VARIATION OF PARAMETERS METHOD FOR THIN FILM FLOW OF A THIRD GRADE FLUID DOWN AN INCLINED PLANE

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Abstract. Thin film Flow of a third grade fluid down an inclined plane is considered. Non-linear differential equation which governs the flow model is obtained by using laws of conservation along with suitable similarity transforms. Variation of Parameters Method (VPM) is then employed to solve aforementioned differential equation. Analytical solution is backed by RK-4 numerical solution and both show excellent agreement. Effects of flow parameters β and m on velocity field are demonstrated graphically with comprehensive discussions.

Keywords: non-Newtonian fluid, thin film flow, nonlinear equation, inclined plane, Variation of Parameters Method (VPM).

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1. Introduction

Recently, the importance of non-Newtonian fluids has become prominent with developments in industries like polymer, petroleum, pulp etc. Many industrial materials fall into this category, such as solutions, melts of polymers, soap, biological solutions, paints, tars, asphalts and glues. Due to complex nature of non-Newtonian fluids it is hard to establish one mathematical model that can describe characteristics of all non-Newtonian fluids. So many mathematical models are used to discuss flow of non-Newtonain flows, third grade fluids lie in one of such class and are papular among researchers due to their simpler mathematical simulation. Denson [1], Waters and Keeley [2] studied some problems of third grade fluids. Siddiqui et al. [3] discussed the solution for thin film flow of third grade fluid down an inclined plane using perturbation method and Homotopy Perturbation Method (HPM). Hayat et al. [4] obtained exact solutions to same problem under certain assumptions. Sajid et al. [5] used Homotopy Analysis Method (HAM) to solve thin film flow of third grade fluid down an inclined plane and proved that HAM works better as compared to HPM.

Most of the physical problems involving non-Newtonian fluids are in form of nonlinearities, so exact solutions may not be possible. Therefore, over the years different analytical techniques have been developed. Variation of Parameters Method (VPM) is one of these techniques which do not require any linearization, discretization or existence of small and large parameters for expansion. Ma et al. [6]-[8] used VPM for solving non-homogeneous partial differential equations. Mohyud-Din et al. [9, 10, 11]–[11] further applied this technique for solving initial and boundary value problems of diversified physical nature.

Inspired and motivated by the ongoing research in this direction, we apply VPM to study thin film flow of a third grade fluid down an inclined plane. This technique is comparatively novel in fluid mechanics and has a good scope in further studies.

2. Governing equation

The basic equations for the flow of an incompressible third grade fluid flow down an inclined plane in absence of body force are

$$\nabla \cdot V = 0 \tag{1}$$

$$\rho \frac{DV}{Dt} = -\nabla p + div\tau \tag{2}$$

Where $\mathbf{V}$, ρ, p, τ denote velocity vector, constant density, pressure and stress tensor respectively, further $\frac{D}{Dt}$ is the material derivative and defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (V \cdot \nabla).$$

The stress tensor defining third grade fluid is given by

$$\tau = S_1 + S_2 + S_3, \tag{3}$$

where

$$S_1 = \mu A_1 \tag{4}$$
$$S_2 = \alpha_1 A_2 + \alpha_2 A_1^2 \tag{5}$$
$$S_3 = \beta_1 A_3 + \beta_2 \left(A_1 A_2 + A_2 A_1\right) + \beta_3 \left(tr A_2\right) A_1. \tag{6}$$

Here μ is the coefficient of viscosity and $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \beta_3$, are material constants. The Rivilin-Ericksen tensor, A_n, are define by $A_0 = I$, the identity tensor, and

$$A_n = \frac{DA_{n-1}}{Dt} + A_{n-1}\left(\nabla V\right) + \left(\nabla V\right)^T A_{n-1},\ \ n \geq 1 \tag{7}$$

3. Problem formulation

Thin film flow of an incompressible third grade fluid down an inclined plane is considered. Ambient air is assumed to be stationary and flow is due to gravity only. It is assumed that the surface tension of the fluid is negligible and film is of uniform thickness δ. We seek a one-dimensional steady velocity field of the form

$$V = (u(y),\ 0,\ 0\ ) \tag{8}$$

Using equation (3) and (8) in equations (1) and (2), we obtain the momentum equation with boundary conditions as

$$\mu\frac{d^2u}{dy^2} + 6\left(\beta_2 + \beta_3\right)\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} + \rho g \sin\alpha = 0 \tag{9}$$

$$u(y) = 0, \quad \text{at} \quad y = 0, \tag{10}$$

$$\frac{du}{dy} = 0, \quad \text{at} \quad y = \delta, \tag{11}$$

where, g is the gravity of the fluid.

Introduce the suitable dimensionless variables as,

$$u^* = \frac{u}{\nu/\delta}, \quad y^* = \frac{y}{\delta}, \quad \beta^* = \frac{\beta}{\rho\delta^4/\nu} \tag{12}$$

and dropping asterisks for simplicity, dimensionless form of the momentum equation (9) with boundary conditions (10) and (11) become

$$\frac{d^2u}{dy^2} + 6\beta\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} + m = 0 \tag{13}$$

$$u(0) = 0, \quad \frac{du}{dy} = 0, \quad \text{at} \quad y = 1 \tag{14}$$

where $m = \dfrac{\delta^3 g \sin\alpha}{\nu^2}$.

4. Variation of Parameter Method

To illustrate the basic concept of the variation of parameter method for differential equations, we consider the general differential equation in operator form

$$Lu(y) + Nu(y) + Ru(y) = g(y), \tag{15}$$

where L is a higher order linear operator, R is a linear operator of order less then L, N is a nonlinear operator, and g is a source term. We have following general solution of equation

$$u(y) = \sum_{i=0}^{n-1} \frac{A_i y^i}{i!} + \int_0^y \lambda\,(y,s)\,(-Nu(s) - Ru(s) + g(s))\,ds, \tag{16}$$

where n is the order of given differential equation and A_i's are unknowns which can be further determined by initial/boundary conditions. Here, $\lambda(y,s)$ is a multiplier which can be obtained with the help of Wronskian technique as

$$\lambda(y,s) = y - s. \tag{17}$$

Hence, we have the following iterative scheme,

$$u_{k+1}(y) = \sum_{i=0}^{n-1} \frac{A_i y^i}{i!} + \int_0^y (y-s)\,(-Nu(s) - Ru(s) + g(s))\,ds, \tag{18}$$

where $k = 0, 1, 2, ...$

It is observed that the fix value of initial guess in each iteration provide the better approximation.

5. Solution of the problem

Using Equation (13) along (14) in (18), we have the follwing iterative scheme

$$u_{k+1}(y) = A - \int_0^y (y-s)\left(6\beta\left(\frac{du}{ds}\right)^2 \frac{d^2u}{ds^2} + m\right) ds,$$

where $A = \frac{du}{dy}(0)$, can be determined by using boundary conditions (14).

Few iterations of the solution are given below

$$\begin{aligned}
u_0 &= Ay \\
u_1 &= Ay - \frac{1}{2}my^2 \\
u_2 &= \frac{1}{2}\beta\, m^3 y^4 - 2\,\beta\, Am^2 y^3 + \left(3\,\beta\, A^2 m - \frac{1}{2}\, m\right) y^2 + Ay
\end{aligned}$$

$$
\begin{aligned}
u_3 \;=\; & Ay + \left(3\,\beta\,A^2 m - \frac{1}{2}m - 18\,\beta^2 A^4 m\right) y^2 \\
& + \left(-2\,\beta\,A m^2 - 72\,\beta^3 A^5 m^2 + 36\,\beta^2 A^3 m^2\right) y^3 \\
& + \left(-108\,\beta^4 A^6 m^3 + 162\,\beta^3 A^4 m^3 - 30\,\beta^2 A^2 m^3 + \frac{1}{2}\beta\,m^3\right) y^4 \\
& + \left(12\,\beta^2 A m^4 - \frac{792}{5}\,\beta^3 A^3 m^4 + \frac{1296}{5}\,\beta^4 A^5 m^4\right) y^5 \\
& + \left(-288\,\beta^4 m^5 A^4 - 2\,\beta^2 m^5 + 84\,\beta^3 m^5 A^2\right) y^6 \\
& + \left(-24\,\beta^3 m^6 A + \frac{1296}{7}\,\beta^4 m^6 A^3\right) y^7 \\
& + \left(-72\,\beta^4 A^2 m^7 + 3\,\beta^3 m^7\right) y^8 + 16\,\beta^4 A m^8 y^9 - \frac{8}{5}\,\beta^4 m^9 y^{10},
\end{aligned}
$$

6. Results and discussion

Fig. 1 depicts the variations in velocity profile for different values of non-Newtonian parameter β and fixed the value of m. It can be observed that as the value of β increases, velocity decreases. For $\beta = 0$ solution reduces to the case of Newtonian fluid, which is represented by solid line in the graph. From Fig. 2 it can be observed that for fixed value of β velocity increases by increasing m. By using RK-4 method and setting tolerance of 0.000001 same problem is solved numerically and results are compared with VPM solution in Fig. 3 and Table 1. It can be seen that both results have excellent agreement.

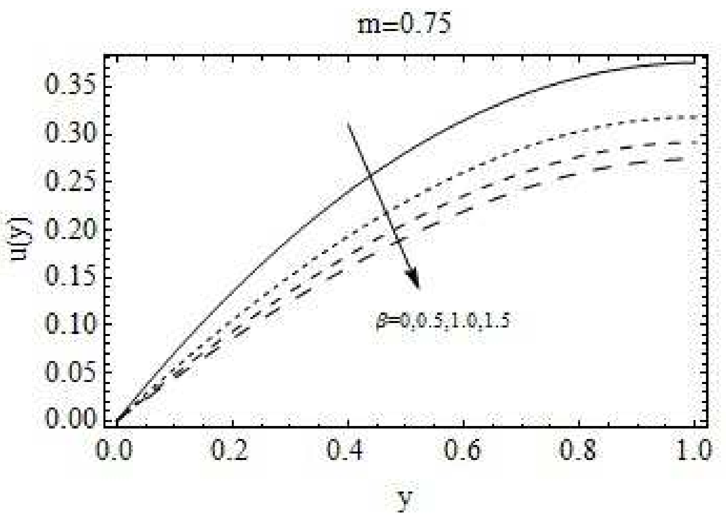

Figure 1. Effects of β on velocity profile

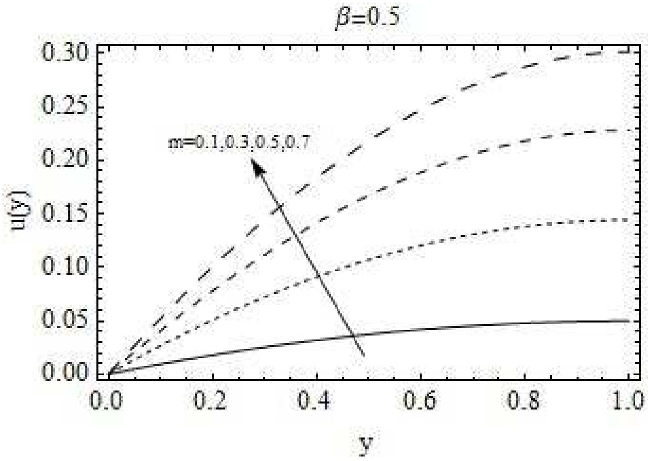

Figure 2. Effects of m on velocity profile

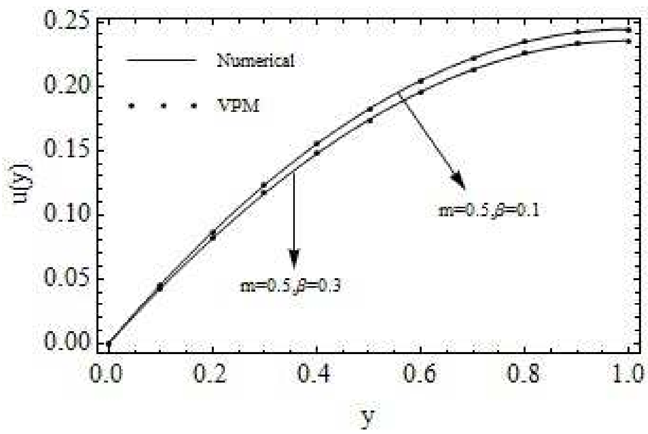

Figure 3. Comparison of VPM and numerical solution

Table 1 presents a comparison of VPM and numerical solutions which is graphically represented in Fig. 3. From Fig. 3 it can be analyzed in even in first glance that VPM has good agreement with numerical results.

	$\beta = 0.1$		$\beta = 0.3$	
y	VPM	Numerical	VPM	Numerical
0.1	0.045599	0.045599	0.044069	0.044069
0.2	0.086705	0.086705	0.084013	0.084013
0.3	0.123228	0.123228	0.11969	0.11969
0.4	0.155078	0.155078	0.150961	0.150961
0.5	0.182177	0.182177	0.177691	0.177691
0.6	0.204453	0.204453	0.199756	0.199756
0.7	0.221846	0.221846	0.217045	0.217045
0.8	0.234305	0.234305	0.229466	0.229466
0.9	0.241796	0.241796	0.236947	0.236947
1.0	0.244296	0.244296	0.239446	0.239446

Table.1: Comparison between VPM and numerical solutions

7. Conclusions

In the present work, VPM is applied to solve nonlinear equation governing thin film flow of a third grade fluid down an inclined plane. Effects of viscoelastic parameter β and dimensionless inclination parameter m on velocity profile are recorded. Velocity is observed to be decreasing for increasing β while it increases with increasing inclination parameter m. Numerical solution affirms the results obtained by VPM and it can be concluded that VPM can successfully be applied to problems in fluid mechanics.

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