ANTIPODAL BIPOLAR FUZZY GRAPHS

Muhammad Akram

Punjab University College of Information Technology
University of the Punjab
Old Campus, Lahore-54000
Pakistan
e-mail: makrammath@yahoo.com
m.akram@pucit.edu.pk

Sheng-Gang Li

College of Mathematics and Information Science
Shaanxi Normal University
710062, Xi'an
China
e-mail: shenggangli@yahoo.com.cn

K.P. Shum

Institute of Mathematics
Yunnan University
Kunming, 650091
China
e-mail: kpshum@ynu.edu.cn

Abstract. The concept of an antipodal bipolar fuzzy graph of a given bipolar fuzzy graph is introduced. Characterizations of antipodal bipolar fuzzy graphs are presented when the bipolar fuzzy graph is complete or strong. Some isomorphic properties of antipodal bipolar fuzzy graph are discussed. The notion of self median bipolar fuzzy graphs of a given bipolar fuzzy graph is also introduced.

Keywords and phrases: antipodal bipolar fuzzy graphs, median bipolar fuzzy graphs, self median bipolar fuzzy graphs.
2000 Mathematics Subject Classification: 05C99.

1. Introduction

Concepts of graph theory have applications in many areas of computer science (such as data mining, image segmentation, clustering, image capturing, networking etc.). For examples, a data structure can be designed in the form of trees, modeling of network topologies can be done using graph concepts. The most important concept of graph coloring is utilized in resource allocation and scheduling. The concepts of paths, walks and circuits in graph theory are used in traveling salesman problem, database design concepts, and resource networking. This leads to the development of new algorithms and new theorems that can be used in tremendous applications.

A notion having certain influence on graph theory is fuzzy set, which is introduced by Zadeh [17] in 1965; actually, the theory of fuzzy sets has already become a vigorous research area which intersects with many research areas, such as medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, artificial intelligence, signal processing, multi-agent systems, pattern recognition, robotics, computer networks, expert systems, decision making and automata theory, etc. A fuzzy set on a given set X is just a mapping $A : X \longrightarrow [0,1]$. Based on the same idea, Zhang [20] defined the notion of bipolar fuzzy set on a given set X in 1994, which is just a mapping $A : X \longrightarrow [-1,1]$, where the membership degree 0 of an element x means that the element x is irrelevant to the corresponding property, the membership degree in $(0,1]$ of an element x indicates that the element somewhat satisfies the property, and the membership degree in $[-1,0)$ of an element x indicates that the element somewhat satisfies the implicit counter-property.

In 1975, Rosenfeld [13] discussed the concept of fuzzy graph whose basic idea was introduced by Kauffmann [11] in 1973. By considering fuzzy relations between fuzzy sets and developing structure of fuzzy graphs, Rosenfeld obtained analogs of several graph theoretical concepts. Bhattacharya [9] gave some remarks on fuzzy graphs. The complement of a fuzzy graph was defined by Mordeson [12] and further studied by Sunitha and Vijayakumar [15]. Ahmed and Gani discussed the concepts of perfect fuzzy graph and self median fuzzy graph in [7]. Based on the notion of intuitionistic fuzzy set [5], Atanassov [5] introduced the concepts of intuitionistic fuzzy relation and intuitionistic fuzzy graphs. Recently, the bipolar fuzzy graphs have been defined and discussed in [1-3] based on the notion of bipolar fuzzy set. The present paper continues to study bipolar fuzzy graphs. We introduce the concepts of antipodal bipolar fuzzy graph and self median bipolar fuzzy graph of a given bipolar fuzzy graph, and prove several characterizations theorems of antipodal bipolar fuzzy graphs whose bipolar fuzzy graph are complete or strong. We also discuss isomorphic properties of antipodal bipolar fuzzy graph.

2. Preliminaries

In this section, we review some elementary concepts whose understanding is necessary fully benefit from this paper.

By a graph $G^* = (V, E)$, we mean a non-trivial, finite, connected and undirected graph without loops or multiple edges. Formally, given a graph $G^* = (V, E)$, two vertices x, $y \in V$ are said to be *neighbors, or adjacent nodes*, if $\{x, y\} \in E$. The *antipodal graph* of a graph G, denoted by $A(G^*)$, has the same vertex set as G^* with an edge joining vertices u and v if $d(u, v)$ is equal to the diameter of G^*. For a graph G^* of order p, the antipodal graph $A(G^*) = G^*$ if and only if $G^* = K_p$. If G^* is a non-complete graph of order p, then $A(G^*) \subset \overline{G^*}$. For a graph G^*, the antipodal graph $A(G^*) = \overline{G^*}$ if and only if (a) G^* is of diameter 2 or (b) G^* is disconnected and the components of G^* are complete graphs. A graph G^* is an antipodal graph if aut only if it is the antipodal graph of its complement. A graph G^* is an antipodal graph if and only if (i) diam$(\overline{G^*}) = 2$ or (ii) $\overline{G^*}$ is disconnected and the components of $\overline{G^*}$ are complete graphs. The self median fuzzy graphs were introduced by Ahmed and Gani in [7] The median of a graph G^* is the set of all vertices v of G^* for which the value $d^*_G(v)$ is minimized. A graph is distance-balanced (also called self-median) if its median is the whole vertex set. Thus, a graph G^* is self-median if and only if the value $d^*_G(v)$ is constant over all vertices v of G^*. The status, or distance sum, of a given vertex v in a graph is defined by $s(v) = \sum\limits_{u \neq v} d(u, v)$, where $d(u, v)$ is the distance from a vertex u to v. In other words, a self median graph G^* is one in which all the nodes have the same status $s(v)$. The graphs C_n, $K_{n,n}$ and K_n are self median. The status of a vertex v_i is denoted by $S(v_i)$ and is defined as $S(v_i) = \sum\limits_{\forall v_j \in V} \delta(v_i, v_j)$. The total status of a fuzzy graph G^* is denoted by $t[S(G^*)]$ and is defined as $t[S(G^*)] = \sum\limits_{\forall v_i \in V} S(v_i)$. The median of a fuzzy graph G^*, denoted, is the set of nodes with minimum status. A fuzzy graph G^* is said to be self-median if all the vertices have the same status. Every self-median fuzzy graph is a self centered fuzzy graph. Every cube Q_n is self-median fuzzy graph.

Definition 2.1 [17, 18] A *fuzzy subset* μ on a set X is a map $\mu : X \to [0, 1]$. A *fuzzy binary relation* on X is a fuzzy subset μ on $X \times X$. By a fuzzy relation we mean a fuzzy binary relation given by $\mu : X \times X \to [0, 1]$.

Definition 2.2 [20] Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \{(x,\, \mu^P_B(x),\, \mu^N_B(x)) \,|\, x \in X\}$$

where $\mu^P_B : X \to [0,\, 1]$ and $\mu^N_B : X \to [-1,\, 0]$ are mappings.

We use the positive membership degree $\mu^P_B(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B, and the negative membership degree $\mu^N_B(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B. If $\mu^P_B(x) \neq 0$ and $\mu^N_B(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B. If $\mu^P_B(x) = 0$ and $\mu^N_B(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B. It is possible for an element x to be such that $\mu^P_B(x) \neq 0$ and $\mu^N_B(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

For the sake of simplicity, we shall use the symbol $B = (\mu_B^P,\, \mu_B^N)$ for the bipolar fuzzy set

$$B = \{(x,\, \mu_B^P(x),\, \mu_B^N(x))\,|\, x \in X\}.$$

A nice application of bipolar fuzzy concept is a political acceptation (map to $[0, 1]$) and non-acceptation (map to $[-1, 0]$).

Definition 2.3 [20] Let X be a nonempty set. Then, we call a mapping $A = (\mu_A^P, \mu_A^N) : X \times X \to [0, 1] \times [-1, 0]$ a *bipolar fuzzy relation* on X such that $\mu_A^P(x, y) \in [0, 1]$ and $\mu_A^N(x, y) \in [-1, 0]$.

Definition 2.4 [1] Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be bipolar fuzzy sets on a set X. If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy relation on a set X, then $A = (\mu_A^P, \mu_A^N)$ is called a *bipolar fuzzy relation* on $B = (\mu_B^P, \mu_B^N)$ if $\mu_A^P(x, y) \leq \min(\mu_B^P(x), \mu_B^P(y))$ and $\mu_A^N(x, y) \geq \max(\mu_B^N(x), \mu_B^N(y))$ for all x, $y \in X$. A bipolar fuzzy relation A on X is called *symmetric* if $\mu_A^P(x, y) = \mu_A^P(y, x)$ and $\mu_A^N(x, y) = \mu_A^N(y, x)$ for all x, $y \in X$.

Definition 2.5 [3] Let G be a connected bipolar fuzzy graph. The μ^P-distance, $\delta_{\mu^P}(v_i, v_j)$, is the smallest μ^P-length of any $v_i - v_j$ path P in G, where $v_i, v_j \in V$. That is, $\delta_{\mu^P}(v_i, v_j) = \min(l_{\mu^P}(P))$. The μ^N-distance, $\delta_{\mu^N}(v_i, v_j)$, is the largest μ^N-length of any $v_i - v_j$ path P in G, where $v_i, v_j \in V$. That is, $\delta_{\mu^N}(v_i, v_j) = \max(l_{\mu^N}(P))$. The distance, $\delta(v_i, v_j)$, is defined as $\delta(v_i, v_j) = (\delta_{\mu^P}(v_i, v_j), \delta_{\mu^N}(v_i, v_j))$. For each $v_i \in V$, the μ^P-eccentricity of v_i, denoted by $e_{\mu^P}(v_i)$ and is defined as $e_{\mu^P}(v_i) = \max\{\delta_{\mu^P}(v_i, v_j) : v_i \in V, v_i \neq v_j\}$. For each $v_i \in V$, the μ^N-eccentricity of v_i, denoted by $e_{\mu^N}(v_i)$ and is defined as $e_{\mu^N}(v_i) = \min\{\delta_{\mu^N}(v_i, v_j) : v_i \in V, v_i \neq v_j\}$. For each $v_i \in V$, the eccentricity of v_i, denoted by $e(v_i)$ and is defined as $e(v_i) = (e_{\mu^P}(v_i), e_{\mu^N}(v_i))$. The μ^P-radius of G is denoted by $r_{\mu^P}(G)$ and is defined as $r_{\mu^P}(G) = \min\{e_{\mu^P}(v_i) : v_i \in V\}$. The μ^N-radius of G is denoted by $r_{\mu^N}(G)$ and is defined as $r_{\mu^N}(G) = \max\{e_{\mu^N}(v_i) : v_i \in V\}$. The radius of G is denoted by $r(G)$ and is defined as $r(G) = (r_{\mu^P}(G), r_{\mu^N}(G))$. The μ^P-diameter of G is denoted by $d_{\mu^P}(G)$ and is defined as $d_{\mu^P}(G) = \max\{e_{\mu^P}(v_i) : v_i \in V\}$. The μ^N-diameter of G is denoted by $d_{\mu^N}(G)$ and is defined as $d_{\mu^N}(G) = \min\{e_{\mu^N}(v_i) : v_i \in V\}$. The diameter of G is denoted by $d(G)$ and is defined as $d(G) = (d_{\mu^P}(G), d_{\mu^N}(G))$. A connected bipolar fuzzy graph G is a self centered graph, if every vertex of G is a central vertex, that is $r_{\mu^P}(G) = e_{\mu^P}(v_i)$ and $r_{\mu^N}(G) = e_{\mu^N}(v_i)$, $\forall v_i \in V$.

3. Antipodal bipolar fuzzy graphs

Definition 3.1 Let $G = (A, B)$ be a bipolar fuzzy graph. An *antipodal bipolar fuzzy graph* $A(G) = (E, F)$ is a bipolar fuzzy graph $G = (A, B)$ in which:

(a) An bipolar fuzzy vertex set of G is taken as bipolar fuzzy vertex set of $A(G)$, that is, $\mu_E^P(x) = \mu_A^P(x)$ and $\mu_E^N(x) = \mu_A^N(x)$ for all $x \in V$,

(b) If $\delta(x,y) = d(G)$, then

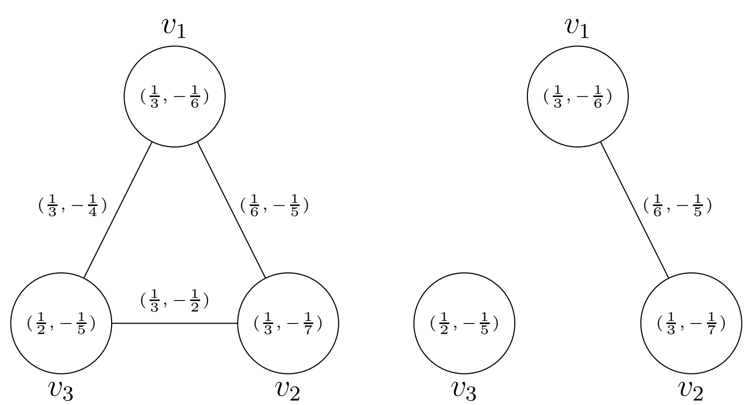
$$\mu_F^P(xy) = \begin{cases} \mu_B^P(xy) & \text{if } x \text{ and } y \text{ are neighbors in } G, \\ \min(\mu_A^P(x), \mu_A^P(y)) & \text{if } x \text{ and } y \text{ are not neighbors in } G, \end{cases}$$

$$\mu_F^N(xy) = \begin{cases} \mu_B^N(xy) & \text{if } x \text{ and } y \text{ are neighbors in } G, \\ \max(\mu_A^N(x), \mu_A^N(y)) & \text{if } x \text{ and } y \text{ are not neighbors in } G. \end{cases}$$

That is, two vertices in $A(G)$ are made as neighborhood if the $\mu^P \mu^N-$distance between them is diameter of G.

Example 3.2 Consider a bipolar fuzzy graph G such that

$$A = \{v_1, v_2, v_3\}, \quad B = \{v_1v_2, v_1v_3, v_2v_3\}.$$



Bipolar Fuzzy Graph Antipodal Bipolar Fuzzy Graph

Figure 1

By routine calculations, we have,

$$\begin{array}{lll} \delta_{\mu^P}(v_1, v_2) = 6, & \delta_{\mu^P}(v_1, v_3) = 3, & \delta_{\mu^P}(v_2, v_3) = 3, \\ \delta_{\mu^N}(v_1, v_2) = -6, & \delta_{\mu^N}(v_1, v_3) = -7, & \delta_{\mu^N}(v_2, v_3) = -9, \\ e_{\mu^P}(v_1) = 6, & e_{\mu^P}(v_2) = 6, & e_{\mu^P}(v_3) = 3, \\ e_{\mu^N}(v_1) = -6, & e_{\mu^N}(v_2) = -6, & e_{\mu^N}(v_3) = -7, \\ d(G) = (6, -6), & \delta(v_1, v_2) = (6, -6) = d(G). & \end{array}$$

Hence $A(G) = (E, F)$, such that $E = \{v_1, v_2, v_3\}$ and $F = \{v_1v_2\}$.

Theorem 3.3 *Let $G = (A, B)$ be a complete bipolar fuzzy graph where (μ_A^P, μ_A^N) is constant function then G is isomorphic to $A(G)$.*

Proof. Given that $G = (A, B)$ be a complete bipolar fuzzy graph with $(\mu_1^P, \mu_1^N) = (k_1, k_2)$, where k_1 and k_2 are constants, which implies that $\delta(v_i, v_j) = (l_1, l_2)$, $\forall v_i, v_j \in V$. Therefore, eccentricity $e(v_i) = (l_1, l_2)$, $\forall v_i \in V$, which implies that $d(G) = (l_1, l_2)$. Hence $\delta(v_i, v_j) = (l_1, l_2) = d(G)$, $\forall v_i, v_j \in V$. Hence every pair of vertices are made as neighbors in $A(G)$ such that

(a) An bipolar fuzzy vertex set of G is taken as bipolar fuzzy vertex set of $A(G)$, that is, $\mu_E^P(v_i) = \mu_A^P(v_i)$ and $\mu_E^N(v_i) = \mu_A^N(v_i)$ for all $v_i \in V$,

(b) $\mu_F^P(v_iv_j) = \mu_B^P(v_iv_j)$, since v_i and v_j are neighbors in G
$\mu_F^N(v_iv_j) = \mu_B^N(v_iv_j)$, since v_i and v_j are neighbors in G.

It has same number of vertices, edges and it preserves degrees of the vertices. Hence $G \cong A(G)$. ∎

Theorem 3.4 *Let $G : (A, B)$ is a connected bipolar fuzzy graph. Every antipodal bipolar fuzzy graph is spanning subgraph of G.*

Proof. By the definition of an antipodal bipolar fuzzy graph, $A(G)$ contains all the vertices of G. That is,

(a) $\mu_E^P(x) = \mu_A^P(x)$ and $\mu_E^N(x) = \mu_A^N(x)$ for all $x \in V$, and

(b) If $\delta(x, y) = d(G)$, then

$$\mu_F^P(xy) = \begin{cases} \mu_B^P(xy) & \text{if } x \text{ and } y \text{ are neighbors in } G, \\ \min(\mu_A^P(x), \mu_A^P(y)) & \text{if } x \text{ and } y \text{ are not neighbors in } G, \end{cases}$$

$$\mu_F^N(xy) = \begin{cases} \mu_B^N(xy) & \text{if } x \text{ and } y \text{ are neighbors in } G, \\ \max(\mu_A^N(x), \mu_A^N(y)) & \text{if } x \text{ and } y \text{ are not neighbors in } G. \end{cases}$$

Hence $A(G)$ is spanning subgraph of G. ∎

Theorem 3.5 *Let G be a bipolar fuzzy graph, where crisp graph G is an even or odd cycle. If alternate edges have same membership values and non-membership values, then G is self centered bipolar fuzzy graph.*

Theorem 3.6 *Let G be a bipolar fuzzy graph, where crisp graph G^* is an even or odd cycle. If alternate edges have same positive and negative values, then $A_{\mu^P}(G)$ and $A_{\mu^N}(G)$ is the edge induced bipolar fuzzy subgraph of $\bar{G}$, whose end vertices of $A_{\mu^P}(G)$ and $A_{\mu^N}(G)$ are with maximum μ^P- eccentricity and minimum μ^N-eccentricity in G.*

Proof. If alternate edges have same positive and negative values, then μ^P-distance between non-adjacent vertices is greater than the adjacent vertices and μ^N-distance between non-adjacent vertices is lesser than the adjacent vertices. Let $\mu_B^P(v_i, v_j)$ be the least among all other edges, then

$$\delta_{\mu^P}(v_i, v_j) = \frac{1}{\mu_B^P(v_i, v_j)}.$$

Claim (i): Neighbors in G are not neighbors in $A(G)$.

Consider an arbitrary path connecting v_k, v_t such that

(1) $$(v_k, v_t) \notin \mu_{B^*}^P$$

If P is a path of length 2 between v_k, v_t, then

$$l_{\mu^P}(P) \geq \frac{1}{\mu_B^P(v_i, v_j)} \tag{2}$$

Hence

$$\delta(v_k, v_t) \geq \frac{1}{\mu_B^P(v_i, v_j)},$$

since by equation (1) and (2), which implies that $\delta(v_i, v_j) < \delta(v_k, v_t) \leq d(G)$, where $\delta(v_k, v_t) \notin \mu_{B^*}^P$ and $(v_i, v_j) \in \mu_{B^*}^P$. That is, $\delta(v_i, v_j) < d(G)$, if $(v_i, v_j) \in \mu_{B^*}^P$. Therefore, if $(v_i, v_j) \in \mu_{B^*}^P$, then v_i and v_j are not neighbors in $A(G)$.

Claim (ii): Edges in $A(G)$ are edges in $\bar{G}$.
If $(v_m, v_n) \in \mu_{F^*}^P$, then by Claim (i), $(v_i, v_j) \notin \mu_{B^*}^P$. So,

$$\mu_F^P(v_m, v_n) = \min(\mu_A^P(v_m), \mu_A^P(v_n)),$$

since by the definition of $A(G)$, which implies that edges in $A(G)$ are edges in $\bar{G}$. Hence $A(G)$ is a bipolar fuzzy subgraph of $\bar{G}$, induced by the edges of $\bar{G}$, whose end vertices are with maximum μ^P eccentricity in G.

Let $(v_i, v_j) \in E$, then $\delta_{\mu^N}(v_i, v_j) = k$.

Claim (i): Neighbors in G are not neighbors in $A(G)$.
Consider an arbitrary path connecting v_k, v_t such that

$$(v_k, v_t) \notin \mu_{B^*}^N \tag{3}$$

If Q is a path of length 2 between v_k, v_t, then

$$\delta_{\mu^P}(P) \leq k \tag{4}$$

Hence

$$\delta_{\mu^N}(v_k, v_t) \leq \delta_{\mu^N}(v_i, v_j),$$

since by equation (3) and (4), which implies that $\delta_{\mu^N}(v_k, v_t) \leq d(G)$, where $(v_k, v_t) \notin \mu_{B^*}^N$ and $(v_i, v_j) \in \mu_{B^*}^N$. That is, $\delta_{\mu^N}(v_i, v_j) \geq d(G)$, if $(v_i, v_j) \in \mu_{B^*}^N$. Therefore, if $(v_i, v_j) \in \mu_{B^*}^N$, then v_i and v_j are not neighbors in $A(G)$.

Claim (ii): Edges in $A(G)$ are edges in $\bar{G}$.
If $(v_m, v_n) \in \mu_{F^*}^N$, then by Claim (i), $(v_i, v_j) \notin \mu_{B^*}^P$. So,

$$\mu_F^N(v_m, v_n) = \max(\mu_A^N(v_m), \mu_A^N(v_n)),$$

since by the definition of $A(G)$, which implies that edges in $A_{\mu^N}(G)$ are edges in $\bar{G}$. Hence $A_{\mu^N}(G)$ is a bipolar fuzzy subgraph of $\bar{G}$, induced by the edges of $\bar{G}$, whose end vertices are with minimum μ^N eccentricity in G. ∎

Theorem 3.7 *Let G be a bipolar fuzzy graph, where crisp graph G^* is an even or odd cycle. If alternate edges have same positive and negative values, then $A(G)$ is a bipartite bipolar fuzzy graph.*

Theorem 3.8 *Let $G : (A, B)$ be a connected strong bipolar fuzzy graph, where crisp graph G^* is an even or odd cycle, such that $(\mu_A^P, \mu_A^N)(v_i) = (k_1, k_2)$, $\forall v_i \in \mu_A^P$ and $\forall v_i \in \mu_A^N$. Then $A(G)$ is the spanning bipolar fuzzy graph subgraph of $\bar{G}$, induced by the edges of $\bar{G}$, whose end vertices are maximum μ^P- eccentricity and minimum μ^N- eccentricity in G.*

Proof. Let $(v_i, v_j) \in \mu_{B^*}^P$, $\delta_{\mu^P}(v_i, v_j) = \frac{1}{k_1}$. But for any $(v_i, v_j) \notin \mu_{B^*}^P$, $\delta(v_k, v_m) \geq \frac{2}{k_1}$. That is $\delta_{\mu^P}(v_i, v_j) = \frac{1}{k_1} < \frac{2}{k_1} \leq \delta_{\mu^P}(v_k, v_m)$, where $(v_i, v_j) \notin \mu_{B^*}^P$, which implies that v_i, v_j are vertices in $A(G)$, but are not neighbors in $A(G)$. Now, let $(v_i, v_j) \in \mu_{B^*}^N$, $\delta_{\mu^N}(v_i, v_j) > \frac{1}{k_1}$. But for any $(v_i, v_j) \notin \mu_{B^*}^N$, $\delta_{\mu^N}(v_i, v_j) \geq \delta_{\mu^N}(v_k, v_m)$, where $(v_i, v_j) \notin \mu_{B^*}^N$, which implies that v_i, v_j are vertices in $A(G)$, but are not neighbors in $A(G)$. The remaining proof is similar to claim (ii) of above Theorem and hence we omit it. ∎

Theorem 3.9 *If G_1 and G_2 are isomorphic to each other, then $A(G_1)$ and $A(G_2)$ are also isomorphic.*

Proof. As G_1 and G_2 are isomorphic, the isomorphism h, between them preserves the edge weights, so the $\mu^P\mu^N$-length and $\mu^P\mu^N$-distance will also be preserved. Hence, if the vertex v_i has the maximum μ^P-eccentricity and minimum μ^N-eccentricity, in G_1, then $h(v_i)$ has the maximum μ^P-eccentricity and minimum μ^N-eccentricity, in G_2. So, G_1 and G_2 will have the same diameter. If the $\mu^P\mu^N$-distance between v_i and v_j is (k_1, k_2) in G_1, then $h(v_i)$ and $h(v_j)$ will also have their $\mu^P\mu^N$-distance as (k_1, k_2). The same mapping h itself is a bijection between $A(G_1)$ and $A(G_2)$ satisfying the isomorphism condition.

(i) $\mu_{E_1}^P(v_i) = \mu_{A_1}^P(v_i) = \mu_{A_2}^P(h(v_i)) = \mu_{E_2}^P(h(v_i))$, $\forall v_i \in G_1$

(ii) $\mu_{E_1}^N(v_i) = \mu_{A_1}^N(v_i) = \mu_{A_2}^N(h(v_i)) = \mu_{E_2}^N(h(v_i))$, $\forall v_i \in G_1$

(iii) $\mu_{F_1}^P(v_i, v_j) = \mu_{B_1}^P(v_i, v_j)$, if v_i and v_j are neighbors in G_1
$\mu_{F_1}^P(v_i, v_j) = \min(\mu_{E_1}^P(v_i), \mu_{E_1}^P(v_j))$, if v_i and v_j are not neighbors in G_1
and

(iv) $\mu_{F_1}^N(v_i, v_j) = \mu_{B_1}^N(v_i, v_j)$, if v_i and v_j are neighbors in G_1
$\mu_{F_1}^N(v_i, v_j) = \max(\mu_{E_1}^N(v_i), \mu_{E_1}^N(v_j))$, if v_i and v_j are not neighbors in G_1

As $h : G_1 \to G_2$ is an isomorphism,

$\mu_{F_1}^N(v_i, v_j) = \mu_{B_2}^N(h(v_i), h(v_j))$, if v_i and v_j are neighbors in G_1
$\mu_{F_1}^N(v_i, v_j) = \max(\mu_{B_2}^N(v_i), \mu_{B_2}^N(v_j))$, if v_i and v_j are not neighbors in G_1
$\mu_{F_1}^N(v_i, v_j) = \mu_{B_2}^N(h(v_i), h(v_j))$, if v_i and v_j are neighbors in G_1
$\mu_{F_1}^N(v_i, v_j) = \max(\mu_{B_2}^N(v_i), \mu_{B_2}^N(v_j))$, if v_i and v_j are not neighbors in G_1

Hence $\mu^P_{F_1}(v_i, v_j) = \mu^P_{F_2}(h(v_i), h(v_j))$ and $\mu^N_{F_1}(v_i, v_j) = \mu^N_{F_2}(h(v_i), h(v_j))$. So, the same h is an isomorphism between $A(G_1)$ and $A(G_2)$. ■

Theorem 3.10 *If G_1 and G_2 are complete bipolar fuzzy graph such that G_1 is co-weak isomorphic to G_2 then $A(G_1)$ is co-weak isomorphic to $A(G_2)$.*

Proof. As G_1 is co-weak isomorphic to G_2, there exists a bijection $h : G_1 \to G_2$ satisfying, $\mu^P_A(v_i) \leq \mu^{P'}_A(h(v_i))$, $\mu^P_B(v_i, v_j) = \mu^{P'}_B(h(v_i), h(v_j))$, $\forall v_i, v_j \in V_1$. If G_1 has n vertices, arrange the vertices of G_1 in such a way that $\mu^P_A(v_1) \leq \mu^P_A(v_2) \leq \mu^P_A(v_3) \ldots \mu^P_A(v_n)$. As G_1 and G_2 are complete, co-weak isomorphic bipolar fuzzy graph, $\mu^P_B(v_i, v_j) = \mu^{P'}_B(h(v_i), h(v_j))$, $\forall v_i, v_j \in V_1$. By Theorem 3.9 and the definition of antipodal bipolar fuzzy graph, we have $A(G_i)$ contains all the vertices of G, where $i = 1, 2$. That is, $\mu^P_E(x) = \mu^P_A(x)$ and $\mu^N_E(x) = \mu^N_A(x)$ for all $x \in V$ and $\mu^P_F(v_i, v_j) = \mu^{P'}_F(h(v_i), h(v_j))$, $\forall v_i, v_j \in V_1$. So, the same bijection h is a co-weak isomorphism between $A(G_1)$ and $A(G_2)$. ■

We state the following Theorem without its proof.

Theorem 3.11 *If G_1 and G_2 are complete bipolar fuzzy graph such that G_1 is co-weak isomorphic to G_2 then $A(G_1)$ is homomorphic to $A(G_2)$.*

Remark 1 If G is a self complementary bipolar fuzzy graph, then its antipodal bipolar fuzzy graph may not be self complementary.

Example 3.12 Consider a bipolar fuzzy graph G

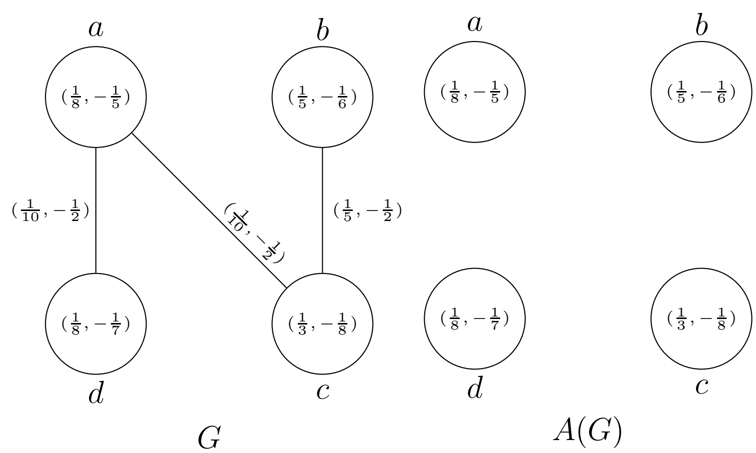


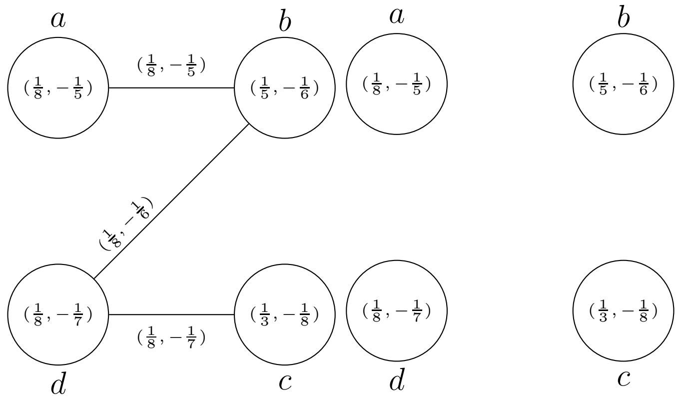
Figure 2

By routine calculations, we have

$$\begin{array}{llll} \delta(a,b) = (15,-4), & \delta(a,c) = (10,-2), & \delta(a,d) = (10,-2), & \delta(b,c) = (5,-2), \\ \delta(b,d) = (25,-6), & \delta(c,d) = (20,-4), & e(a) = (15,-2), & e(b) = (15,-2), \\ e(c) = (20,-2), & e(d) = (25,-2), & d(G) = (25,-2). & \end{array}$$

Since $d(G) \neq \delta(x, y)$ for all x, $y \in V$. Hence $A(G)$ is an antipodal bipolar fuzzy graph of G having same vertices as in G only, and no two vertices in $A(G)$ are made as neighborhood since their $\mu^P\mu^N-$ distance between them is not equal to the diameter of G.

Consider a bipolar fuzzy graph $\overline{G}$



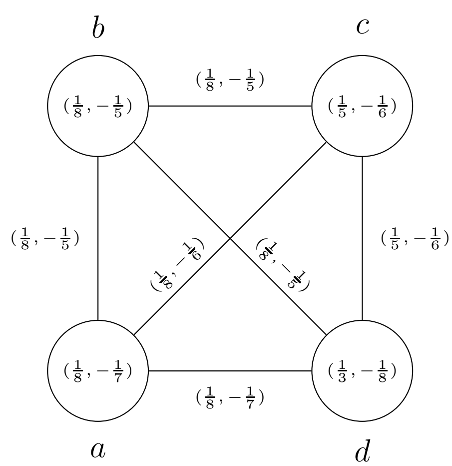
$\overline{G}$ $A(\overline{G})$

Figure 3

By routine calculations, we have

$\delta(a,b) = (8,-5)$, $\delta(a,c) = (24,-18)$, $\delta(a,d) = (16,-11)$, $\delta(b,c) = (16,-13)$,
$\delta(b,d) = (8,-6)$, $\delta(c,d) = (8,-7)$, $e(a) = (24,-5)$, $e(b) = (16,-5)$,
$e(c) = (24,-7)$, $e(d) = (16,-6)$, $d(\overline{G}) = (24,-5)$.

Since $d(\overline{G}) \neq \delta(x,y)$ for all x, $y \in \overline{V}$. Hence $A(\overline{G})$ is an antipodal bipolar fuzzy graph of $\overline{G}$ having same vertices as in $\overline{G}$ only, and no two vertices in $A(\overline{G})$ are made as neighborhood since their $\mu^P\mu^N-$distance between them is not equal to the diameter of $\overline{G}$.



$\overline{A(G)}$

Figure 4

Clearly, $A(G)$ is not isomorphic to $\overline{A(G)}$ Hence G is self complementary , but its antipodal bipolar fuzzy graph $A(G)$ is not a self complementary bipolar fuzzy graph.

Example 3.13 Consider a bipolar fuzzy graph G

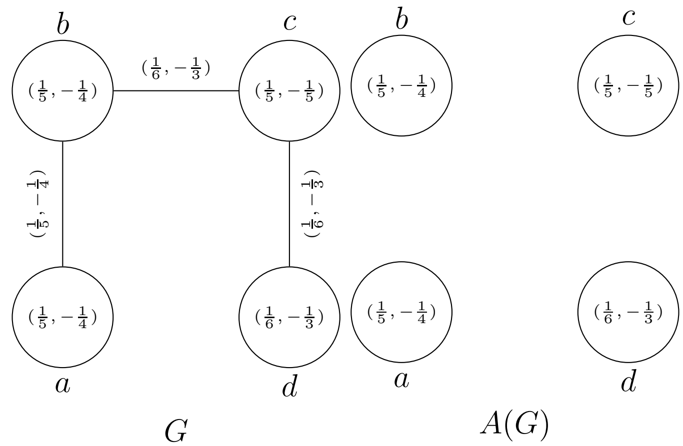


Figure 5

By routine calculations, we have

$$\delta(a,b)=(5,-4),\quad \delta(a,c)=(11,-7),\quad \delta(a,d)=(17,-10),\quad \delta(b,c)=(6,-3),$$
$$\delta(b,d)=(12,-6),\quad \delta(c,d)=(6,-3),\quad e(a)=(17,-4),\quad e(b)=(12,-3),$$
$$e(c)=(11,-3),\quad e(d)=(17,-3),\quad d(G)=(17,-3).$$

Since $d(G)\neq\delta(x,y)$ for all x, $y\in V$. Hence $A(G)$ is an antipodal bipolar fuzzy graph of G having same vertices as in G only, and no two vertices in $A(G)$ are made as neighborhood since their $\mu^P\mu^N-$distance between them is not equal to the diameter of G.

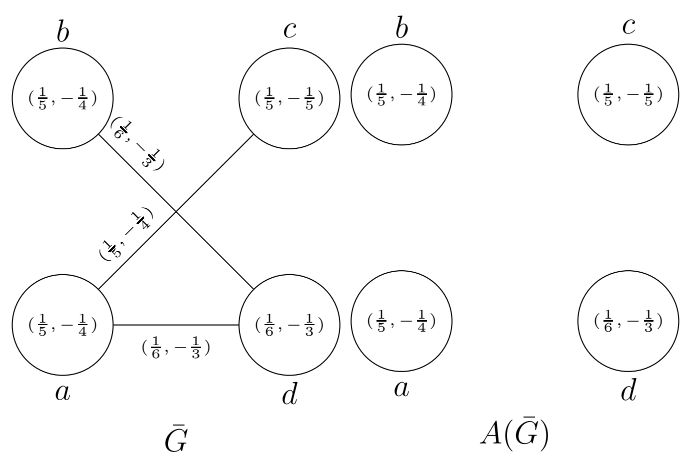
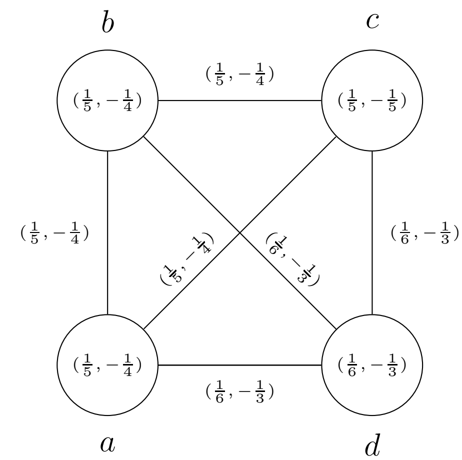


Figure 6

By routine calculations, we have

$$\delta(a,b)=(12,-6),\quad \delta(a,c)=(5,-4),\quad \delta(a,d)=(6,-3),\quad \delta(b,c)=(17,-10),$$
$$\delta(b,d)=(6,-3),\quad \delta(c,d)=(11,-7),\quad e(a)=(12,-3),\quad e(b)=(17,-3),$$
$$e(c)=(17,-4),\quad e(d)=(11,-3),\quad d(\overline{G})=(17,-3).$$

Since $d(\overline{G}) \neq \delta(x, y)$ for all x, $y \in \overline{V}$. Hence $A(\overline{G})$ is an antipodal bipolar fuzzy graph of $\overline{G}$ having same vertices as in $\overline{G}$ only, and no two vertices in $A(\overline{G})$ are made as neighborhood since their $\mu^P\mu^N-$distance between them is not equal to the diameter of $\overline{G}$.



$\overline{A(G)}$
Figure 7

Clearly, $A(G)$ is not isomorphic to $\overline{A(G)}$, though $A(G)$ is isomorphic to $A(\overline{G})$. Hence G is self complementary bipolar fuzzy graph but $A(G)$ is not self complementary bipolar fuzzy graph.

We now present the concept of median bipolar fuzzy graphs.

Definition 3.14 Let G be a connected bipolar fuzzy graph. The μ^P*-status* of a vertex v_i is denoted by $S_{\mu^P}(v_i)$ and is defined as $S_{\mu^P}(v_i) = \sum\limits_{\forall v_j \in V} \delta_{\mu^P}(v_i, v_j)$. The μ^N*-status* of a vertex v_i is denoted by $S_{\mu^N}(v_i)$ and is defined as $S_{\mu^N}(v_i) = \sum\limits_{\forall v_j \in V} \delta_{\mu^N}(v_i, v_j)$. The *minimum* μ^P*-status* of G is denoted by $m[S_{\mu^P}(G)]$ and is defined as $m[S_{\mu^P}(G)] = \min(S_{\mu^P}(v_i), \forall v_i \in V)$. The *minimum* μ^N*-status* of G is denoted by $m[S_{\mu^N}(G)]$ and is defined as $m[S_{\mu^N}(G)] = \min(S_{\mu^N}(v_i), \forall v_i \in V)$. The *minimum* $\mu^P\mu^N$ *status* of G is denoted by $m[S_{\mu^P\mu^N}(G)]$ and is defined as $m[S_{\mu^P\mu^N}(G)] = (m[S_{\mu^P}(G)], m[S_{\mu^N}(G)])$. The *maximum* μ^P*-status* of G is denoted by $M[S_{\mu^P}(G)]$ and is defined as $M[S_{\mu^P}(G)] = \max(S_{\mu^P}(v_i), \forall v_i \in V)$. The *maximum* μ^N*-status* of G is denoted by $M[S_{\mu^N}(G)]$ and is defined as $M[S_{\mu^N}(G)] = \max(S_{\mu^N}(v_i), \forall v_i \in V)$. The *maximum* $\mu^P\mu^N$ status of G is denoted by $M[S_{\mu^P\mu^N}(G)]$ and is defined as $M[S_{\mu^P\mu^N}(G)] = (M[S_{\mu^P}(G)], M[S_{\mu^N}(G)])$. The total μ^P*-status* of a bipolar fuzzy graph G is denoted by $t[S_{\mu^P}(G)]$ and is defined as $t[S_{\mu^P}(G)] = \sum\limits_{\forall v_i \in V} S_{\mu^P}(v_i)$. The *total* μ^N*-status* of a bipolar fuzzy graph G is denoted by $t[S_{\mu^N}(G)]$ and is defined as $t[S_{\mu^N}(G)] = \sum\limits_{\forall v_i \in V} S_{\mu^N}(v_i)$. The *total* μ^N*-status* of a bipolar fuzzy graph G is denoted by $t[S_{\mu^P\mu^N}(G)]$ and is defined as $t[S_{\mu^P\mu^N}(G)] = (t[S_{\mu^P}(G)], t[S_{\mu^N}(G)])$. The *median* of a bipolar fuzzy graph G is denoted by $M(G)$ and is defined as the set of nodes with minimum $\mu^P\mu^N$ status. An bipolar fuzzy graph G is said to be *self-median* if all the vertices have the same status. In other words, G is self-median if and only if $m[S_{\mu^P\mu^N}(G)] = M[S_{\mu^P\mu^N}(G)]$.

Example 3.15

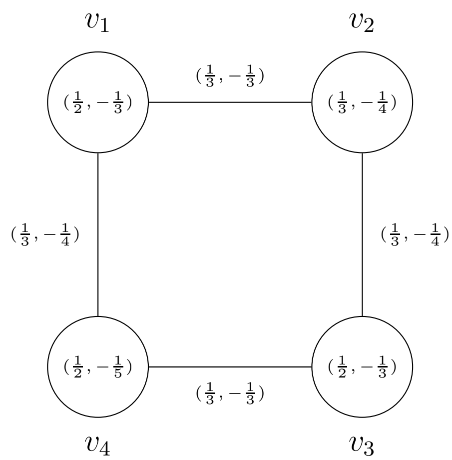


Figure 8. Selfmedianbipolarfuzzygraph.

By routine calculations, we have,

$$\begin{array}{llll}
\delta_{\mu^P}(v_1, v_2) = 3, & \delta_{\mu^P}(v_1, v_3) = 6, & \delta_{\mu^P}(v_1, v_4) = 3, & \\
\delta_{\mu^P}(v_2, v_3) = 3, & \delta_{\mu^P}(v_2, v_4) = 6, & \delta_{\mu^P}(v_3, v_4) = 3, & \\
\delta_{\mu^N}(v_1, v_2) = -11, & \delta_{\mu^N}(v_1, v_3) = -7, & \delta_{\mu^N}(v_1, v_4) = -10, & \\
\delta_{\mu^N}(v_2, v_3) = -10, & \delta_{\mu^N}(v_2, v_4) = -7, & \delta_{\mu^N}(v_3, v_4) = -11, & \\
S_{\mu^P}(v_1) = 12, & S_{\mu^P}(v_2) = 12, & S_{\mu^P}(v_3) = 12, & S_{\mu^P}(v_4) = 12, \\
S_{\mu^N}(v_1) = -28, & S_{\mu^N}(v_2) = -28, & S_{\mu^N}(v_3) = -28, & S_{\mu^N}(v_4) = -28.
\end{array}$$

Therefore, $S_{\mu^P\mu^N}(v_1) = (12, -28)$, $S_{\mu^P\mu^N}(v_2) = (12, -28)$, $S_{\mu^P\mu^N}(v_3) = (12, -28)$, $S_{\mu^P\mu^N}(v_4) = (12, -28)$ and $t[S_{\mu^P\mu^N}(G)] = (48, -112)$. Here, $S_{\mu^P\mu^N}(v_i) = (12, -28)$, $\forall v_i \in V$. Hence G is self median bipolar fuzzy graph.

Theorem 3.15 *Let G be a bipolar fuzzy graph, where crisp graph G^* is an even cycle. If alternate edges have same positive values and negative values, then G is self median bipolar fuzzy graph.*

Proof. Given that G is a bipolar fuzzy graph. Since crisp graph G^* is an even cycle. Also, alternate edges of G have same positive values and negative values, we have, $\delta(v_1, v_2) = \delta(v_3, v_4) = \delta(v_1, v_2) = \ldots = \delta(v_{n-1}, v_n)$ and, similarly, $\delta(v_2, v_3) = \delta(v_4, v_5) = \ldots = \delta(v_n, v_1)$, $\delta(v_1, v_3) = \delta(v_2, v_4) = \delta(v_3, v_5) = \ldots = l$, so on. Hence $S_{\mu^P}(v_i) = k$ and $S_{\mu^N}(v_i) = m$, $\forall v_i \in V$. Hence G is a self median bipolar fuzzy graph. ∎

Remark 2 Let G be a bipolar fuzzy graph, where crisp graph G^* is an odd cycle. If alternate edges have same positive and negative values, then G may not be self median bipolar fuzzy graph.

References

[1] Akram, M., *Bipolar fuzzy graphs*, Information Sciences, 181 (2011), 5548-5564.

[2] Akram, M. and Dudek, W.A., *Regular bipolar fuzzy graphs*, Neural Computing & Applications, 21, (2012), 197-205.

[3] Akram, M. and Karunambigai, M.G., *Metric in bipolar fuzzy graphs*, World Applied Sciences Journal, 14 (2011), 1920-1927.

[4] Akram, M., Saeid, A.B., Shum, K.P. and Meng, B.L., *Bipolar fuzzy K-algebras*, International Journal of Fuzzy System, 10 (3) (2010, 252-258.

[5] Atanassov, K.T., *Intuitionistic fuzzy sets: Theory and applications*, Studies in fuzziness and soft computing, Heidelberg, New York, Physica-Verl., 1999.

[6] Acharya, B.D. and Acharya, M., *On self-antipodal graphs*, Nat. Acad. Sci. Lett., 8 (1985), 151-153.

[7] Ahamed, B. and Gani, A.N., *Perfect Fuzzy Graphs*, Bulletin of Pure and Applied Sciences, 28 (2009), 83-90.

[8] Aravamudhan, R. and Rajendran, B., *On antipodal graphs*, Discrete Math., 49 (1984), 193-195.

[9] Bhattacharya, P., *Some remarks on fuzzy graphs*, Pattern Recognition Letter, 6 (1987), 297-302.

[10] Gani, A.N., and Malarvizhi, J., *On antipodal fuzzy graph*, Applied Mathematical Sciences, 4 (43) (2010), 2145-2155.

[11] Kauffman, A., *Introduction a la Theorie des Sous-emsembles Flous*, Masson et Cie, vol. 1, 1973.

[12] Mordeson, J.N. and Nair, P.S., *Fuzzy graphs and fuzzy hypergraphs*, Physica Verlag, Heidelberg 1998; Second Edition 2001.

[13] Rosenfeld, A., *Fuzzy graphs*, Fuzzy Sets and Their Applications (L.A. Zadeh, K.S. Fu, M. Shimura, Eds.), Academic Press, New York, 1975, 77-95.

[14] Shannon, A., Atanassov, K.T., *A first step to a theory of the intuitionistic fuzzy graphs*, Proceeding of FUBEST (D. Lakov, Ed.), Sofia, 1994, 59-61.

[15] Sunitha, M.S. and Vijayakumar, A., *Complement of a fuzzy graph*, Indian Journal of Pure and Applied Mathematics, 33 (9), 1451-1464.

[16] Yang, H.-L., Li, S.-G., Guo, Z.-L., Ma, C.-H., *Transformation of bipolar fuzzy rough set models*, Knowl.Based Syst., 27 (2012), 60-68.

[17] Zadeh, L.A., *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

[18] Zadeh, L.A., *Similarity relations and fuzzy orderings*, Information Sciences, 3 (2) (1971), 177-200.

[19] Zhang, W.-R., *Bipolar fuzzy sets and relations: a computational framework forcognitive modeling and multiagent decision analysis*, Proc. of IEEE Conf., 1994, 305-309.

[20] Zhang, W.-R., *Bipolar fuzzy sets*, Proc. of FUZZ-IEEE, 1998, 835-840.

Accepted: 10.12.2012