A NOVEL APPROACH IN HOMOTOPY PERTURBATION METHOD FOR THE EXPANSION OF NON-LINEAR TERMS

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Abstract. Homotopy Perturbation Method (HPM) is widely used in approximate calculation. The complexity of the method is to expand the nonlinear terms in series with higher powers. To make the calculation of nonlinear terms easier and friendly, we have used new algorithm based on derivatives. The same approach can be introduced to expand the higher power nonlinear terms in place of He's polynomial in HPM.

Keywords: Nonlinear, Partial differential equations, Homotopy perturbation method, Zaidian modification.

1. Introduction

Recently, the interest of scientists and engineers in finding the analytic solution to nonlinear problems has increased, for this purpose many new techniques and methods have been developed. Traditional perturbation methods have their own limitations like, presence of a very large or very small parameter inside the problem is essential, so that the solution of the problem may be expressed as a series expansion in terms of that small parameter. Choosing the small parameter is not an easy task and requires special skills. A proper and good choice of small

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parameter will make results more accurate, while on the other hand a wrong choice may lead to inaccurate results.

In 1998, J.H. He [1]–[3] introduced the Homotopy Perturbation Method (HPM) with the advantage that it absorbs all the positive traits, in which, the presence of small parameter was not necessary. J.H. He used the idea of Homotopy along with traditional perturbation method to develop this technique. Many mathematicians and engineers used this method to solve highly nonlinear differential equations, Integral equations and boundary value problem. The solution is assumed to be the summation of infinite series in HPM. Noor and Mohyud-Din [4], [5] applied HPM to fifth, sixth order boundary value problems and many nonlinear problems to show the efficiency of the method. Also, there are analytical techniques to deal with local fractional differential equations [6], [7], such as Local fractional variational iteration method [8], local fractional Fourier series method [9], Yang-Fourier and Yang-Laplace transform method [10]. Recently by introducing He's polynomial in homotopy perturbation method, especially for the nonlinear terms Ghorbani et al. [11], [12] proved the compatibility of He's polynomials with Adomian's polynomials which are easy to calculate. He's polynomials and correction functional of the VIM was combined with HPM by Noor and Mohyud-Din [13]–[14] and used to solve many real world problems as an application by VIMHP. A new method was exploited by E. Babolian [15] to calculate the Adomian's polynomials which was the easier way to handle the higher power nonlinear terms. He has utilized this method only to find the Adomian's polynomials.

It has been observed that in HPM the complex and difficult task is to split and expand the nonlinear terms like $(v_0+pv_1+p^2v_2+\cdots)^i,\ i=2,3,4,...$ with conventional Homotopy Perturbation Method or by using He's polynomials. Manually it is always difficult to expand the fourth and above power of the series having more than four terms or product of two or more terms. In the present study, we have widened the horizon of the method by introducing approach based on derivative [15] in Homotopy Perturbation Method for the highly nonlinear terms. Hence, there was a dire need to develop some new algorithm to tackle the nonlinearities. Inspired from the wide range of applications of He's polynomials in various analytical techniques which not only include Homotopy Perturbation Method but also many other well-known solution techniques like Variation Iteration Method (VIM), Variation of Parameters Method (VPM) and Adomian's Decomposition Method (ADM). The basic inspiration of this paper is that we have split nonlinear terms in two parts, i.e., $N(u)=Z_0+Z_k,\ k=1,2,3,...$, then applied action of T on $N(u)$, which distributes the highly nonlinear terms in the coefficient of $p_0,p_1,p_2,...$. We introduce a new scheme to cope with the complex nonlinear terms. Our proposed technique is easier to implement and the results so obtained are fully compatible with the traditional versions.

2. Homotopy Perturbation Method

To explain the basic idea of Homotopy Perturbation Method for solving nonlinear differential equations, we consider the following nonlinear differential equation

(1) $$A(u) - f(\eta) = 0,\ \eta \in \Omega^*,$$

with boundary conditions

(2) $$B'\left(u,\, \frac{\partial u}{\partial \zeta}\right) = 0,\ \eta \in \partial\Omega^*$$

where A is a general non-linear operator, B' is a boundary operator, $f(\eta)$ is known as analytic function, $\partial\Omega^*$ is the boundary domain and $\frac{\partial u}{\partial \zeta}$ is the directional derivative. The non-linear operator A can further be divided into two parts, linear L and non-linear N, so equation (1) can be expressed as

(3) $$L(u) + N(u) - f(\eta) = 0.$$

He constructed a homotopy as

(4) $$v(\eta, r) : \Omega^* \times [0, 1] \to R,$$

that satisfies the equation

$$(1 - r)\,[L(v) - L(u_0)] + r\,[A(v) - f(\eta)] = 0,$$

Where, $r \in [0,\ 1]$ is an embedding parameter, u_0 is the initial approximation. Therefore, equation (1) can be written as

$$
\begin{aligned}
(5) \qquad H\,[\eta,\, r] &= L(v) - L(u_0) + rL[u_0] + r\,[N(v) - f(\eta)]\,, \\
(6) \qquad H\,[\eta,\, r] &= L(v) - L(u_0) + rL[u_0] + r\,[Z_0 + Z_k - f(\eta)]\,,\ k \geq 1
\end{aligned}
$$

On setting $r = 0$ in equation (5), we get

$$H(v,\, 0) = L[v] - L[u_0] = 0,$$

which provides us initial guess u_0. Similarly by setting $r = 1$ in equation (5), we have

$$
\begin{aligned}
H(v, 1) &= L[v] - L[u_0] + L[u_0] + Z_0 + Z_k - f(\eta), k \geq 1 \\
&= L[v] + Z_0 + Z_k - f(\eta) \\
&= A[v] - f(\eta),
\end{aligned}
$$

where $Z_0 = N\,(v_0)$, for Z_k, $k \geq 1$, applying action of T on Z_0, Z_1,...Z_{n+1} and, by using the result of Lemma [15], it can be prove that $Z_{n+1} = \frac{1}{n+1}\, T\,(Z_n)$

$$\begin{aligned}
Z_1 &= T\left(Z_0\right) = v_1 N^1\left(v_0\right), \\
Z_2 &= \frac{1}{2} T\left(Z_1\right) = \frac{1}{2} T\left(v_1 N^1\left(v_0\right)\right) = \frac{1}{2}\left(T\left(v_1\right) N^1\left(v_0\right) + v_1 T\left(N^1\left(v_0\right)\right)\right) \\
&= v_2 N^1\left(v_0\right) + \frac{1}{2} v_1^2 N^2\left(v_0\right), \\
Z_3 &= \frac{1}{3} T\left(Z_2\right) = \frac{1}{3} T\left(v_2 N^{(1)}\left(v_0\right) + \frac{1}{2} v_1^2 N^{(2)}\left(v_0\right)\right) \\
&= \frac{1}{3}\left(T\left(v_2\right) N^{(1)}\left(v_0\right) + v_2 T\left(N^{(1)}\left(v_0\right)\right)\right. \\
&\quad \left.+\frac{1}{2}\left(T\left(v_1^2\right) N^{(2)}\left(v_0\right) + v_1^2 T\left(N^{(2)}\left(v_0\right)\right)\right)\right) \\
&= v_3 N^1\left(v_0\right) + v_2 v_1 N^{(2)}\left(v_0\right) + \frac{1}{3!} v_1^3 N^{(3)}\left(v_0\right), \\
&\vdots
\end{aligned}$$

and so on we can find $Z_4, Z_5, \ldots$

Thus, the changing process of r from zero to unity is essentially the change in of $v(\eta, r)$ from $u_o(\eta)$ to $u(\eta)$. In topology, this is called deformation. We expand $v(\eta, r)$ in terms of a series in the form

$$v(\eta, r) = v_0 + r v_1 + r^2 v_2 + \cdots,$$

and then taking limit $r \to 1$, equation (**??**) yields

$$u(\eta) = \lim_{r \to 1} v(\eta,\ r) = v_0 + v_1 + v_2 + \cdots,$$

The question of convergence of series has been discussed extensively by J.H. He [2].

3. Properties of the T operator

From [15], consider the operator T with the following properties:

$$T((w_{i_1} w_{i_2} \ldots w_{i_l}) G^{(k)}(u_0)) = T(w_{i_1} w_{i_2} \ldots w_{i_l}) G^{(k)}(u_0), +(w_{i_1} w_{i_2} \ldots w_{i_l}) T(G^{(k)}(u_0)),$$
$$(k \in \mathbf{N} \cup \{0\},\ l \in \mathbf{N},\ l \geq 2),$$

$$T(w_i) = (i+1)(w_{i+1}),$$

$$T(G^{(k)}(u_0)) = u_1 G^{(k+1)} t(u_0)(k \in \mathbf{N} \cup \{0\}),$$

$$T(w_{i_1} w_{i_2} \ldots w_{i_j}) = T(w_{i_1})(w_{i_2} \ldots w_{i_j}) + w_{i_1} T(w_{i_2} \ldots w_{i_j}),\ (j \in \mathbf{N},\ j \geq 2),$$

$$\begin{aligned}
&T(\alpha w_{i_1} w_{i_2} \ldots w_{i_l} G^{(k)}(u_0) + \beta w_{j_1} w_{j_2} \ldots w_{j_m} G^{(k^{'})}(u_0)) \\
&\quad = \alpha T(w_{i_1} w_{i_2} \ldots w_{i_l} G^{(k)}(u_0)) + \beta T(w_{j_1} w_{j_2} \ldots w_{j_m} G^{(k^{'})}(u_0)), \\
&\quad\quad \left(k, k^{'} \in \mathbf{N} \cup \{0\},\ l, m \in \mathbf{N},\ m \geq 2\right)
\end{aligned}$$

where $w_i \in S$, $i \in \mathbf{N} \cup \{0\}$. Define $u(\lambda) = \sum_{j=0}^{\infty} \mathbf{u_j}\left(\lambda^{\mathbf{j}}\right)$ the i-th derivative of $u(\lambda)$ is $u^{(i)}(\lambda) = \sum_{j=1}^{\infty} \mathbf{j}\alpha_{\mathbf{j}}\mathbf{u_j}\lambda^{j-1}$, where $\alpha_j = (j-1)\ldots(j-i+1)$. Putting $\lambda = 0$ gives $u_i = \left(u^{(i)}(0)/i!\right)$.

4. Applications

Example 5.1. $N(u) = u^2 u_x$

Expansion of nonlinear term by using conventional HPM.

$$\begin{aligned} N(u) &= \left(u_0 + p^1u_1 + p^2u_2 + p^3u_3 ...\right)^2 \left(u_{0x} + p^1u_{1x} + p^2u_{2x} + p^3u_{3x} ...\right) \\ &\quad \left(u_0^2 + \left(p^1u_1 + p^2u_2 + p^3u_3 ...\right)^2 + 2u_0\left(p^1u_1 + p^2u_2 + p^3u_3 ...\right)\right) \\ &\quad \left(u_{0x} + p^1u_{1x} + p^2u_{2x} + p^3u_{3x} ...\right) \\ &= p^0\left(u_0^2u_{0x}\right) + p^1\left(u_0^2u_{1x} + 2u_0u_1u_{0x}\right) \\ &\quad + p^2\left(2u_0u_1u_{1x} + u_0^2u_{2x} + u_1^2u_{0x} + 2u_0u_2u_{0x}\right) ... \end{aligned}$$

Expansion by using the new approach we get like powers of p_0, p_1, p_2, ... without expanding the series.

$$\begin{aligned} N(u) &= N(u_0) + T(Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + ...), \\ &= N(u_0) + T(Z_0) + \frac{1}{2}T(Z_1) + \frac{1}{3}T(Z_2) + \frac{1}{4}T(Z_3) + ... \\ &= u_0^2u_{0x} + T\left(u_0^2u_{0x}\right) + \frac{1}{2}T\left(2u_0u_1u_{0x} + u_0^2u_{1x}\right) \\ &\quad + \frac{1}{3}T\left(2u_0u_2u_{0x} + u_1^2u_{0x} + 2u_0u_1u_{1x} + u_0^2u_{2x}\right) + ... \\ &= \underbrace{u_0^2u_{0x}}_{p_0} + \underbrace{2u_0u_1u_{0x} + u_0^2u_{1x}}_{p_1} + \underbrace{2u_0u_2u_{0x} + u_1^2u_{0x} + 2u_0u_1u_{1x} + u_0^2u_{2x}}_{p_2} + ... \end{aligned}$$

Example 5.2. $N(u) = uu_xu_{xx}^2$

Expanding the nonlinear term by using series expansion method in HPM, we get,

$$\begin{aligned} N(u) &= \left(u_0 + p^1u_1 + p^2u_2 + p^3u_3 ...\right)\left(u_{0x} + p^1u_{1x} + p^2u_{2x} + p^3u_{3x} ...\right) \\ &\quad \left(u_{0xx} + p^1u_{1xx} + p^2u_{2xx} + p^3u_{3xx} ...\right)^2 \\ &= \left(u_0(u_{0x} + p^1u_{1x} + p^2u_{2x} + p^3u_{3x} ...) + p^1u_1(u_{0x} + p^1u_{1x} + p^2u_{2x} + p^3u_{3x} ...) + ...\right) \\ &\quad \left(u_{0xx}^2 + t(p^1u_{1xx} + p^2u_{2xx} + p^3u_{3xx} ...)^2 + 2u_{0xx}(p^1u_{1xx} + p^2u_{2xx} + p^3u_{3xx} ...)\right) \\ &= p^0(u_0u_{0x}u_{0xx}^2) + p^1(2u_0u_{0x}u_{0xx}u_{1xx} + u_1u_{0x}u_{0xx}^2 + u_0u_{1x}u_{0xx}^2t) \\ &\quad + p^2 2u_0u_{0x}u_{0xx}u_{2xx} + u_1u_{1x}u_{0xx}^2 + u_0u_{0x}u_{1xx}^2 + 2u_0u_{1x}u_{0xx}u_{1xx} \\ &\quad + u_0u_{2x}u_{0xx}^2 + 2u_1u_{0x}u_{0xx}u_{1xx} + u_2u_{0x}u_{0xx}^2 \end{aligned}$$

Expansion of $N(u)$ by using the new approach we get $p_0, p_1, p_2, ...$ without expanding the series

$$\begin{aligned}
N(u) &= N(u_0) + T(Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + ...) \\
&= N(u_0) + T(Z_0) + \frac{1}{2}T(Z_1) + \frac{1}{3}T(Z_2) + \frac{1}{4}T(Z_3) + ... \\
&= u_0 u_{0x} u_{0xx}^2 + T(u_0 u_{0x} u_{0xx}^2) \\
&\quad + \frac{1}{2}T(2u_0 u_{0x} u_{0xx} u_{1xx} + u_0 u_{1x} u_{0xx}^2 + u_1 u_{0x} u_{0xx}^2) \\
&\quad + \frac{1}{3}T\left(2u_0 u_{0x} u_{0xx} u_{2xx} + u_0 u_{0xx} u_{1xx}^2 + 2u_0 u_{1x} u_{0xx} u_{1xx} + u_2 u_{0x} u_{0xx}^2\right) + ... \\
&= \underbrace{u_0 u_{0x} u_{0xx}^2}_{p_0} + \underbrace{2u_0 u_{0x} u_{0xx} u_{1xx} + u_0 u_{1x} u_{0xx}^2 + u_1 u_{0x} u_{0xx}^2}_{p_1} + \underbrace{2u_0 u_{0x} u_{0xx} u_{2xx}}_{p_2} \\
&\quad \underbrace{+u_0 u_{0xx} u_{1xx}^2 + 2u_0 u_{1x} u_{0xx} u_{1xx} + u_2 u_{0x} u_{0xx}^2}_{p_2} + ...
\end{aligned}$$

Example 5.3. $N(u) = u^4 + uu_x$.

Expansion of nonlinear term by using conventional HPM

$$\begin{aligned}
N(u) &= (u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 ...)^4 + (u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 ...) \\
&\quad (u_{0x} + p^1 u_{1x} + p^2 u_{2x} + p^3 u_{3x} ...) \\
N(u) &= (u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 ...)^2 + (u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 ...)^2 \\
&\quad + u_0 (u_{0x} + p^1 u_{1x} + p^2 u_{2x} + p^3 u_{3x} ...) + p^1 u_1 u_{0x} + p^1 u_{1x} + p^2 u_{2x} \\
&\quad + p^3 u_{3x} ... + .. \\
&= p^0 (u_0^4 + u_0 u_{0x}) + p^1 (u_0 u_{1x} + u_1 u_{0x} + 4u_0^3 u_{1x}) \\
&\quad + p^2 (u_1 u_{1x} + 6u_0^2 u_1^2 + u_2 u_{0x} + u_0 u_{2x} + 4u_0^3 u_2) + ..
\end{aligned}$$

Expansion of given nonlinear term by using the new approach we get $p_0, p_1, p_2, ...$ without expanding the series

$$\begin{aligned}
N(u) &= N(u_0) + T(Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + ...) \\
&= N(u_0) + T(Z_0) + \frac{1}{2}T(Z_1) + \tfrac{1}{3}T(Z_2) + \tfrac{1}{4}T(Z_3) + ... \\
&= u_0^4 + u_0 u_{0x} + T(u_0^4 + u_0 u_{0x}) + \frac{1}{2}T(4u_0^3 u_1 + u_1 u_{0x} + u_0 u_{1x}) + ... \\
&= = \underbrace{u_0^4 + u_0 u_{0x}}_{p_0} + \underbrace{4u_0^3 u_1 + u_1 u_{0x} + u_0 u_{1x}}_{p_1} + \underbrace{4u_0^3 u_2 + 6u_0^2 u_1^2 + u_2 u_{0x}}_{p_2} \\
&\quad \underbrace{+u_1 u_{1x} + u_0 u_{2x}}_{p_2} + ...
\end{aligned}$$

5. Conclusions

The Homotopy Perturbation Method is very powerful and efficient method to solve the linear and nonlinear equations. For the physical problems it provides more realistic series solution but the complexity is to expand the terms with higher powers. The *Zaidian* approach presented in this paper to handle such situation made the application of HPM much easier and friendly. Moreover, application of this paper is very wide in a way that it can be used to expand nonlinear terms in Variation Iteration Method, Variation of Parameters Method, perturbation method and series solution method.

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