{-1}(\tau_\beta\text{-cl}\,(G))$ for every β-preopen set G of Y.*

Proof. (i)⇒(ii): Let G be any β-β-open set of Y. Since $\tau_\beta\text{-cl}\,(G)$ is β-regular closed, $\tau_\gamma\text{-cl}\,(f^{-1}(\tau_\beta\text{-cl}\,(G))) = f^{-1}(\tau_\beta\text{-cl}\,(G))$. Thus, $\tau_\gamma\text{-cl}\,(f^{-1}(G)) \subset \tau_\gamma\text{-cl}\,(f^{-1}(\tau_\beta\text{-cl}\,(G))) = f^{-1}(\tau_\beta\text{-cl}\,(G))$. (ii)⇒(iii): It is obvious since every β-semiopen set is β-β-open. (iii)⇒(i): Let F be any β-regular closed set of Y; then since F is β-semiopen, we have $\tau_\gamma\text{-cl}\,(f^{-1}(G)) \subset f^{-1}(\tau_\beta\text{-cl}\,(G)) = f^{-1}(F)$. Thus, from Theorem 4.5 (iii), f is almost (γ, β)-continuous. (i)⇒(iv): Let V be any β-preopen set of Y; then $V \subset \tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V))$ and $\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V))$ is β-regular open. By Theorem 4.5 (ii), $f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V))) = \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V))))$. Thus, we have $f^{-1}(V) \subset f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V))) = \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V))))$. (iv)⇒(i): Let V be any β-regular open set of Y; then since F is β-preopen and $f^{-1}(V) \subset \tau_\gamma\text{-cl}\,(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}\,(V)))) = \tau_\gamma\text{-Int}(f^{-1}(V))$. Hence by Theorem 4.5 (ii), f is almost (γ, β)-continuous. ∎

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Accepted: 13.09.2013