PROPERTIES OF BIPOLAR FUZZY HYPERGRAPHS

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Abstract. In this article, we apply the concept of bipolar fuzzy sets to hypergraphs and investigate some properties of bipolar fuzzy hypergraphs. We introduce the notion of $A-$ tempered bipolar fuzzy hypergraphs and present some of their properties. We also present application examples of bipolar fuzzy hypergraphs.

Keywords: Bipolar fuzzy hypergraph, bipolar fuzzy partition, dual bipolar fuzzy hypergraph, $A-$ tempered bipolar fuzzy hypergraphs, clustering problem.

Mathematics Subject Classification 2010: 05C99.

1. Introduction

In 1994, Zhang [27] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets [25]. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets [20]. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [15], because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places and negative information expressed as a set of impossible places.

At present, graph theoretical concepts are highly utilized by computer science applications. Especially in research areas of computer science including data mining, image segmentation, clustering, image capturing and networking, for example a data structure can be designed in the form of tree which in turn utilized vertices and edges. Similarly, modeling of network topologies can be done using graph concepts. In the same way the most important concept of graph coloring is utilized in resource allocation, scheduling. Also, paths, walks and circuits in graph theory are used in tremendous applications say traveling salesman problem, database design concepts, resource networking. This leads to the development of new algorithms and new theorems that can be used in tremendous applications. Hypergraphs are the generalization of graphs (cf. [10]) in case of set of multiarity relations. It means the expansion of graph models for the modeling complex systems. In case of modeling systems with fuzzy binary and multiarity relations between objects, transition to fuzzy hypergraphs, which combine advantages both fuzzy and graph models, is more natural. It allows to realize formal optimization and logical procedures. However, using of the fuzzy graphs and hypergraphs as the models of various systems (social, economic systems, communication networks and others) leads to difficulties. The graph isomorphic transformations are reduced to redefinition of vertices and edges. This redefinition does not change properties the graph determined by an adjacent and an incidence of its vertices and edges. Fuzzy independent set, domination fuzzy set, fuzzy chromatic set are invariants concerning the isomorphism transformations of the fuzzy graphs and fuzzy hypergraph and allow make theirs structural analysis [11]. Lee-kwang et al. [21] generalized and redefined the concept of fuzzy hypergraphs whose basic idea was given by Kaufmann [18]. Further, the concept of fuzzy hypergraphs was discussed in [17]. Chen [14] introduced the concept of interval-valued fuzzy hypergraphs. Parvathi *et al.*[23] introduced the concept of intuitionistic fuzzy hypergraphs. Samanta and Pal [20] introduced the concept of a bipolar fuzzy hypergraph and studied some of its elementary properties. In this article, we first investigate some interesting properties of bipolar fuzzy hypergraphs. We introduce the regularity of bipolar fuzzy hypergraphs. We then introduce the notion of $A-$ tempered bipolar fuzzy hypergraphs and present some of their properties. Finally, we present an example of a bipolar fuzzy partition on the digital image processing.

We used standard definitions and terminologies in this paper. For notations, terminologies and applications are not mentioned in the paper, the readers are referred to [1]-[9].

2. Preliminaries

A *hypergraph* is a pair $H^* = (V, E^*)$, where V is a finite set of nodes (vertices) and E^* is a set of edges (or hyperedges) which are arbitrary nonempty subsets of V such that $\bigcup_j E_j^* = V$. A hypergraph is a generalization of an ordinary undirected graph, such that an edge need not contain exactly two nodes, but can instead contain an arbitrary nonzero number of vertices. An ordinary undirected graph (without self-loops) is, of course, a hypergraph where every edge has exactly two nodes (vertices). A hypergraph is *simple* if there are no repeated edges and no edge properly contains another. Hypergraphs are often defined by an incidence matrix with columns indexed by the edge set and rows indexed by the vertex set. The *rank* $r(H)$ of a hypergraph is defined as the maximum number of nodes in one edge, $r(H)$ =max$_j(|Ej|)$, and the *anti-rank* $s(H)$ is defined likewise, i.e., $s(H)$ =min$_j(|Ej|)$. We say that a hypergraph is *uniform* if $r(H) = s(H)$. A uniform hypergraph of rank k is called k-uniform hypergraph. Hence a simple graph is a 2-uniform hypergraph, and thus all simple graphs are also hypergraphs. A hypergraph is *vertex* (resp. *hyperedge*) *symmetric* if for any two vertices (resp. hyperedges) v_i and v_j (resp. e_i and e_j), there is an automorphism of the hypergraph that maps v_i to v_j (resp. e_i to e_j). The *dual of a hypergraph* $H^* = (V, E^*)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and hyperedge set $E^* = \{e_1^*, e_2^*, \ldots, e_m^*\}$ is a hypergraph $H^d = (V^d, E^{*d})$ with vertex set $V^d = \{v_1^d, v_2^d, \ldots, v_m^d\}$ and hyperedge set $E^d = \{(e_1^*)^d, (e_2^*)^d, \ldots, (e_n^*)^d\}$ such that v_j^d corresponds to e_j^* with hyperedges $(e_i^*)^d = \{v_j^d \mid v_i \in e_j^* \text{ and } e_j^* \in E^*\}$. In other words, H^d is obtained from H^* by interchanging of vertices and hyperedges in H^*. The incidence matrix of H^d is the transpose of the incidence matrix of H^*. Thus, $(H^d)^d = H^*$.

Definition 2.1. [25, 26]A *fuzzy set* μ on a nonempty set X is a map $\mu : X \to [0, 1]$. In the clustering, the fuzzy set μ, is called a *fuzzy class*. We define the *support* of μ by supp $(\mu) = \{x \in X \mid \mu(x) \neq 0\}$ and say μ is nontrivial if supp(μ) is nonempty. The *height* of μ is $h(\mu) = \max\{\mu(x) \mid x \in X\}$. We say μ is *normal* if $h(\mu)$ =1. A map $\nu : X \times X \to [0, 1]$ is called a *fuzzy relation* on X if $\nu(x, y) \leq \min(\mu(x), \mu(y))$ for all $x, y \in X$. A *fuzzy partition* of a set X is a family of nontrivial fuzzy sets $\{\mu_1, \mu_2, \mu_3, \ldots, \mu_m\}$ such that

(1) $\bigcup_i \text{supp}(\mu_i) = X, \quad i = 1, 2, \ldots, m$

(2) $\sum_{i=1}^m \mu_i(x) = 1$ for all $x \in X$.

We call a family $\{\mu_1, \mu_2, \mu_3, \ldots, \mu_m\}$ a *fuzzy covering* of X if it verifies only the above conditions (1) and (2).

Definition 2.2. [20]Let V be a finite set and let E be a finite family of nontrivial fuzzy sets on V such that $V = \bigcup_j \operatorname{supp}(\mu_j)$, where μ_j is membership function defined on $E_j \in E$. Then the pair $H = (V, E)$ is a *fuzzy hypergraph* on V, E is the family of fuzzy edges of H and V is the (crisp) vertex set of H.

Definition 2.3. [19, 27] Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \{(x,\, \mu^P(x),\, \mu^N(x)) \,|\, x \in X\}$$

where $\mu^P : X \to [0,\,1]$ and $\mu^N : X \to [-1,\,0]$ are mappings.

We use the positive membership degree $\mu^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B, and the negative membership degree $\mu^N(x)$ to denote the satisfaction degree of an element x to some explicit or implicit property corresponding to a bipolar fuzzy set B. If $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B. If $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B . It is possible for an element x to be such that $\mu^P(x) \neq 0$ and $\mu^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

For the sake of simplicity, we shall use the symbol $B = (\mu^P,\, \mu^N)$ for the bipolar fuzzy set $B = \{(x,\, \mu^P(x),\, \mu^N(x)) \,|\, x \in X\}$.

Definition 2.4. [27] Let X be a nonempty set. Then we call a mapping $A = (\mu_A^P, \mu_A^N) : X \times X \to [0,1] \times [-1,0]$ a *bipolar fuzzy relation* on X such that $\mu_A^P(x,y) \in [0,1]$ and $\mu_A^N(x,y) \in [-1,0]$.

Definition 2.5. [19] The *support* of a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$, denoted by supp(A), is defined by

$$\operatorname{supp}(A) = \operatorname{supp}^P(A) \cup \operatorname{supp}^N(A), \quad \operatorname{supp}^P(A) = \{x \,|\, \mu_A^P(x) > 0\},$$
$$\operatorname{supp}^N(A) = \{x \,|\, \mu_A^N(x) < 0\}.$$

We call $\operatorname{supp}^P(A)$ as positive support and $\operatorname{supp}^N(A)$ as negative support.

Definition 2.6. [19] Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set on X and let $\alpha \in [0,1]$. α-cut A_α of A can be defined as

$$A_\alpha = A_\alpha^P \cup A_\alpha^N,\ A_\alpha^P = \{x \mid \mu_\alpha^P(x) \geq \alpha\},\ A_\alpha^P = \{x \mid \mu_\alpha^N(x) \leq -\alpha\}.$$

We call A_α^P as positive α-cut and A_α^N as negative α-cut.

Definition 2.7. The *height* of a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is defined as $h(A) = \max\{\mu_A^P(x)|x \in X\}$. The *depth* of a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is defined as $d(A) = \min\{\mu_A^N(x)|x \in X\}$. We shall say that bipolar fuzzy set A is *normal*, if there is at least one $x \in X$ such that $\mu_A^P(x) =1$ or $\mu_A^N(x) =-1$.

3. Bipolar fuzzy hypergraphs

Definition 3.1. [24] Let V be a finite set and let $E = \{E_1, E_2, \ldots, E_m\}$ be a finite family of nontrivial bipolar fuzzy subsets of V such that

$$V = \bigcup_j \mathrm{supp}(\mu_j^P, \mu_j^N), \quad j = 1, 2 \ldots, m,$$

where μ_j^P and μ_j^N are positive and negative membership functions defined on $E_j \in E$. Then the pair $H = (V, E)$ is a *bipolar fuzzy hypergraph* on V, E is the family of bipolar fuzzy edges of H and V is the (crisp) vertex set of H. The order of H (number of vertices) is denoted by $|V|$ and the number of edges is denoted by $|E|$.

Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of V and let E be a collection of bipolar fuzzy subsets of V such that for each $B = (\mu_B^P, \mu_B^N) \in E$ and $x \in V$, $\mu_B^P(x) \le \mu_A^P(x)$, $\mu_B^N(x) \ge \mu_A^N(x)$. Then the pair (A, B) is a bipolar fuzzy hypergraph on the bipolar fuzzy set A. The bipolar fuzzy hypergraph (A, B) is also a bipolar fuzzy hypergraph on $V = \mathrm{supp}(A)$, the bipolar fuzzy set A defines a condition for positive membership and negative membership in the edge set E. This condition can be stated separately, so without loss of generality we restrict attention to bipolar fuzzy hypergraphs on crisp vertex sets.

Example 3.2. Consider a bipolar fuzzy hypergraph $H = (V, E)$ such that $V = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3\}$, where

$$E_1 = \left\{ \frac{a}{(0.2, -0.3)}, \frac{b}{(0.4, -0.5)} \right\},$$
$$E_2 = \left\{ \frac{b}{(0.4, -0.5)}, \frac{c}{(0.5, -0.2)} \right\},$$
$$E_3 = \left\{ \frac{a}{(0.2, -0.3)}, \frac{d}{(0.2, -0.4)} \right\}.$$

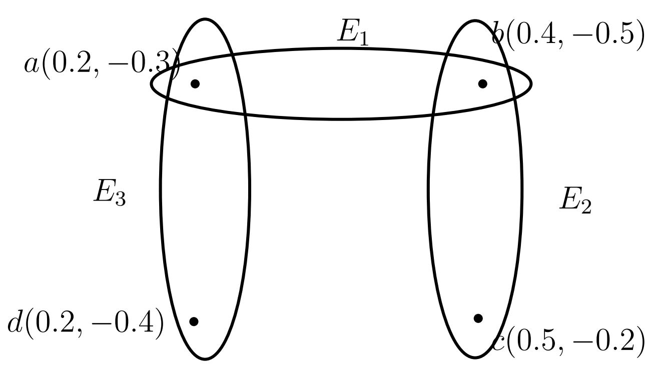

Figure 1: Bipolar fuzzy hypergraph

Table 1: The corresponding incidence matrix is given below:

M_H	E_1	E_2	E_3
a	$(0.2, -0.3)$	$(0, 0)$	$(0.2, -0.3)$
b	$(0.4, -0.5)$	$(0.4, -0.5)$	$(0, 0)$
c	$(0, 0)$	$(0.5, -0.2)$	$(0, 0)$
d	$(0, 0)$	$(0, 0)$	$(0.2, -0.4)$

Definition 3.3. A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N) : X \to [0,1] \times [-1,0]$ is an *elementary bipolar fuzzy set* if A is single valued on supp(A). An elementary bipolar fuzzy hypergraph $H = (V, E)$ is a bipolar fuzzy hypergraph whose edges are elementary.

We explore the sense in which a bipolar fuzzy graph is a bipolar fuzzy hypergraph.

Proposition 3.4. *Bipolar fuzzy graphs are special cases of the bipolar fuzzy hypergraphs.*

A bipolar fuzzy multigraph is a multivalued symmetric mapping $D = (\mu_D^P, \mu_D^N) : V \times V \to [0,1] \times [-1,0]$. A bipolar fuzzy multigraph can be considered to be the "disjoint union" or "disjoint sum" of a collection of simple bipolar fuzzy graphs, as is done with crisp multigraphs. The same holds for multidigraphs. Therefore, these structures can be considered as "disjoint unions" or "disjoint sums" of bipolar fuzzy hypergraphs.

Definition 3.5. A bipolar fuzzy hypergraph $H = (V, E)$ is *simple* if $A = (\mu_A^P, \mu_A^N)$, $B = (\mu_B^P, \mu_B^N) \in E$ and $\mu_A^P \leq \mu_B^P$, $\mu_A^N \geq \mu_B^N$ imply that $\mu_A^P = \mu_B^P$, $\mu_A^N = \mu_B^N$. In particular, a (crisp) hypergraph $H^* = (V, E^*)$ is simple if X, $Y \in E^*$ and $X \subseteq Y$ imply that $X = Y$. A bipolar fuzzy hypergraph $H = (V, E)$ is *support simple* if $A = (\mu_A^P, \mu_A^N)$, $B = (\mu_B^P, \mu_B^N) \in E$, supp$(A) =$ supp(B), and $\mu_A^P \leq \mu_B^P$, $\mu_A^N \geq \mu_B^N$ imply that $\mu_A^P = \mu_B^P$, $\mu_A^N = \mu_B^N$. A bipolar fuzzy hypergraph $H = (V, E)$ is *strongly support simple* if $A = (\mu_A^P, \mu_A^N)$, $B = (\mu_B^P, \mu_B^N) \in E$ and supp$(A) =$ supp(B) imply that $A = B$.

Remark. The definition 3.5 reduces to familiar definitions in the special case where H is a crisp hypergraph. The bipolar fuzzy definition of simple is identical to the crisp definition of simple. A crisp hypergraph is support simple and strongly support simple if and only if it has no multiple edges. For bipolar fuzzy hypergraphs all three concepts imply no multiple edges. Simple bipolar fuzzy hypergraphs are support simple and strongly support simple bipolar fuzzy hypergraphs are support simple. Simple and strongly support simple are independent concepts.

Definition 3.6. Let $H = (V, E)$ be a bipolar fuzzy hypergraph. Suppose that $\alpha \in [0,1]$, $\beta \in$ [-1, 0]. Let

- $E_{(\alpha,\beta)} = \{A_{(\alpha,\beta)}|$ where A is positive and negative membership function defined on $E_j \in E\}$, $A_{(\alpha,\beta)} = \{x \,|\, \mu_A^P(x) \geq \alpha$ or $\mu_A^N(x) \leq \beta\}$, and
- $V_{(\alpha,\beta)} = \bigcup_{A \in E} A_{(\alpha,\beta)}$.

If $E_{(\alpha,\beta)} \neq \emptyset$, then the crisp hypergraph $H_{(\alpha,\beta)} = (V_{(\alpha,\beta)}, E_{(\alpha,\beta)})$ is the $(\alpha,\beta)-$ level hypergraph of H.

Clearly, it is possible that $A_{(\alpha,\beta)} = B_{(\alpha,\beta)}$ for $A \neq B$, by using distinct markers to identity the various members of E a distinction between $A_{(\alpha,\beta)}$ and $B_{(\alpha,\beta)}$ to represent multiple edges in $H_{(\alpha,\beta)}$. However, we do not take this approach unless otherwise stated, we will always regard $H_{(\alpha,\beta)}$ as having no repeated edges.

The families of crisp sets (hypergraphs) produced by the (α,β)-cuts of a bipolar fuzzy hypergraph share an important relationship with each other, as expressed below: suppose $\mathbb{X}$ and $\mathbb{Y}$ are two families of sets such that for each set X belonging to $\mathbb{X}$ there is at least one set Y belonging to $\mathbb{Y}$ which contains X. In this case we say that $\mathbb{Y}$ *absorbs* $\mathbb{X}$ and symbolically write $\mathbb{X} \sqsubseteq \mathbb{Y}$ to express this relationship between $\mathbb{X}$ and $\mathbb{Y}$. Since it is possible for $\mathbb{X} \sqsubseteq \mathbb{Y}$ while $\mathbb{X} \cap \mathbb{Y} = \emptyset$, we have that $\mathbb{X} \subseteq \mathbb{Y} \Rightarrow \mathbb{X} \sqsubseteq \mathbb{Y}$, whereas the converse is generally false. If $\mathbb{X} \sqsubseteq \mathbb{Y}$ and $\mathbb{X} \neq \mathbb{Y}$, then we write $\mathbb{X} \sqsubset \mathbb{Y}$.

Definition 3.7. Let $H = (V, E)$ be a bipolar fuzzy hypergraph. Let $H_{(s,t)}$ be the $(s,t)-$ level hypergraph of H. The sequence of real numbers

$$\{(s_1, r_1), (s_2, r_2), ..., (s_n, r_n)\},\ 0 < s_1 < s_2 < \cdots < s_n \text{ and } 0 > r_1 > r_2 > \cdots > r_n,$$
$$\text{where } (s_n, r_n) = h(H),$$

which satisfies the properties:

- if $(s_{i-1}, r_{i-1}) < (u, v) \leq (s_i, r_i)$, then $E_{(u,v)} = E_{(s_i,r_i)}$, and
- $E_{(s_i,r_i)} \sqsubset E_{(s_{i+1},r_{i+1})}$,

is called the *fundamental sequence* of H, and is denoted by $F(H)$ and the set of (s_i, r_i)-level hypergraphs $\{H_{(s_1,r_1)}, H_{(s_2,r_2)}, \ldots, H_{(s_n,r_n)}\}$ is called the *set of core hypergraphs* of H or, simply, the *core set* of H, and is denoted by $C(H)$.

Definition 3.8. Suppose $H = (V, E)$ is a bipolar fuzzy hypergraph with

$$F(H) = \{(s_1, r_1), (s_2, r_2), \ldots, (s_n, r_n)\},$$

and $s_{n+1} = 0$, $r_{n+1} = 0$, then H is called *sectionally elementary* if for each edge $A = (\mu_A^P, \mu_A^N) \in E$, each $i = \{1, 2, \ldots, n\}$, and $(s_i, r_i) \in F(H)$, $A_{(s,t)} = A_{(s_i,r_i)}$ for all $(s,t) \in ((s_{i-1}, r_{i-1}), (s_i, r_i)]$.

Clearly H is sectionally elementary if and only if $A(x) = (\mu_A^P(x), \mu_A^N(x)) \in F(H)$ for each $A \in E$ and each $x \in X$.

Definition 3.9. A sequence of crisp hypergraphs $H_i = (V_i, E_i^*)$, $1 \leq i \leq n$, is said to be *ordered* if $H_1 \subset H_2 \subset \ldots \subset H_n$. The sequence $\{H_i \,|\, 1 \leq i \leq n\}$ is *simply ordered* if it is ordered and if whenever $E^* \in E_{i+1}^* - E_i^*$, then $E^* \nsubseteq V_i$.

Definition 3.10. A bipolar fuzzy hypergraph H is *ordered* if the H induced fundamental sequence of hypergraphs is ordered. The bipolar fuzzy hypergraph H is *simply ordered* if the H induced fundamental sequence of hypergraphs is simply ordered.

Example 3.11. Consider the bipolar fuzzy hypergraph $H = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3, E_4, E_5\}$ which is represented by the following incidence matrix:

Table 2: Incidence matrix of H

H	E_1	E_2	E_3	E_4	E_5
a	(0.7, −0.2)	(0.9, −0.2)	(0, 0)	(0, 0)	(0.4, −0.3)
b	(0.7, −0.2)	(0.9, −0.2)	(0.9, −0.2)	(0.7, −0.2)	(0, 0)
c	(0, 0)	(0, 0)	(0.9, −0.2)	(0.7, −0.2)	(0.4, −0.3)
d	(0, 0)	(0.4, −0.3)	(0, 0)	(0.4, −0.3)	(0.4, −0.3)

Clearly, $h(H) = (0.9, -0.1)$.
Now

$$E_{(0.9,-0.1)} = \{\{a, b\}, \{b, c\}\}$$
$$E_{(0.7,-0.2)} = \{\{a, b\}, \{b, c\}\}$$
$$E_{(0.4,-0.3)} = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

Thus for $0.4 < s \leq 0.9$ and $-0.1 > t \geq -0.3$, $E_{(s,t)} = \{\{a, b\}, \{b, c\}\}$, and for $0 < s \leq 0.4$ and $-1 < t \geq -0.3$,

$$E_{(s,t)} = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

We note that $E_{(0.9,-0.1)} \subseteq E_{(0.4,-0.3)}$. The fundamental sequence is

$$F(H) = \{(s_1, r_1) = (0.9, -0.1),\ (s_2, r_2) = (0.4, -0.3)\}$$

and the set of core hypergraph is

$$C(H) = \{H_1 = (V_1, E_1) = H_{(0.9,-0.1)},\ H_2 = (V_2, E_2) = H_{(0.4,-0.3)}\},$$

where

$$V_1 = \{a, b, c\}, \qquad E_1 = \{\{a, b\}, \{b, c\}\}$$
$$V_2 = \{a, b, c, d\}, \quad E_2 = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

H is support simple, but not simple. H is not sectionally elementary since $E_{1(s,t)} \neq E_{1(0.9,-0.1)}$ for $s = 0.7$, $t = -0.2$. Clearly, bipolar fuzzy hypergraph H is simply ordered.

Proposition 3.12. *Let $H = (V, E)$ be an elementary bipolar fuzzy hypergraph. Then H is support simple if and only if H is strongly support simple.*

Proof. Suppose that H is elementary, support simple and that $\mathrm{supp}(A) = \mathrm{supp}(B)$. We assume without loss of generality that $h(A) \leq h(B)$. Since H is elementary, it follows that $\mu_A^P \leq \mu_B^P$, $\mu_A^N \geq \mu_B^N$ and since H is support simple that $\mu_A^P = \mu_B^P$, $\mu_A^N = \mu_B^N$. Therefore, H is strongly support simple. The proof of converse part is obvious. ∎

The complexity of a bipolar fuzzy hypergraph depends in part on how many edges it has. The natural question arises: is there an upper bound on the number of edges of a bipolar fuzzy hypergraph of order n?

Proposition 3.13. *Let $H = (V, E)$ be a simple bipolar fuzzy hypergraph of order n. Then there is no upper bound on $|E|$.*

Proof. Let $V = \{x, y\}$, and define $E_N = \{A_i = (\mu_{A_i}^P, \mu_{A_i}^N) \mid i = 1, 2, \ldots, N\}$, where

$$\mu_{A_i}^P(x) = \frac{1}{i+1}, \quad \mu_{A_i}^N(x) = -1 + \frac{1}{i+1},$$

$$\mu_{A_i}^P(y) = \frac{1}{i+1}, \quad \mu_{A_i}^N(y) = -\frac{i}{i+1}.$$

Then $H_N = (V, E_N)$ is a simple bipolar fuzzy hypergraph with N edges. This ends the proof. ∎

Proposition 3.14. *Let $H = (V, E)$ be a support simple bipolar fuzzy hypergraph of order n. Then there is no upper bound on $|E|$.*

Proof. The class of support simple bipolar fuzzy hypergraphs contains the class of simple bipolar fuzzy hypergraphs, thus the result follows from Proposition 3.13. ∎

Proposition 3.15. *Let $H = (V, E)$ be an elementary simple bipolar fuzzy hypergraph of order n. Then there is no upper bound on $|E| \leq 2^n - 1$ if and only if $\{\mathrm{supp}(A) \mid A \in E\} = P(V) - \emptyset$.*

Proof. Since H is elementary and simple, each nontrivial $W \subseteq V$ can be the support of at most one $A = (\mu_A^P, \mu_A^N) \in E$. Therefore, $|E| \leq 2^n - 1$. To show there exists an elementary, simple H with $|E| = 2^n - 1$, let $E = \{A = (\mu_A^P, \mu_A^N) \mid W \subseteq V\}$ be the set of functions defined by

$$\mu_A^P(x) = \frac{1}{|W|}, \quad \text{if } x \in W, \qquad \mu_A^P(x) = 0, \quad \text{if } x \notin W,$$

$$\mu_A^N(x) = -1 + \frac{1}{|W|}, \quad \text{if } x \in W, \qquad \mu_A^N(x) = -1, \quad \text{if } x \notin W.$$

Then each one element has height $(1, -1)$, each two elements has height $(0.5, -0.5)$, and so on. Hence H is an elementary and simple, and $|E| = 2^n - 1$. ∎

We state the following proposition without proof.

Proposition 3.16. (a) *If $H = (V, E)$ is an elementary bipolar fuzzy hypergraph, then H is ordered.*

(b) *If H is an ordered bipolar fuzzy hypergraph with simple support hypergraph, then H is elementary.*

Definition 3.17. The *dual of a bipolar fuzzy hypergraph* $H = (V, E)$ is a bipolar fuzzy hypergraph $H^D = (E^D, V^D)$ whose vertex set is the edge set of H and with edges $V^D : E^D \to [0,1] \times [-1,0]$ by $V^D(A^D) = (\mu_A^D(x), \nu_A^D(x))$. H^D is a bipolar fuzzy hypergraph whose incidence matrix is the transpose of the incidence matrix of H, thus $H^{DD} = H$.

Example 3.18. Consider a bipolar fuzzy hypergraph $H = (V, E)$ such that

$$V = \{x_1, x_2, x_3, x_4\}, \;\; E = \{E_1, E_2, E_3, E_4\},$$

where

$$E_1 = \left\{ \frac{x_1}{(0.5, -0.3)}, \frac{x_2}{(0.4, -0.2)} \right\}, \quad E_2 = \left\{ \frac{x_2}{(0.4, -0.2)}, \frac{x_3}{(0.3, -0.6)} \right\},$$
$$E_3 = \left\{ \frac{x_3}{(0.3, -0.6)}, \frac{x_4}{(0.5, -0.1)} \right\}, \quad E_4 = \left\{ \frac{x_4}{(0.5, -0.1)}, \frac{x_1}{(0.5, -0.3)} \right\}.$$

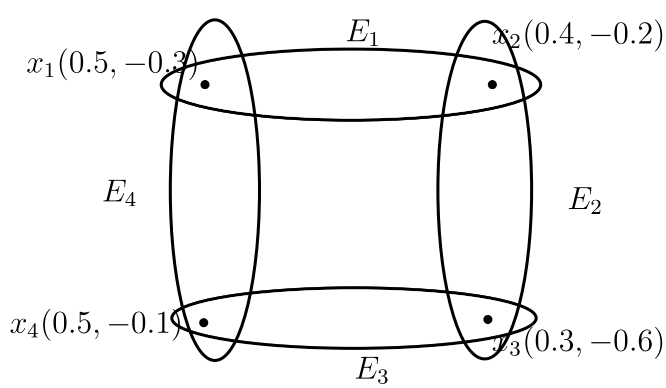

Figure 2: Bipolar fuzzy hypergraph

Table 3: The corresponding incidence matrix of H is given below:

M_H	E_1	E_2	E_3	E_4
x_1	(0.5, −0.3)	(0, 0)	(0, 0)	(0.5, −0.3)
x_2	(0.4, −0.2)	(0.4, −0.2)	(0, 0)	(0, 0)
x_3	(0, 0)	(0.3, −0.6)	(0.3, −0.6)	(0, 0)
x_4	(0, 0)	(0, 0)	(0.5, −0.1)	(0.5, −0.1)

Consider the dual bipolar fuzzy hypergraph $H^D = (E^D, V^D)$ of H such that

$$E^D = \{e_1, e_2, e_3, e_4\}, \;\; V^D = \{A, B, C, D\}$$

where

$$A = \left\{\frac{e_1}{(0.5, -0.3)}, \frac{e_4}{(0.5, -0.3)}\right\}, \quad B = \left\{\frac{e_1}{(0.4, -0.2)}, \frac{e_2}{(0.4, -0.2)}\right\},$$
$$C = \left\{\frac{e_2}{(0.3, -0.6)}, \frac{e_3}{(0.3, -0.6)}\right\}, \quad D = \left\{\frac{e_3}{(0.5, -0.1)}, \frac{e_4}{(0.5, -0.1)}\right\}.$$

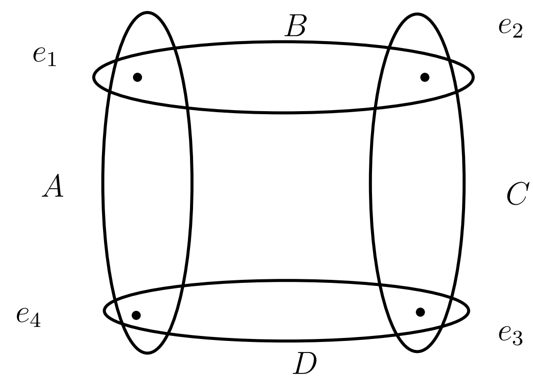

Figure 3: Dual bipolar fuzzy hypergraph

Table 4: The corresponding incidence matrix of H^D is given below:

M_{H^D}	A	B	C	D
e_1	(0.5, −0.3)	(0.4, −0.2)	(0, 0)	(0, 0)
e_2	(0, 0)	(0.4, −0.2)	(0.3, −0.6)	(0, 0)
e_3	(0, 0)	(0, 0)	(0.3, −0.6)	(0.5, −0.1)
e_4	(0.5, −0.3)	(0, 0)	(0, 0)	(0.5, −0.1)

We see that some edges contain only vertices having high positive membership degree and high negative membership degree. We define here the concept of strength of an edge.

Definition 3.19. The *strength* η of an edge E is the maximum positive membership $\mu^P(x)$ of vertices and maximum negative membership $\mu^N(x)$ of vertices in the edge E. That is, $\eta(E_j) = \{\max(\mu_j^P(x) \mid \mu_j^P(x) > 0)\ ,\ \max(\mu_j^N(x) \mid \mu_j^N(x) < 0)\}$.

Its interpretation is that the edge E_j groups elements having participation degree at least $\eta(E_j)$ in the hypergraph.

Example 3.20. Consider a bipolar fuzzy hypergraph $H = (V, E)$ such that $V = \{a, b, c, d\}$, $E = \{E_1, E_2, E_3, E_4\}$.

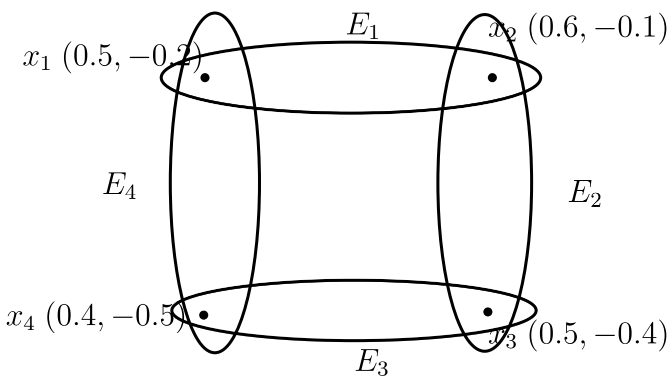

Figure 4: Bipolar fuzzy hypergraph

It is easy to see that E_1 is strong than E_3, and E_2 is strong than E_4. We call the edges with high strength the strong edges because the cohesion in them is strong.

Definition 3.21. The $(\alpha, \beta)-$*cut of a bipolar fuzzy hypergraph* H, denoted by $H_{(\alpha,\beta)}$, is defined as an ordered pair

$$H_{(\alpha,\beta)} = (V_{(\alpha,\beta)}, E_{(\alpha,\beta)}),$$

where

(i) $V_{(\alpha,\beta)} = \{x_1, x_2, \ldots, x_n\} = V$,

(ii) $E_{j(\alpha,\beta)} = \{x_i \mid \mu_j^P(x_i) \geq \alpha \text{ and } \mu_j^N(x_i) \leq \beta,\ j = 1, 2, 3, \ldots, m\}$,

(iii) $E_{m+1(\alpha,\beta)} = \{x_i \mid \mu_j^P(x_i) < \alpha \text{ and } \mu_j^N(x_i) > \beta,\ \forall\ j\}$.

The edge $E_{m+1(\alpha,\beta)}$ is added to the group of elements which are not contained in any edge $E_{j(\alpha,\beta)}$ of $H_{(\alpha,\beta)}$. The edges in the $(\alpha, \beta)-$cut hypergraph are now crisp sets.

Example 3.22. Consider the bipolar fuzzy hypergraph $H = (V, E)$, where $V = \{x, y, z\}$ and $E = \{E_1, E_2\}$, which is represented by the following incidence matrix:

Table 5: Incidence matrix of H

H	E_1	E_2
x	$(0.4, -0.2)$	$(0,\ 0)$
y	$(0.5, -0.3)$	$(0.6, -0.2)$
z	$(0,\ 0)$	$(0.2, -0.05)$

From this matrix we understand that, for example, $E_1 = (\mu_1^P, \mu_1^N) : V \to [0, 1] \times [-1, 0]$ satisfies:

$$\mu_1^P(x) = 0.4,\ \mu_1^N(x) = -0.2;\ \ \mu_1^P(y) = 0.5,\ \mu_1^N(y) = -0.3;\ \ \mu_1^P(z) = 0,\ \mu_1^N(z) = 0.$$

$(0.3, -0.1)$-cut of bipolar fuzzy hypergraph H is

$$E_{1(0.3,-0.1)} = \{x, y\},\ E_{2(0.3,-0.1)} = \{y\},\ E_{3(0.3,-0.1)} = \{z\}.$$

The incidence matrix of $H_{(0.3,-0.1)}$ is given below.

Table 6: Incidence matrix of $H_{(0.3,-0.1)}$

$H_{(0.3,-0.1)}$	$E_{1(0.3,-0.1)}$	$E_{2(0.3,-0.1)}$	$E_{3(0.3,-0.1)}$
x	1	0	0
y	1	1	0
z	0	0	1

Definition 3.23. A bipolar fuzzy hypergraph $H = (V, E)$ is called a $A = (\mu_A^P, \mu_A^N)$-tempered bipolar fuzzy hypergraph of $H = (V, E)$ if there is a crisp hypergraph $H^* = (V, E^*)$ and a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N) : V \to [0, 1] \times [-1, 0]$ such that $E = \{B_F = (\mu_{B_F}^P, \mu_{B_F}^N) \mid F \in E^*\}$, where

$$\mu_{B_F}^P(x) = \begin{cases} \min(\mu_A^P(y) \mid y \in F) & \text{if } x \in F, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_{B_F}^N(x) = \begin{cases} \max(\mu_A^N(y) \mid y \in F) & \text{if } x \in F, \\ -1 & \text{otherwise.} \end{cases}$$

Let $A \otimes H$ denote the A-tempered bipolar fuzzy hypergraph of H determined by the crisp hypergraph $H = (V, E^*)$ and the bipolar fuzzy set $A : V \to [0, 1] \times [-1, 0]$.

Example 3.24. Consider the bipolar fuzzy hypergraph $H = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3, E_4\}$ which is represented by the following incidence matrix:

Table 7: Incidence matrix of H

H	E_1	E_2	E_3	E_4
a	(0.2, −0.7)	(0, 0)	(0, 0)	(0.2, −0.7)
b	(0.2, −0.7)	(0.3, −0.4)	(0.0, −0.9)	(0, 0)
c	(0, 0)	(0, 0)	(0, −0.9)	(0.2, −0.7)
d	(0, 0)	(0.3, −0.4)	(0, 0)	(0, 0)

Then

$$\begin{aligned} E_{(0,-0.1)} &= \{\{b, c\}\}, \\ E_{(0.2,-0.7)} &= \{\{a, b\}, \{a, c\}, \{b, c\}\}. \\ E_{(0.3,-0.1)} &= \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}\}. \end{aligned}$$

Define $A = (\mu_A^P, \mu_A^N) : V \to [0, 1] \times [-1, 0]$ by

$$\mu_A^P(a) = 0.2, \quad \mu_A^P(b) = \mu_A^P(c) = 0.0, \quad \mu_A^P(d) = 0.3, \quad \mu_A^N(a) = -0.7,$$
$$\mu_A^N(b) = \mu_A^N(c) = -0.9, \quad \mu_A^N(d) = -0.4.$$

Note that

$$\mu^P_{B_{\{a,b\}}}(a) = \min(\mu^P_A(a), \quad \mu^P_A(b)) = 0.0, \qquad \mu^P_{B_{\{a,b\}}}(b) = \min(\mu^P_A(a),$$
$$\mu^P_A(b)) = 0.0, \qquad \mu^P_{B_{\{a,b\}}}(c) = 0.0, \quad \mu^P_{B_{\{a,b\}}}(d) = 0.0,$$
$$\mu^N_{B_{\{a,b\}}}(a) = \max(\mu^N_A(a), \quad \mu^N_A(b)) = -0.9, \quad \mu^N_{B_{\{a,b\}}}(b) = \max(\mu^N_A(a),$$
$$\mu^N_A(b)) = -0.9, \qquad \mu^N_{B_{\{a,b\}}}(c) = -1, \quad \mu^N_{B_{\{a,b\}}}(d) = -1.$$

Thus

$$E_1 = (\mu^P_{B_{\{a,b\}}}, \mu^P_{B_{\{a,b\}}}), \quad E_2 = (\mu^P_{B_{\{b,d\}}}, \mu^N_{B_{\{b,d\}}}),$$
$$E_3 = (\mu^P_{B_{\{b,c\}}}, \mu^N_{B_{\{b,c\}}}), \quad E_4 = (\mu^P_{B_{\{a,c\}}}, \mu^N_{B_{\{a,c\}}}).$$

Hence H is A-tempered hypergraph.

Theorem 3.25. *A bipolar fuzzy hypergraph H is a $A = (\mu^P_A, \mu^N_A)$-tempered bipolar fuzzy hypergraph of some crisp hypergraph H^* if and only if H is elementary, support simple and simply ordered.*

Proof. Suppose that $H = (V, E)$ is a A-tempered bipolar fuzzy hypergraph of some crisp hypergraph H^*. Clearly, H is elementary and support simple. We show that H is simply ordered. Let

$$C(H) = \{(H^*_1)^{r_1} = (V_1, E^*_1),\ (H^*_2)^{r_2} = (V_2, E^*_2), \cdots,\ (H^*_n)^{r_n} = (V_n, E^*_n)\}.$$

Since H is elementary, it follows from Proposition 3.16 that H is ordered. To show that H is simply ordered, suppose that there exists $F \in E^*_{i+1} \setminus E^*_i$. Then there exists $x^* \in F$ such that $\mu^P_A(x^*) = r_{i+1}$, $\mu^N_A(x^*) = \acute{r}_{i+1}$. Since $\mu^P_A(x^*) = r_{i+1} < r_i$ and $\mu^N_A(x^*) = \acute{r}_{i+1} < \acute{r}_i$, it follows that $x^* \notin V_i$ and $F \nsubseteq V_i$, hence H is simply ordered.

Conversely, suppose $H = (V, E)$ is elementary, support simple and simply ordered. Let

$$C(H) = \{(H^*_1)^{r_1} = (V_1, E^*_1),\ (H^*_2)^{r_2} = (V_2, E^*_2), \cdots,\ (H^*_n)^{r_n} = (V_n, E^*_n)\}$$

where $D(H) = \{r_1, r_2, \cdots, r_n\}$ with $0 < r_n < \cdots < r_1$. Since $(H^*)^{r_n} = H^*_n = (V_n, E^*_n)$ and define $A = (\mu^P_A, \mu^N_A) : V_n \to [0,1] \times [-1,0]$ by

$$\mu^P_A(x) = \begin{cases} r_1 & \text{if } x \in V_1, \\ r_i & \text{if } x \in V_i \setminus V_{i-1}, i = 1, 2, \cdots, n \end{cases}$$

$$\mu^N_A(x) = \begin{cases} s_1 & \text{if } x \in V_1, \\ s_i & \text{if } x \in V_i \setminus V_{i-1}, i = 1, 2, \cdots, n \end{cases}$$

We show that $E = \{B_F = (\mu^P_{B_F}, \mu^N_{B_F}) \mid F \in E^*\}$, where

$$\mu^P_{B_F}(x) = \begin{cases} \min(\mu^P_A(y) \mid y \in F) & \text{if } x \in F, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu^N_{B_F}(x) = \begin{cases} \max(\mu^N_A(y) \mid y \in F) & \text{if } x \in F, \\ -1 & \text{otherwise.} \end{cases}$$

Let $F \in E_n^*$. Since H is elementary and support simple, there is a unique bipolar fuzzy edge $C_F = (\mu^P_{C_F}, \mu^N_{C_F})$ in E having support E^*. Indeed, distinct edges in E must have distinct supports that lie in E_n^*. Thus, to show that $E = \{B_F = (\mu^P_{B_F}, \mu^N_{B_F}) \mid F \in E_n^*\}$, it suffices to show that for each $F \in E_n^*$, $\mu^P_{C_F} = \mu^P_{B_F}$ and $\mu^N_{C_F} = \mu^N_{B_F}$. As all edges are elementary and different edges have different supports, it follows from the definition of fundamental sequence that $h(C_F)$ is equal to some number r_i of $D(H)$. Consequently, $E^* \subseteq V_i$. Moreover, if $i > 1$, then $F \in E^* \setminus E^*_{i-1}$. Since $F \subseteq V_i$, it follows from the definition of $A = (\mu^P_A, \mu^N_A)$ that for each $x \in F$, $\mu^P_A(x) \geq r_i$ and $\mu^N_A(x) \leq s_i$. We claim that $\mu^P_A(x) = r_i$ and $\mu^N_A(x) = s_i$, for some $x \in F$. If not, then by definition of $A = (\mu^P_A, \mu^N_A)$, $\mu^P_A(x) \geq r_i$ and $\mu^N_A(x) \leq s_i$ for all $x \in F$ which implies that $F \subseteq V_{i-1}$ and so $F \in E^* \setminus E^*_{i-1}$ and since H is simply ordered $F \subsetneq V_{i-1}$, a contradiction. Thus it follows from the definition of B_F that $B_F = C_F$. This completes the proof. ■

As a consequence of the above theorem we obtain.

Proposition 3.26. *Suppose that H is a simply ordered bipolar fuzzy hypergraph and $F(H) = \{r_1, r_2, \cdots, r_n\}$. If H^{r_n} is a simple hypergraph, then there is a partial bipolar fuzzy hypergraph $\acute{H}$ of H such that the following assertions hold:*

(1) *$\acute{H}$ is a $A = (\mu^P_A, \mu^N_A)$-tempered bipolar fuzzy hypergraph of H_n.*

(2) *$E \sqsubseteq \acute{E}$.*

(3) *$F(\acute{H}) = F(H)$ and $C(\acute{H}) = C(H)$.*

4. Application examples of bipolar fuzzy hypergraphs

Definition 4.1. Let X be a reference set. Then, a family of nontrivial bipolar fuzzy sets $\{A_1, A_2, A_3, \ldots, A_m\}$ where $A_i = (\mu_i^P, \mu_i^N)$ is a *bipolar fuzzy partition* if

(1) $\bigcup_i \mathrm{supp}(A_i) = X, \quad i = 1, 2, ..., m,$

(2) $\sum_{i=1}^{m} \mu_i^P(x) = 1$ for all $x \in X$,

(3) $\sum_{i=1}^{m} \mu_i^N(x) = -1$ for all $x \in X$.

Note that this definition generalizes fuzzy partitions because the definition is equivalent to a fuzzy partition when for all x, $\nu_i(x)$ =0. We call a family $\{A_1, A_2, A_3, \ldots, A_m\}$ a *bipolar fuzzy covering* of X if it satisfies above conditions $(1) - (3)$.

A bipolar fuzzy partition can be represented by a bipolar fuzzy matrix $[a_{ij}]$ where a_{ij}, is the positive membership degree and negative membership degree of element x_i in class j. We see that the matrix is the same as the incidence matrix in bipolar fuzzy hypergraph. Then we can represent a bipolar fuzzy partition by a bipolar fuzzy hypergraph $H = (V, E)$ such that

(1) V: a set of elements x_i, $i = 1, ..., n$,

(2) $E = \{E_1, E_2, ..., E_m\}$= a set of nontrivial bipolar fuzzy classes,

(3) $V = \bigcup_j \text{supp}(E_j)$, $\quad j = 1, 2, ..., m$,

(4) $\displaystyle\sum_{i=1}^{m} \mu_i^P(x) = 1$ for all $x \in X$,

(5) $\displaystyle\sum_{i=1}^{m} \mu_i^N(x) = -1$ for all $x \in X$.

Note that conditions (4)–(5) are added to the bipolar fuzzy hypergraph for bipolar fuzzy partition. If these conditions are added, the bipolar fuzzy hypergraph can represent a bipolar fuzzy covering. Naturally, we can apply the (α, β)-cut to the bipolar fuzzy partition.

Example 4.2. [Radio coverage network] In telecommunications, the coverage of a radio station is the geographic area where the station can communicate.

Let V be a finite set of radio receivers (vertices); perhaps a set of representative locations at the centroid of a geographic region. For each of m radio transmitters we define the bipolar fuzzy set "listening area of station j" where $A_j(x) = (\mu^P_{Aj}(x), \mu^N_{Aj}(x))$ represents the "quality of reception of station j at location x." Positive membership value near to 1, could signify "very clear reception on a very sensitive radio" while negative membership value near to -1, could signify "very poor reception on a very poor radio". Since graphy affects signal strength, each "listening area" is an bipolar fuzzy set. Also, for a fixed radio the reception will vary between different stations. Thus this model uses the full definition of an bipolar fuzzy hypergraph. The model could be used to determine station programming or marketing strategies or to establish an emergency broadcast network. Further variables could relate signal strength to changes in time of day, weather and other conditions.

Example 4.3. [Clustering problem] We consider here the clustering problem, which is a typical example of a bipolar fuzzy partition on the digital image processing.

There are five objects and they are classified into two classes: tank and house. To cluster the elements x_1, x_2, x_3, x_4, x_5 into A_t (tank) and B_h (house), a bipolar fuzzy partition matrix is given as the form of incidence matrix of bipolar fuzzy hypergraph.

We can apply the (α, β)-cut to the hypergraph and obtain a hypergraph $H_{(\alpha,\beta)}$ which is not bipolar fuzzy hypergraph. We denote the edge (class) in (α, β)-cut hypergraph $H_{(\alpha,\beta)}$ as $E_{j(\alpha,\beta)}$. This hypergraph H, represents generally the covering

Table 8: Bipolar fuzzy partition matrix

H	A_t	B_h
x_1	$(0.96, -0.04)$	$(0.04, -0.96)$
x_2	$(0.95, -0.5)$	$(0.5, -0.95)$
x_3	$(0.61, -0.39)$	$(0.39, -0.61)$
x_4	$(0.05, -0.95)$	$(0.95, -0.05)$
x_5	$(0.03, -0.97)$	$(0.97, -0.03)$

because the conditions: (4) $\sum_{i=1}^{m} \mu_i^P(x) = 1$ for all $x \in X$, (5)$\sum_{i=1}^{m} \mu_i^N(x) = -1$ for all $x \in X$, is not always guaranteed. The hypergraph $H_{(0.61,-0.03)}$ is shown in Table 8.

Table 9: Hypergraph $H_{(0.61,-0.03)}$

$H_{(0.61,-0.03)}$	$A_{t(0.61,-0.03)}$	$B_{h(0.61,-0.03)}$
x_1	1	0
x_2	1	0
x_3	1	0
x_4	0	1
x_5	0	1

We obtain dual bipolar fuzzy hypergraph $H^D_{(0.61,-0.03)}$ of $H_{(0.61,-0.03)}$ which is given in Table 9.

Table 10: Dual bipolar fuzzy hypergraph

$H^D(0.61, -0.03)$	X_1	X_2	X_3	X_4	X_5
A_t	1	1	1	0	0
B_h	0	0	0	1	1

We consider the strength of edge (class) $E_{j(\alpha,\beta)}$, or in the (α, β)-cut hypergraph $H_{(\alpha,\beta)}$. It is necessary to apply Definition 3.19 to obtain the strength of edge $E_{j(\alpha,\beta)}$ in $H_{(\alpha,\beta)}$.
The possible interpretations of $\eta(E_{j(\alpha,\beta)})$ are:

- the edge (class) in the hypergraph (partition) $H_{(\alpha,\beta)}$, groups elements having at least η positive and negative memberships,
- the strength (cohesion) of edge (class) $E_{j(\alpha,\beta)}$ in $H_{(\alpha,\beta)}$ is η.

Thus we can use the strength as a measure of the cohesion or strength of a class in a partition. For example, the strengths of classes $A_t(0.61, -0.03)$ and $B_h(0.61, -0.03)$ at s=0.61, t=−0.03 are $\eta(A_t(0.61, -0.03))$=$(0.96, -0.04)$, $\eta(B_h(0.61, -0.03))$=$(0.97, -0.03)$. Thus we say that the class $\eta(B_h(0.61, -0.03))$ is stronger than $\eta(A_t(0.61, -0.03))$ because $\eta(B_h(0.61, -0.03)) > \eta(A_t(0.61, -0.03))$.

From the above discussion on the hypergraph $H_{(0.61,-0.03)}$ and $H^D_{(0.61,-0.03)}$, we can state that:

- The bipolar fuzzy hypergraph can represent the fuzzy partition visually. The (α,β)-cut hypergraph also represents the (α,β)-cut partition.
- The dual hypergraph $H^D_{(0.61,-0.03)}$ can represent elements X_i, which can be grouped into a class $E_{j(\alpha,\beta)}$. For example, the edges X_1, X_2, X_3 of the dual hypergraph in Table 9 represent that the elements x_1, x_2, x_3 that can be grouped into A_t at level (0.61, -0.03).
- At (α,β)=(0.61, -0.03) level, the strength of class $B_h(0.61,-0.03)$ is the highest (0.95, -0.05), so it is the strongest class. It means that this class can be grouped independently from the other parts. Thus we can eliminate the class B_h from the others and continue clustering. Therefore, the discrimination of strong classes from the others can allow us to decompose a clustering problem into smaller ones. This strategy allows us to work with the reduced data in a clustering problem.

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Accepted: 30.01.2013