A PRACTICAL METHOD FOR MATRIX INVERSION

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Abstract. This report presents a matrix inversion method that has the following features: general applicability for any non-singular $(n \times n)$ matrix $\mathbf{A}$ with either real or complex elements a_{ij}, $i, j = 1, 2, ..., n$; univocally defined matrix operations; analytical representation of the sought-for inverse matrix A^{-1} as a product of three uniquely specified non-singular triangular matrices ($A^{-1} = \mathbf{PGV}$, $\mathbf{P}$ and $\mathbf{V}$ lower triangular, $\mathbf{G}$ upper triangular); and convenient (minimal) number, n^3, of required multiplication/division operations.

The inversion procedure is carried out in two stages: I) transformation of matrix $\mathbf{A}$ into an upper triangular matrix $\mathbf{T}$ having unit diagonal elements; II) transformation of matrix $\mathbf{T}$ into the $(n \times n)$ unit matrix $\mathbf{U}^{(n)}$ having elements $u_{ij}^{(n)} = 0$ if $i \neq j$, and $= 1$ if $i = j$.

The first stage conforms with a Gaussian elimination procedure that can be carried out in the natural order, through n consecutive ordered steps, since at each step univocally prescribed non-zero diagonal elements (leading pivots) are made available. A sequence of transformed non-singular $(n \times n)$ matrices $\{\mathbf{A}^{(k)}\}$, $k = 1, 2, ..., n$, having elements $a_{ij}^{(k)}$, is obtained with pivots $a_{kk}^{(k)} = 1$. The viability of the procedure is assured by the ordered use of univocally defined, very simple non-singular lower triangular $(n \times n)$ operational matrices $\mathbf{P}^{(k)}$ which post-multiply $\mathbf{A}^{(k-1)}$ ($\mathbf{A}^{(0)}$ coinciding with $\mathbf{A}$) and transform diagonal element $a_{kk}^{(k-1)}$ into a leading pivot having a positive non-zero value given by the sum of the absolute values of all the elements in its row.

A simple numerical example is detailed for illustrating practical aspects.

The present matrix inversion method is free from any operational ambiguity. The simplicity and univocal definiteness of its transformations are expected to provide operational advantages for the development of related numerical algorithms both for finding matrix inverses and for solving systems of linear algebraic equations. Further useful features are related with the final triangular matrix factorization achieved which, in particular, allows an immediate computation of the determinants of matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$.

General presentation

The determination of the unique inverse matrix $\mathbf{A}^{-1}$ of a given non-singular $(n \times n)$ matrix $\mathbf{A}$, having n^2 elements a_{ij}, $i, j = 1, 2, ..., n$, that are real or complex numbers, represents a basic problem in linear algebra with implications for many theories, and applications in various fields.

Gaussian elimination has been recognized as the most widely used method for inverting matrices [1] and it is known that it may need suitable matrix transformations such as row (column) interchanges or combinations, for achieving adequate non-zero diagonal elements (pivots). The focus of the present inversion method consists in prescribing all required transformations at the outset, in view of avoiding any operational ambiguity and potentially critical situations. With such a formulation:

i) A numerical algorithm based on the present method is of general use with no operational ambiguity, exploits a very simple logic, and can be applied through consecutive steps taken in the natural order.

ii) In the successive ordered matrix transformations which achieve the triangular factorization of $\mathbf{A}$, a leading pivot has a positive non-zero value given by the sum of the absolute values of all the elements in its row, which tends to diminish the possible negative effects of round-off errors.

iii) The sought-for inverse matrix $\mathbf{A}^{-1}$ has an analytical representation, of great practical interest in some contexts, as a product of three uniquely specified non-singular triangular matrices, $\mathbf{A}^{-1} = \mathbf{PGV}$, $\mathbf{P}$ and $\mathbf{V}$ lower triangular, $\mathbf{G}$ upper triangular. In particular, an immediate computation of the determinants of matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ is made available.

iv) The inversion of a matrix $\mathbf{A}$ having complex elements is obtained in a straightforward way in analogy with the case of real elements.

In general, for solving a given problem, preference is given to the numerical method which requires a minimum number of operations and is logically simpler and, thus, is realized more rapidly on a computer [2]. As for the number of operations, a specific count for the present inversion method is detailed in Appendix 1 where the minimal count, n^3, for all required multiplication/division operations is obtained [1]. The development of different numerical methods for solving specific problems should turn out to be a useful endeavour in many cases, particularly if various difficulties and complexities are at work.

The present inversion method is described for the case of a matrix $\mathbf{A}$ having real elements a_{ij}. The case of complex elements can be handled by introducing simple variations that are detailed below.

The inverse matrix $\mathbf{A}^{-1}$ is achieved through two consecutive transformations: $\tau 1$, which transforms $\mathbf{A}$ into an upper triangular matrix $\mathbf{T}$ having all diagonal elements $t_{ii} = 1$; and $\tau 2$, which transforms $\mathbf{T}$ into the $(n \times n)$ unit matrix $\mathbf{U}^{(n)}$ having elements $u_{ij}^{(n)} = 0$ if $i \neq j$, and $= 1$ if $i = j$.

Transformation $\tau 1$ conforms with a Gaussian elimination procedure that can be carried out in the natural order, through n consecutive ordered steps, since at each step univocally prescribed non-zero diagonal elements (leading pivots) are made available. A sequence of transformed non-singular $(n \times n)$ matrices $\mathbf{A}^{(k)}$, $k = 1, 2, ..., n$, having elements $a_{ij}^{(k)}$, is obtained with pivots $a_{kk}^{(k)} = 1$. The feasibility of the procedure is assured by the ordered use of univocally defined, very simple non-singular lower triangular $(n \times n)$ operational matrices $\mathbf{P}^{(k)}$ which post-multiply $\mathbf{A}^{(k-1)}$ ($\mathbf{A}^{(0)}$ coinciding with$\mathbf{A}$) and transform diagonal element $a_{kk}^{(k-1)}$ into the positive sum $\sum_{q=k}^{n} |a_{kq}^{(k-1)}|$.

A matrix $\mathbf{P}^{(k)}$ differs (or may differ) from $\mathbf{U}^{(n)}$ only in the diagonal and lower elements of the k-th column, prescribed by the relations: $p_{ik}^{(k)} = \sigma(a_{ki}^{(k-1)})$, $i = k$, $k+1, ..., n$, where symbol $\sigma(r)$ indicates, for any real number r, the values: -1 if $r \leq 0$, and 1 if $r > 0$. A matrix $\mathbf{P}^{(k)}$ has thus a very simple structure with only two possible absolute values, 0 or 1, for its elements. This structural simplicity provides a convenient matrix multiplication property: the matrix product $\mathbf{P}^{(k)}$ $\mathbf{P}^{(s)}$, with $s > k$, is a matrix having the same elements as the unit matrix $\mathbf{U}^{(n)}$ except for the k-th and s-th columns which equal, respectively, the k-th column of $\mathbf{P}^{(k)}$ and the s-th column of $\mathbf{P}^{(s)}$.

Transformation $\tau 1$

This transformation requires n consecutive ordered steps. At the k-th step, $k = 1, 2, ..., n-1$, matrix $\mathbf{A}$ is transformed into a matrix $\mathbf{A}^{(k)}$ that admits the following four-block representation:

$$\mathbf{A}^{(k)} = \begin{bmatrix} \mathbf{Z}^{(k)} & \mathbf{W}^{(k)} \\ \mathbf{R}^{(k)} & \mathbf{S}^{(k)} \end{bmatrix}; \tag{1.0}$$

where $\mathbf{Z}^{(k)}$ is an upper triangular square sub-matrix of order k with all diagonal elements $z_{ii}^{(k)} = 1$; $\mathbf{R}^{(k)}$ is an $(n-k) \times k)$ null matrix; $\mathbf{S}^{(k)}$ is a non-singular square sub-matrix of order $(n-k)$; and $\mathbf{W}^{(k)}$ is a $(k \times (n-k))$ matrix. At the last step, $k = n$, matrix $\mathbf{A}^{(n)} = \mathbf{T}$ is obtained by transforming element $a_{nn}^{(n-1)}$ into unity.

At the k-th step, $k = 1, 2, ..., n$, the following transformations apply:

$$\mathbf{A}^{(k-1)}\mathbf{P}^{(k)} = \mathbf{A}^{(k-1)\oplus}; \tag{1.1}$$

$$\mathbf{V}^{(k)}\mathbf{A}^{(k-1)\oplus} = \mathbf{A}^{(k)}; \tag{1.2}$$

where:

- $\mathbf{A}^{(k-1)}$ with elements $a_{ij}^{(k-1)}$, $\mathbf{A}^{(k-1)\oplus}$ with elements $a_{ij}^{(k-1)\oplus}$, and $\mathbf{A}^{(k)}$ with elements $a_{ij}^{(k)}$, are $(n \times n)$ non-singular matrices and $\mathbf{A}^{(0)} = \mathbf{A}$ so that elements $a_{ij}^{(0)} = a_{ij}$;

- $\mathbf{P}^{(k)}$ and $\mathbf{V}^{(k)}$ are non singular $(n \times n)$ lower triangular operational matrices which differ (or may differ) from $\mathbf{U}^{(n)}$ only in the diagonal and lower elements of the k-th column. For $\mathbf{P}^{(k)}$, these elements are: $p_{ik}^{(k)} = \sigma(a_{ki}^{(k-1)})$; $i = k$, $k+1, ..., n$; while for $\mathbf{V}^{(k)}$, they are: $v_{kk}^{(k)} = (a_{kk}^{(k-1)\oplus})^{-1}$; $v_{ik}^{(k)} = -a_{ik}^{(k-1)\oplus}(a_{kk}^{(k-1)\oplus})^{-1}$; $i = k+1, k+2, ..., n$.

Operational matrices $\mathbf{P}^{(k)}$ and $\mathbf{V}^{(k)}$ play key roles and allow transformation $\tau 1$ to be carried out through all its consecutive steps taken in the natural order, so that the sequence of transformed matrices $\{\mathbf{A}^{(k)}\}$ achieves the sought-for triangular factorization of $\mathbf{A}$.

The univocal algebraic relations among matrix elements in equations (1.1) and (1.2) may be visualized with the help of the following matrix representations:

$$\mathbf{A}^{(k-1)} =$$

$$\begin{bmatrix}
1 & a_{12}^{(k-1)} & a_{13}^{(k-1)} & \cdot & a_{1k-1}^{(k-1)} & a_{1k}^{(k-1)} & a_{1k+1}^{(k-1)} & a_{1k+2}^{(k-1)} & a_{1k+3}^{(k-1)} & \cdot & a_{1n-1}^{(k-1)} & a_{1n}^{(k-1)} \\
0 & 1 & a_{23}^{(k-1)} & \cdot & a_{2k-1}^{(k-1)} & a_{2k}^{(k-1)} & a_{2k+1}^{(k-1)} & a_{2k+2}^{(k-1)} & a_{2k+3}^{(k-1)} & \cdot & a_{2n-1}^{(k-1)} & a_{2n}^{(k-1)} \\
0 & 0 & 1 & \cdot & a_{3k-1}^{(k-1)} & a_{3k}^{(k-1)} & a_{3k+1}^{(k-1)} & a_{3k+2}^{(k-1)} & a_{3k+3}^{(k-1)} & \cdot & a_{3n-1}^{(k-1)} & a_{3n}^{(k-1)} \\
\cdots & & & & & & & & & & & \cdots \\
0 & 0 & 0 & \cdot & 1 & a_{k-1k}^{(k-1)} & a_{k-1k+1}^{(k-1)} & a_{k-1k+2}^{(k-1)} & a_{k-1k+3}^{(k-1)} & \cdot & a_{k-1n-1}^{(k-1)} & a_{k-1n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & a_{kk}^{(k-1)} & a_{kk+1}^{(k-1)} & a_{kk+2}^{(k-1)} & a_{kk+3}^{(k-1)} & \cdot & a_{kn-1}^{(k-1)} & a_{kn}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & a_{k+1k}^{(k-1)} & a_{k+1k+1}^{(k-1)} & a_{k+1k+2}^{(k-1)} & a_{k+1k+3}^{(k-1)} & \cdot & a_{k+1n-1}^{(k-1)} & a_{k+1n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & a_{k+2k}^{(k-1)} & a_{k+2k+1}^{(k-1)} & a_{k+2k+2}^{(k-1)} & a_{k+2k+3}^{(k-1)} & \cdot & a_{k+2n-1}^{(k-1)} & a_{k+2n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & a_{k+3k}^{(k-1)} & a_{k+3k+1}^{(k-1)} & a_{k+3k+2}^{(k-1)} & a_{k+3k+3}^{(k-1)} & \cdot & a_{k+3n-1}^{(k-1)} & a_{k+3n}^{(k-1)} \\
\cdots & & & & & & & & & & & \cdots \\
0 & 0 & 0 & \cdot & 0 & a_{n-1k}^{(k-1)} & a_{n-1k+1}^{(k-1)} & a_{n-1k+2}^{(k-1)} & a_{n-1k+3}^{(k-1)} & \cdot & a_{n-1n-1}^{(k-1)} & a_{n-1n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & a_{nk}^{(k-1)} & a_{nk+1}^{(k-1)} & a_{nk+2}^{(k-1)} & a_{nk+3}^{(k-1)} & \cdot & a_{nn-1}^{(k-1)} & a_{nn}^{(k-1)}
\end{bmatrix}$$

$$\mathbf{P}^{(k)} = \begin{bmatrix}
1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
\cdots & & & & & & & & & & & \cdots \\
0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kk}^{(k-1)}\right) & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kk+1}^{(k-1)}\right) & 1 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kk+2}^{(k-1)}\right) & 0 & 1 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kk+3}^{(k-1)}\right) & 0 & 0 & 1 & \cdot & 0 & 0 \\
\cdots & & & & & & & & & & & \cdots \\
0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kn-1}^{(k-1)}\right) & 0 & 0 & 0 & \cdot & 1 & 0 \\
0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kn}^{(k-1)}\right) & 0 & 0 & 0 & \cdot & 0 & 1
\end{bmatrix}$$

$$\mathbf{A}^{((k-1)\oplus)} =$$

$$\begin{bmatrix}
1 & a_{12}^{(k-1)} & a_{13}^{(k-1)} & \cdot & a_{1k-1}^{(k-1)} & \sum_{q=k}^{n}(a_{1q}^{(k-1)}p_{qk}^{(k)}) & a_{1k+1}^{(k-1)} & a_{1k+2}^{(k-1)} & a_{1k+3}^{(k-1)} & \cdot & a_{1n-1}^{(k-1)} & a_{1n}^{(k-1)} \\
0 & 1 & a_{23}^{(k-1)} & \cdot & a_{2k-1}^{(k-1)} & \sum_{q=k}^{n}(a_{2q}^{(k-1)}p_{qk}^{(k)}) & a_{2k+1}^{(k-1)} & a_{2k+2}^{(k-1)} & a_{2k+3}^{(k-1)} & \cdot & a_{2n-1}^{(k-1)} & a_{2n}^{(k-1)} \\
0 & 0 & 1 & \cdot & a_{3k-1}^{(k-1)} & \sum_{q=k}^{n}(a_{3q}^{(k-1)}p_{qk}^{(k)}) & a_{3k+1}^{(k-1)} & a_{3k+2}^{(k-1)} & a_{3k+3}^{(k-1)} & \cdot & a_{3n-1}^{(k-1)} & a_{3n}^{(k-1)} \\
\cdots & & & & & & & & & & & \\
0 & 0 & 0 & \cdot & 1 & \sum_{q=k}^{n}(a_{k-1q}^{(k-1)}p_{qk}^{(k)}) & a_{k-1k+1}^{(k-1)} & a_{k-1k+2}^{(k-1)} & a_{k-1k+3}^{(k-1)} & \cdot & a_{k-1n-1}^{(k-1)} & a_{k-1n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & \sum_{q=k}^{n}|a_{kq}^{(k-1)}| & a_{kk+1}^{(k-1)} & a_{kk+2}^{(k-1)} & a_{kk+3}^{(k-1)} & \cdot & a_{kn-1}^{(k-1)} & a_{kn}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & \sum_{q=k}^{n}(a_{k+1q}^{(k-1)}p_{qk}^{(k)}) & a_{k+1k+1}^{(k-1)} & a_{k+1k+2}^{(k-1)} & a_{k+1k+3}^{(k-1)} & \cdot & a_{k+1n-1}^{(k-1)} & a_{k+1n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & \sum_{q=k}^{n}(a_{k+2q}^{(k-1)}p_{qk}^{(k)}) & a_{k+2k+1}^{(k-1)} & a_{k+2k+2}^{(k-1)} & a_{k+2k+3}^{(k-1)} & \cdot & a_{k+2n-1}^{(k-1)} & a_{k+2n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & \sum_{q=k}^{n}(a_{k+3q}^{(k-1)}p_{qk}^{(k)}) & a_{k+3k+1}^{(k-1)} & a_{k+3k+2}^{(k-1)} & a_{k+3k+3}^{(k-1)} & \cdot & a_{k+3n-1}^{(k-1)} & a_{k+3n}^{(k-1)} \\
\cdots & & & & & & & & & & & \\
0 & 0 & 0 & \cdot & 0 & \sum_{q=k}^{n}(a_{n-1q}^{(k-1)}p_{qk}^{(k)}) & a_{n-1k+1}^{(k-1)} & a_{n-1k+2}^{(k-1)} & a_{n-1k+3}^{(k-1)} & \cdot & a_{n-1n-1}^{(k-1)} & a_{n-1n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & \sum_{q=k}^{n}(a_{nq}^{(k-1)}p_{qk}^{(k)}) & a_{nk+1}^{(k-1)} & a_{nk+2}^{(k-1)} & a_{nk+3}^{(k-1)} & \cdot & a_{nn-1}^{(k-1)} & a_{nn}^{(k-1)}
\end{bmatrix}$$

$$\mathbf{V}^{(k)} = \begin{bmatrix}
1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
\cdots & & & & & & & & & & & \\
0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & (a_{kk}^{(k-1)\oplus})^{-1} & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & -a_{k+1k}^{(k-1)\oplus}\left(a_{kk}^{(k-1)\oplus}\right)^{-1} & 1 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & -a_{k+2k}^{(k-1)\oplus}\left(a_{kk}^{(k-1)\oplus}\right)^{-1} & 0 & 1 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & -a_{k+3k}^{(k-1)\oplus}\left(a_{kk}^{(k-1)\oplus}\right)^{-1} & 0 & 0 & 1 & \cdot & 0 & 0 \\
\cdots & & & & & & & & & & & \\
0 & 0 & 0 & \cdot & 0 & -a_{n-1k}^{(k-1)\oplus}\left(a_{kk}^{(k-1)\oplus}\right)^{-1} & 0 & 0 & 0 & \cdot & 1 & 0 \\
0 & 0 & 0 & \cdot & 0 & -a_{nk}^{(k-1)\oplus}\left(a_{kk}^{(k-1)\oplus}\right)^{-1} & 0 & 0 & 0 & \cdot & 0 & 1
\end{bmatrix}$$

$$\mathbf{A}^{((k))}=\begin{bmatrix}
1 & a_{12}^{k-1)\oplus} & a_{13}^{(k-1)\oplus} & \cdot & a_{1k-1}^{(k-1)\oplus} & a_{1k}^{(k-1)\oplus} & a_{1k+1}^{(k-1)\oplus} & a_{1k+2}^{(k-1)\oplus} & a_{1k+3}^{(k-1)\oplus} 0 & \cdot & a_{1n-1}^{(k-1)\oplus} & a_{1n}^{(k-1)\oplus} \\
0 & 1 & a_{23}^{(k-1)\oplus} & \cdot & a_{2k-1}^{(k-1)\oplus} & a_{2k}^{(k-1)\oplus} & a_{2k+1}^{(k-1)\oplus} & a_{2k+2}^{(k-1)\oplus} & a_{2k+3}^{(k-1)\oplus} & \cdot & a_{2n-1}^{(k-1)\oplus} & a_{2n}^{(k-1)\oplus} \\
0 & 0 & 1 & \cdot & a_{3k-1}^{(k-1)\oplus} & a_{3k}^{(k-1)\oplus} & a_{3k+1}^{(k-1)\oplus} & a_{3k+2}^{(k-1)\oplus} & a_{3k+3}^{(k-1)\oplus} & \cdot & a_{3n-1}^{(k-1)\oplus} & a_{3n}^{(k-1)\oplus} \\
\cdots & & & & & & & & & & & \cdots \\
0 & 0 & 0 & \cdot & 1 & a_{k-1k}^{(k-1)\oplus} & a_{k-1k+1}^{(k-1)\oplus} & a_{k-1k+2}^{(k-1)\oplus} & a_{k-1k+3}^{(k-1)\oplus} & \cdot & a_{k-1n-1}^{(k-1)\oplus} & a_{k-1n}^{(k-1)\oplus} \\
0 & 0 & 0 & \cdot & 0 & 1 & a_{kk+1}^{(k)} & a_{kk+2}^{(k)} & a_{kk+3}^{(k)} & \cdot & a_{kn-1}^{(k)} & a_{kn}^{(k)} \\
0 & 0 & 0 & \cdot & 0 & 0 & a_{k+1k+1}^{(k)} & a_{k+1k+2}^{(k)} & a_{k+1k+3}^{(k)} & \cdot & a_{k+1n-1}^{(k)} & a_{k+1n}^{(k)} \\
0 & 0 & 0 & \cdot & 0 & 0 & a_{k+2k+1}^{(k)} & a_{k+2k+2}^{(k)} & a_{k+2k+3}^{(k)} & \cdot & a_{k+2n-1}^{(k)} & a_{k+2n}^{(k)} \\
0 & 0 & 0 & \cdot & 0 & 0 & a_{k+3k+1}^{(k)} & a_{k+3k+2}^{(k)} & a_{k+3k+3}^{(k)} & \cdot & a_{k+3n-1}^{(k)} & a_{k+3n}^{(k)} \\
\cdots & & & & & & & & & & & \cdots \\
0 & 0 & 0 & \cdot & 0 & 0 & a_{n-1k+1}^{(k)} & a_{n-1k+2}^{(k)} & a_{n-1k+3}^{(k)} & \cdot & a_{n-1n-1}^{(k)} & a_{n-1n}^{(k)} \\
0 & 0 & 0 & \cdot & 0 & 0 & a_{nk+1}^{(k)} & a_{nk+2}^{(k)} & a_{nk+3}^{(k)} & \cdot & a_{nn-1}^{(k)} & a_{nn}^{(k)}
\end{bmatrix}$$

In the representation of $\mathbf{A}^{(k)}$, transformed elements $a_{ij}^{(k)}$ are given by the positions:

$$a_{kj}^{(k)} = v_{kk}^{(k)} a_{kj}^{(k-1)\oplus}; \qquad j = k+1, k+2, ..., n.$$

$$a_{ij}^{(k)} = a_{ij}^{(k-1)\oplus} + v_{ik}^{(k)} a_{kj}^{(k-1)\oplus}; \quad i, j = k+1, k+2, ..., n.$$

It is important to notice that in the representation of $\mathbf{A}^{(k-1)\oplus}$, the key pivotal element $a_{kk}^{(k-1)\oplus}$ is non-zero since it is given by the expression:

$$a_{kk}^{(k-1)\oplus} = \sum_{q=k}^{n} |a_{kq}^{(k-1)}|.$$

At the end of the n-th step, the transformation of matrix $\mathbf{A}$ is expressed by the following equivalences:

(2.0) $$\mathbf{VAP} = \mathbf{T};$$

(2.1) $$\mathbf{V} = \prod_{q=1}^{n} \mathbf{V}^{(n+1-q)};$$

(2.2) $$\mathbf{P} = \prod_{q=1}^{n} \mathbf{P}^{(q)};$$

(2.3) $$\mathbf{T} = \mathbf{A}^{(n)};$$

so that the sought-for inverse $\mathbf{A}^{-1}$ may be represented by the formula:

(3.0) $$\mathbf{A}^{-1} = \mathbf{PT}^{-1}\mathbf{V}.$$

Transformation $\tau 2$

Matrix $\mathbf{T}$ is transformed into the unit matrix $\mathbf{U}^{(n)}$ through $(n-1)$ consecutive ordered steps in such a way that at the k-th step, $k = 1, 2, ..., n-1$, it is transformed into an upper triangular matrix $\mathbf{T}^{(k)}$ that admits the following four-block representation:

$$\mathbf{T}^{(k)} = \begin{bmatrix} \mathbf{U}^{(k)} & \mathbf{D}^{(k)} \\ \mathbf{Q}^{(k)} & \mathbf{H}^{(k)} \end{bmatrix}; \tag{4.0}$$

where $\mathbf{D}^{(k)}$ is a $(k \times (n-k))$ null matrix; $\mathbf{Q}^{(k)}$ is an $((n-k) \times k)$ null matrix; and $\mathbf{H}^{(k)}$ is a square upper triangular sub-matrix of order $(n-k)$ with all its diagonal elements $h_{ii}^{(k)} = 1$.

At the k-th step, $k = 1, 2, ..., n-1$, the following transformation applies:

$$\mathbf{T}^{(k-1)}\mathbf{G}^{(k)} = \mathbf{T}^{(k)}; \tag{4.1}$$

where:

- $\mathbf{T}^{(0)} = \mathbf{T}$, so that elements $t_{ij}^{(0)} = t_{ij}$;
- $\mathbf{G}^{(k)}$ is a non-singular $(n \times n)$ upper triangular operational matrix which differs (or may differ) from $\mathbf{U}^{(n)}$ only in the k-th row for the elements $g_{kj}^{(k)}$, with $j > k$, given by the expressions: $g_{kj}^{(k)} = -t_{kj}^{(k-1)}$, $j = k+1, k+2, ..., n$.

The following matrix representations help in visualizing the operations required by (4.1)

$$\mathbf{T}^{(k-1)} = \left[\begin{array}{ccccccccccc}
1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot\ 0 & 0 \\
0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot\ 0 & 0 \\
0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot\ 0 & 0 \\
\hdashline
0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot\ 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 1 & t_{kk+1}^{(k-1)} & t_{kk+2}^{(k-1)} & t_{kk+3}^{(k-1)} & \cdot\ t_{kn-1}^{(k-1)} & t_{kn}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & 0 & 1 & t_{k+1k+2}^{(k-1)} & t_{k+1k+3}^{(k-1)} & \cdot\ t_{k+1n-1}^{(k-1)} & t_{k+1n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 1 & t_{k+2k+3}^{(k-1)} & \cdot\ t_{k+2n-1}^{(k-1)} & t_{k+2n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 1 & \cdot\ t_{k+3n-1}^{(k-1)} & t_{k+3n}^{(k-1)} \\
\hdashline
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot\ 1 & t_{n-1n}^{(k-1)} \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot\ 0 & 1
\end{array}\right]$$

$$\mathbf{G}^{(k)} = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ \multicolumn{12}{c}{\dotfill} \\ 0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1 & -t^{(k-1)}_{kk+1} & -t^{(k-1)}_{kk+2} & -t^{(k-1)}_{kk+3} & \cdot & -t^{(k-1)}_{kn-1} & -t^{(k-1)}_{kn} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 1 & \cdot & 0 & 0 \\ \multicolumn{12}{c}{\dotfill} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}^{(k)} = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ \multicolumn{12}{c}{\dotfill} \\ 0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 & t^{(k-1)}_{k+1k+2} & t^{(k-1)}_{k+1k+3} & \cdot & t^{(k-1)}_{k+1n-1} & t^{(k-1)}_{k+1n} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 1 & t^{(k-1)}_{k+2k+3} & \cdot & t^{(k-1)}_{k+2n-1} & t^{(k-1)}_{k+2n} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 1 & \cdot & t^{(k-1)}_{k+3n-1} & t^{(k-1)}_{k+3n} \\ \multicolumn{12}{c}{\dotfill} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 1 & t^{(k-1)}_{n-1n} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 1 \end{bmatrix}$$

At the end of the $(n-1)$-th step, the transformation of matrix $\mathbf{T}$ is expressed by the following equivalences:

$$\mathbf{TG} = \mathbf{U}^{(n)}; \tag{5.0}$$

$$\mathbf{G} = \prod_{q=1}^{n-1} \mathbf{G}^{(q)}. \tag{5.1}$$

Since $\mathbf{G}$ is, according to (5.0), the inverse of $\mathbf{T}$, expression (3.0) for $\mathbf{A}^{-1}$ is rewritten as follows:

$$\mathbf{A}^{-1} = \mathbf{PGV}. \tag{6.0}$$

A numerical example

The practical application of the present method for matrix inversion is illustrated through an elementary example for which, it is to be noticed, the simplest Gaussian elimination procedure cannot be carried out in the natural order.

It is required to find the inverse $\mathbf{A}^{-1}$ of the matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

Transformation $\tau 1$

Step 1:

$$\mathbf{A}^{(0)}\mathbf{P}^{(1)} = \mathbf{A}^{(0)\oplus}; \quad \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & 1 & -1 & -2 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 \end{bmatrix}$$

$$\mathbf{V}^{(1)}\mathbf{A}^{(0)\oplus} = \mathbf{A}^{(1)}; \quad \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{-3}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & 1 & -1 & -2 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} & 0 \\ 0 & \frac{-1}{2} & \frac{1}{2} & -2 \\ 0 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}$$

Step 2:

$$\mathbf{A}^{(1)}\mathbf{P}^{(2)} = \mathbf{A}^{(1)\oplus}; \quad \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} & 0 \\ 0 & \frac{-1}{2} & \frac{1}{2} & -2 \\ 0 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 3 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -1 \end{bmatrix}$$

$$\mathbf{V}^{(2)}\mathbf{A}^{(1)\oplus} = \mathbf{A}^{(2)}; \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 3 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{1}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-1}{3} \end{bmatrix}$$

Step 3:

$$\mathbf{A}^{(2)}\mathbf{P}^{(3)} = \mathbf{A}^{(2)\oplus}; \quad \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{1}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{-1}{3} \end{bmatrix}$$

$$\mathbf{V}^{(3)}\mathbf{A}^{(2)\oplus} = \mathbf{A}^{(3)}; \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{-1}{3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{-1}{3} \end{bmatrix}$$

Step 4:

$$\mathbf{A}^{(3)}\mathbf{P}^{(4)} = \mathbf{A}^{(3)\oplus}; \quad \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\mathbf{V}^{(4)}\mathbf{A}^{(3)\oplus}=\mathbf{A}^{(4)}=\mathbf{T}; \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation $\tau 2$

Step 1:

$$\mathbf{T}^{(0)}\mathbf{G}^{(1)} = \mathbf{T}^{(1)}; \quad \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2:

$$\mathbf{T}^{(1)}\mathbf{G}^{(2)} = \mathbf{T}^{(2)}; \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3:

$\mathbf{T}^{(2)}\mathbf{G}^{(3)} = \mathbf{T}^{(3)} = \mathbf{U}^{(4)}$;

Since $\mathbf{T}^{(2)} = \mathbf{U}^{(4)}$, also $\mathbf{G}^{(3)} = \mathbf{U}^{(4)}$ and $\mathbf{T}^{(3)} = \mathbf{U}^{(4)}$.

$\mathbf{A}^{-1} = \mathbf{PGV}$.

$$\mathbf{P} = \prod_{q=1}^{4} \mathbf{P}^{(q)};\ \mathbf{G} = \prod_{q=1}^{3} \mathbf{G}^{(q)};\ \mathbf{V} = \prod_{q=1}^{4} \mathbf{V}^{(5-q)}.$$

$$\begin{aligned}\mathbf{P} &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}\end{aligned}$$

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{-1}{3} & \frac{-2}{3} \\ 0 & 1 & \frac{-5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}\mathbf{V} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{-3}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & -1 & -2 & 3 \end{bmatrix}\end{aligned}$$

$$\mathbf{A}^{-1}=\mathbf{PGV}=\begin{bmatrix}-1&0&0&0\\1&-1&0&0\\-1&1&1&0\\-1&-1&-1&-1\end{bmatrix}\begin{bmatrix}1&1&\frac{-1}{3}&\frac{-2}{3}\\0&1&\frac{-5}{6}&\frac{-2}{3}\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0&0&0\\\frac{-1}{2}&\frac{1}{3}&0&0\\0&0&1&0\\3&-1&-2&3\end{bmatrix}$$

$$=\begin{bmatrix}-1&-1&\frac{1}{3}&\frac{2}{3}\\1&0&\frac{1}{2}&0\\-1&0&\frac{1}{2}&0\\-1&-2&\frac{1}{6}&\frac{1}{3}\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0&0&0\\\frac{-1}{2}&\frac{1}{3}&0&0\\0&0&1&0\\3&-1&-2&3\end{bmatrix}=\begin{bmatrix}2&-1&-1&2\\\frac{1}{2}&0&\frac{1}{2}&0\\\frac{-1}{2}&0&\frac{1}{2}&0\\\frac{3}{2}&-1&\frac{-1}{2}&1\end{bmatrix}$$

Complex elements

If the elements of matrix $\mathbf{A}$ are complex numbers: $a_{ij}=b_{ij}+\mathbf{i}c_{ij}$, $\mathbf{i}$ being the imaginary unit such that $\mathbf{i}^2=-1$, in transformation $\tau 1$ matrices $\mathbf{A}^{(k-1)}$ (with $\mathbf{A}^{(0)}=\mathbf{A}$), $k=1,2,...,n$, have complex elements $a_{ij}^{(k-1)}=b_{ij}^{(k-1)}+\mathbf{i}c_{ij}^{(k-1)}$, so that the specification of operational matrix $\mathbf{P}^{(k)}$ must be such as to achieve a suitable non-zero leading pivot $a_{kk}^{(k-1)\oplus}$.

Specification of $\mathbf{P}^{(k)}$, $k=1,2,...,n$

$\mathbf{P}^{(k)}$ is a non singular $(n\times n)$ lower triangular matrix which differs (or may differ) from $\mathbf{U}^{(n)}$ only in the diagonal and lower elements of the k-th column. These elements are:

$$p_{ik}^{(k)}=\sigma(b_{ki}^{(k-1)})-\mathbf{i}\sigma(c_{ki}^{(k-1)});\ i=k,k+1,...,n;$$

Specification of $a_{kk}^{(k-1)\oplus}$, $k=1,2,...,n$

$$a_{kk}^{(k-1)\oplus}=\sum_{q=k}^{n}\left(\left|b_{kq}^{(k-1)}\right|+\left|c_{kq}^{(k-1)}\right|\right)+\mathbf{i}\sum_{q=k}^{n}\left(\sigma\left(b_{kq}^{(k-1)}\right)c_{kq}^{(k-1)}-\sigma\left(c_{kq}^{(k-1)}\right)b_{kq}^{(k-1)}\right)$$

Inversion by partitioning

A given square $(n\times n)$ matrix $\mathbf{A}$ may be partitioned into four blocks $\mathbf{A}_q$, $q=1,2,3,4$, in view of operating with matrices of reduced dimensions [3]:

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_1&\mathbf{A}_2\\\mathbf{A}_3&\mathbf{A}_4\end{bmatrix},\qquad (7.0)$$

where $\mathbf{A}_1$ is a square sub-matrix of order s, $s=1,2,...,n-1$; $\mathbf{A}_2$ is an $(s\times(n-s))$ sub-matrix; $\mathbf{A}_3$ is an $((n-s)\times s)$ sub-matrix; and $\mathbf{A}_4$ is a square sub-matrix of order $n-s$.

If $\mathbf{A}$ is non-singular, the possibility of inverting it by operating on its blocks is, however, not granted since it is necessary that all required operations be feasible. In this connection, it is to be noticed that the present $\tau 1$ transformations provide feasibility.

In the four-block representation of the matrix $\mathbf{A}^{(k)}$:

$$\mathbf{A}^{(k)} = \begin{bmatrix} \mathbf{Z}^{(k)} & \mathbf{W}^{(k)} \\ \mathbf{R}^{(k)} & \mathbf{S}^{(k)} \end{bmatrix}, \tag{1.0}$$

the matrix $\mathbf{Z}^{(k)}$ is non-singular so that $\mathbf{A}^{(k)}$ can be easily inverted once the inverses $(\mathbf{Z}^{(k)})^{-1}$ and $(\mathbf{S}^{(k)})^{-1}$ have been computed. Inverse $(\mathbf{A}^{(k)})^{-1}$ is provided by the following expression:

$$(\mathbf{A}^{(k)})^{-1} = \begin{bmatrix} (\mathbf{Z}^{(k)})^{-1} & -(\mathbf{Z}^{(k)})^{-1}\mathbf{W}^{(k)}(\mathbf{S}^{(k)})^{-1} \\ \mathbf{R}^{(k)} & (\mathbf{S}^{(k)})^{-1} \end{bmatrix} \tag{8.0}$$

In view of the relationship between $\mathbf{A}$ and $\mathbf{A}^{(k)}$:

$$\mathbf{V}^{\{k\}}\mathbf{A}\mathbf{P}^{[k]} = \mathbf{A}^{(k)}; \tag{8.1}$$

where

$$\mathbf{V}^{\{k\}} = \prod_{q=1}^{k} \mathbf{V}^{(k+1-q)}; \tag{8.2}$$

$$\mathbf{P}^{[k]} = \prod_{q=1}^{k} \mathbf{P}^{(q)}; \tag{8.3}$$

the required inverse $\mathbf{A}^{-1}$ is represented by the expression:

$$\mathbf{A}^{-1} = \mathbf{P}^{[k]}(\mathbf{A}^{(k)})^{-1}\mathbf{V}^{\{k\}}. \tag{9.0}$$

Appendix 1

Operational counts

Logical simplicity and number of required arithmetic operations are important features of numerical methods.

If operations of multiplication and division are considered equivalent and operations of subtraction and addition are neglected, for the present matrix inversion method the number of required multiplications/divisions turns out to be n^3, i.e., the best result available [1].

The overall number of operations, N_0, required for achieving $\mathbf{A}^{-1}$, is expressed by the sum:

$$N_0 = N_{\tau 1} + N_{\tau 2} + N_{\nu} + N_p + N_g + N_{gv} + N_{p(gv)}; \tag{10.0}$$

where $N_{\tau 1}$, $N_{\tau 2}$, N_{ν}, N_p, N_g, N_{gv} and $N_{p(gv)}$ represent, respectively, the maximal numbers of operations required for carrying out transformations $\tau 1$ and $\tau 2$; for computing $\mathbf{V}$ according to (2.1); $\mathbf{P}$ according to (2.2); $\mathbf{G}$ according to (5.1); $\mathbf{GV}$; and $\mathbf{P}(\mathbf{GV})$.

Evaluation of $N_{\tau 1}$

$$N_{\tau 1} = \sum_{k=1}^{n} N_{1k}, \tag{10.1}$$

where N_{1k} represents the maximal number of operations required for carrying out the k-th step in transformation $\tau 1$.

Evaluation of $N_{1k},\ k = 1, 2, ..., n$

- Operations for $\mathbf{P}^{(k)} : 0$;
- Operations for $\mathbf{A}^{(k-1)}\mathbf{P}^{(k)} : 0$;
- Operations for $\mathbf{V}^{(k)} : n - k + 1$;
- Operations for $\mathbf{V}^{(k)}\mathbf{A}^{(k-1)\oplus} : (n - k + 1)(n - k)$;

$$N_{1k} = (n + 1)^2 - 2(n + 1)k + k^2.$$

$N_{\tau 1}$ is evaluated by taking into account the formulae:

$$\begin{aligned}
\sum_{k=1}^{n} k &= 2^{-1}n(n + 1); \quad \sum_{k=1}^{n} k^2 = 6^{-1}n(n + 1)(2n + 1). \\
N_{\tau 1} &= \sum_{k=1}^{n} N_{1k} = \sum_{k=1}^{n}((n + 1)^2 - 2(n + 1)k + k^2) \\
&= n(n + 1)^2 - n(n + 1)^2 + 3^{-1}n^3 + 2^{-1}n^2 + 6^{-1}n \\
&= 3^{-1}n^3 + 2^{-1}n^2 + 6^{-1}n.
\end{aligned}$$

Evaluation of $N_{\tau 2}$

$$N_{\tau 2} = \sum_{k=1}^{n-1} N_{2k}, \tag{10.2}$$

where N_{2k} represents the maximal number of operations required for carrying out the k-th step in transformation $\tau 2$.

In view of the specifications of the elements of upper triangular matrices $\mathbf{T}^{(k-1)}$ and $\mathbf{G}^{(k)}$ in (4.1), one finds: $N_{2k} = 0$, and consequently $N_{\tau 2} = 0$.

Evaluation of N_v

By introducing the matrix $\mathbf{V}^{[s]},\ s = 1, 2, ..., n-1$, through the position:

$$\mathbf{V}^{[s]} = \prod_{q=1}^{s} \mathbf{V}^{(n+1-q)}; \tag{10.3}$$

N_v may be expressed by the sum:

$$N_v = \sum_{s=1}^{n-1} N_{vs}, \tag{10.4}$$

where N_{vs} represents the maximal number of operations for $\mathbf{V}^{[s]}\mathbf{V}^{(n-s)}$.

Evaluation of $N_{vs},\ s = 1, 2, ..., n-1$.

$$N_{vs} = \sum_{q=1}^{s} q = 2^{-1}s(s+1).$$

The value of N_v is found to be:

$$\begin{aligned} N_v &= \sum_{s=1}^{n-1} 2^{-1}s(s+1) = 4^{-1}(n-1)n + 6^{-1}n^3 - 4^{-1}n^2 + 12^{-1}n \\ &= 6^{-1}n^3 - 6^{-1}n. \end{aligned}$$

Evaluation of N_p

By introducing matrix $\mathbf{P}^{[s]},\ s = 1, 2, ..., n-1$, through the position:

$$\mathbf{P}^{[s]} = \prod_{q=1}^{s} \mathbf{P}^{(q)}. \tag{10.5}$$

N_p may be expressed by the sum:

$$N_p = \sum_{s=1}^{n-1} N_{ps}, \tag{10.6}$$

where N_{ps} represents the maximal number of operations for $\mathbf{P}^{[s]}\mathbf{P}^{(s+1)}$.

In view of the specifications of the elements of lower triangular matrices $\mathbf{P}^{(k)}$ in (1.1), one finds: $N_{ps} = 0$, and consequently $N_p = 0$.

Evaluation of N_g

By introducing the matrix $\mathbf{G}^{[s]}$, $s = 1, 2, ..., n-2$, through the position:

$$\mathbf{G}^{[s]} = \prod_{q=1}^{s} G^{(q)}, \tag{10.7}$$

N_g may be expressed by the sum:

$$N_g = \sum_{s=1}^{n-2} N_{gs}, \tag{10.8}$$

where N_{gs} represents the maximal number of operations for $\mathbf{G}^{[s]}\mathbf{G}^{(s+1)}$.

Evaluation of N_{gs}, $s = 1, 2, ..., n-2$

$$N_{gs} = s(n - s - 1)$$

The value of N_g is found to be:

$$\begin{aligned} N_g &= \sum_{s=1}^{n-2} s(n-s-1) = 2^{-1}(n-2)(n-1)^2 - 6^{-1}(n-2)(n-1)(2n-4+1) \\ &= 2^{-1}(n^3 - 4n^2 + 5n - 2) - 6^{-1}(2n^3 - 9n^2 + 13n - 6) \\ &= 6^{-1}n^3 - 2^{-1}n^2 + 3^{-1}n. \end{aligned}$$

Evaluation of N_{gv}

N_{gv} is expressed by the sum:

$$N_{gv} = \sum_{s=1}^{n-1} N_{gvs}, \tag{10.9}$$

where N_{gvs} represents the maximal number of operations required for multiplying the s-th row, $s = 1, 2, ..., n-1$, of $\mathbf{G}$ with $\mathbf{V}$.

$$\begin{aligned} N_{gvs} &= s(n-s) + \sum_{q=1}^{n-s} q = s(n-s) + 2^{-1}(n-s)(n-s+1) \\ &= ns - s^2 + 2^{-1}(n^2 - 2ns + n + s^2 - s) = 2^{-1}n^2 + 2^{-1}n - 2^{-1}s^2 - 2^{-1}s. \end{aligned}$$

The value of N_{gv} is found to be:

$$\begin{aligned} N_{gv} &= 2^{-1}n^2(n-1) + 2^{-1}n((n-1) - 2^{-1}(3^{-1}n^3 - 2^{-1}n^2 + 6^{-1}n) - 2^{-2}(n-1)n \\ &= 3^{-1}n^3 - 3^{-1}n. \end{aligned}$$

Evaluation of $N_{p(gv)}$

Since the lower triangular matrix $\mathbf{P}$ has elements having absolute values either 0 or 1, the evaluation of $N_{p(gv)}$ is immediate:

$$N_{p(gv)} = 0.$$

Evaluation of N_0

The value of N_0, as defined by (10.0), is found to be:

$$N_0 = 3^{-1}n^3+2^{-1}n^2+6^{-1}n+6^{-1}n^3-6^{-1}n+6^{-1}n^3-2^{-1}n^2+3^{-1}n+3^{-1}n^3-3^{-1}n = n^3.$$

Acknowledgements. I dedicate this work to the memory of my father, Odino Reali, who could see me through a university education in difficult times. At the international colloquium on applications of mathematics in memoriam Lothar Collatz held in Hamburg, 4 and 5 July 1997, and for which no proceedings were published, I suggested the possibility of inverting $\mathbf{A}$ by representing it as a sum $\mathbf{B}+\mathbf{C}$ ($\mathbf{B}$ and $\mathbf{C}$ being $(n \times n)$ non-singular matrices respectively lower and upper triangular), and by applying Scarborough's inversion method [4] to the auxiliary $(2n \times 2n)$ matrix

$$\begin{bmatrix} \mathbf{B} & -\mathbf{U}^{(n)} \\ \mathbf{C} & \mathbf{U}^{(n)} \end{bmatrix}$$

having the inverse

$$\begin{bmatrix} \mathbf{A}^{-1} & \mathbf{A}^{-1} \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbf{B}\mathbf{A}^{-1} \end{bmatrix}.$$

The diagonal elements b_{ii} of $\mathbf{B}$ played the role of free parameters to be suitably specified in view of avoiding the occurrence of null diagonal elements. A specific application for supporting a block solution of a linear algebraic system was envisaged.

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Accepted: 30.05.2013