ON HYPER EQ-ALGEBRAS[1]

R.A. Borzooei

Department of Mathematics
Shahid Beheshti University
Tehran
Iran
e-mail: borzooei@sbu.ac.ir

B. Ganji Saffar

Department of Mathematics
Payamnour University
Tehran
Iran
e-mail: bganji@alzahra.ac.ir

R. Ameri

Department of Mathematics
Tehran University
Tehran
Iran
e-mail: rameri@tu.ac.ir

Abstract. In this paper, we applied the hyper structures theory to EQ-algebras and we introduced the notion of hyper EQ-algebra which is a generalization of EQ-algebra. In the following, we define the notions of good and separated hyper EQ-algebras and state and prove some properties of (good, separated) hyper EQ-algebras. Moreover, by define the concept of (pre)filter, we construct the quotient hyper EQ-algebra. Finally, we investigate the relation between hyper EQ-algebras and hyper BCK-algebras and (weak) hyper residuated lattices.

Keywords: hyper EQ-algebra, EQ-algebra, good and separated hyper EQ-algebras, (pre-)filter, hyper BCK-algebra, (weak) hyper residuated lattice.

1. Introduction

Recently, a special algebra called EQ-algebra has been introduced by Vilém Novák in [6]. These algebras are intended to become algebras of truth values for a higher-order fuzzy logic (a fuzzy type theory, FTT). An EQ-algebra has three basic binary operations (meet, multiplication and a fuzzy equality) and a top

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element. The implication is defined from the fuzzy equality "$\sim$" by the formula $a \to b = (a \wedge b) \sim a$. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes commutative residuated lattice. From the point of view of potential application, it seems very interesting that unlike (Hájek 2003), we can have non-commutativity without necessity to introduce, two kinds of implication. The concept of hyper structure (called also multialgebra) was introduced by Marty in [5], at first. That was 8th Congress of Scandinavian mathematician 1934. Till now, the hyper structures are studied from the theoretical point of view for their applications to many subject of pure and applied mathematics. Now, in this paper, we follow the study of EQ-algebras and get some results as mentioned in the abstract.

2. Preliminaries

Definition 2.1. [8] An *EQ-algebra* is an algebra $\varepsilon = (E, \wedge, \otimes, \sim, 1)$ of type (2,2,2,0) such that, for all $x, y, z, t \in E$:

$(E1)$ $\langle E, \wedge, 1\rangle$ is a commutative idempotent monoid (i.e. $\wedge$-semilattice with top element 1);

$(E2)$ $\langle E, \otimes, 1\rangle$ is a commutative monoid and $\otimes$ is isotone w.r.t. "$\leq$" (where $x \leq y$ is defined as $x \wedge y = x$);

$(E3)$ $x \sim x = 1$; (reflexivity axiom)

$(E4)$ $((x \wedge y) \sim z) \otimes (t \sim x) \leq z \sim (t \wedge y)$; (substitution axiom)

$(E5)$ $(x \sim y) \otimes (z \sim t) \leq (x \sim z) \sim (y \sim t)$; (congruence axiom)

$(E6)$ $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$; (monotonicity axiom)

$(E7)$ $(x \wedge y) \sim x \leq (x \wedge y \wedge z) \sim (x \wedge z)$; (monotonicity axiom)

$(E8)$ $x \otimes y \leq x \sim y$. (boundedness axiom)

Remark 2.2. In Definition 2.1, we can omit (E3). Since by (E8), we have $1 \otimes 1 \leq 1 \sim 1$. Now, since 1 is a top element of E, then $1 \sim 1 = 1$. Hence by (E7), for any $x \in E$,

$$1 = 1 \sim 1 = (1 \wedge 1) \sim 1 \leq (1 \wedge 1 \wedge x) \sim (1 \wedge x) = x \sim x$$

and so $x \sim x = 1$.

Proposition 2.3. [8] *Let ε be an EQ-algebra, $x \to y := (x \wedge y) \sim x$ and $\bar{x} = x \sim 1$. Then the following properties hold, for all $x, y, z \in E$:*

(i) $x \otimes y \leq x, y, \quad x \otimes y \leq x \wedge y$;

(ii) $z \otimes (x \wedge y) \leq (z \otimes x) \wedge (z \otimes y)$;

(iii) $x \sim y \leq x \to y$;

(iv) $x \to x = 1$;

(v) $(x \sim y) \otimes (y \sim z) \leq x \sim z$;

(vi) $(x \to y) \otimes (y \to z) \leq x \to z$;

(vii) $x \leq \bar{x}, \quad \bar{1}=1$;

(viii) $x \otimes (x \sim y) \leq \bar{y}$;
(ix) $(z \to (x \wedge y)) \otimes (x \sim t) \leq z \to (t \wedge y)$;
(x) $(x \to y) \otimes (y \to x) \leq x \sim y$;
(xi) *if* $x \leq y \to z$, *then* $x \otimes y \leq \bar{z}$;
(xii) *if* $x \leq y \leq z$, *then* $z \sim x \leq z \sim y$ *and* $x \sim z \leq x \sim y$.

The hyperstructure theory was introduced by Marty [5], at the 8th Congress of Scandinavian Mathematicians. In his definition, a function $\circ : A \times A \longrightarrow P^*(A)$, of the set $A \times A$ into the set of all non-empty subsets of A, is called a binary hyperoperation, and the pair $(A, \circ)$ is called a hypergroupoid. If $\circ$ is associative, then A is called a semihypergroup, and it is said to be commutative if $\circ$ is commutative. Also, an element $1 \in A$ is called an identity element if $x \in 1 \circ x$, for all $x \in A$.

Note that if $A, B \subseteq H$, then

(i) $x \circ B = \bigcup_{b \in B} (x \circ b)$, $B \circ x = \bigcup_{b \in B} (b \circ x)$,

(ii) $A \circ B = \bigcup_{a \in A} \left(\bigcup_{b \in B} a \circ b \right)$.

3. Hyper EQ-algebra

In this section, by considering the notion of EQ-algebra, we define the notion of hyper EQ-algebra which is a generalization of EQ-algebra. Moreover, we give some examples and some important properties of this structure.

Definition 3.1. A *hyper EQ-algebra* $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a non-empty set H with a binary operations $\wedge$ and two binary hyper operations $\otimes$, $\sim$ and top element "1" satisfying the following conditions, for all $x, y, z, t \in H$:
$(HEQ1)$ $\langle H, \wedge, 1 \rangle$ is a commutative idempotent monoid with top element "1",
$(HEQ2)$ $\langle H, \otimes, 1 \rangle$ is a commutative semihypergroup with "1" as an identity and $\otimes$ is isotone w.r.t. $\leq$, i.e. if $x \leq y$, then $x \otimes z \ll y \otimes z$(where $x \leq y$ if and only if $x \wedge y = x$),
$(HEQ3)$ $((x \wedge y) \sim z) \otimes (t \sim x) \ll z \sim (t \wedge y)$,
$(HEQ4)$ $(x \sim y) \otimes (z \sim t) \ll (x \sim z) \sim (y \sim t)$,
$(HEQ5)$ $(x \wedge y \wedge z) \sim x \ll (x \wedge y) \sim x$,
$(HEQ6)$ $(x \wedge y) \sim x \ll (x \wedge y \wedge z) \sim (x \wedge z)$,
$(HEQ7)$ $x \otimes y \ll x \sim y$,
where $A \ll B$, means that, for all $a \in A$ there exists $b \in B$ such that $a \leq b$.

Example 3.2.

(i) In any hyper EQ-algebra $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$, if $x \sim y$ and $x \otimes y$ are singleton, for any $x, y \in H$, then $\mathcal{H}$ is an EQ-algebra.

(ii) Let $H = \{0, 1\}$ such that, $0 < 1$. Define $\wedge, \otimes$, and $\sim$ on H as follows:

$x \wedge y = \min\{x, y\}$, $\quad x \otimes y = \min\{x, y\}$,

$\sim$	0	1
0	{1}	{0,1}
1	{0,1}	{1}

Then $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra.

(iii) Let $H = \{0, a, 1\}$, which $0 < a < 1$, and $\wedge, \otimes$ and $\sim$ are defined on H as follows,

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\otimes$	0	a	1
0	{0}	{0}	{0}
a	{0}	{0,a}	{0,a}
1	{0}	{0,a}	{1}

$\sim$	0	a	1
0	{1}	{a}	{a}
a	{a}	{1}	{1,a}
1	{a}	{1,a}	{1}

Routine calculation shows that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra.

(iv) Let $H = \{0, a, b, 1\}$, such that $0 < a < b < 1$. Define $\wedge, \otimes$ and $\sim$ on H as follows:

$\otimes$	0	a	b	1
0	{0}	{0}	{0}	{0}
a	{0}	{0,a}	{0,a}	{0,a}
b	{0}	{0,a}	{0,b}	{0,b}
1	{0}	{0,a}	{0,b}	{0,1}

$\sim$	0	a	b	1
0	{1}	{a,b,1}	{a,1}	{0,1}
a	{a,b,1}	{a,1}	{a,b,1}	{a,1}
b	{a,1}	{a,b,1}	{b,1}	{b,1}
1	{0,1}	{a,1}	{b,1}	{1}

$$x \wedge y = \min\{x, y\}$$

It is easy to see that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra.

(v) Let $H = [0, 1]$. Then we define $\wedge, \otimes, \sim$, on H as follows:

$$x \otimes y = \begin{cases} 0, & x + y \leq 1 \\ x \wedge y, & o.w. \end{cases}, \quad x \sim y = \{1, x \wedge y\}, \quad x \wedge y = \min\{x, y\}$$

Then $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra.

(vi) Let $H = [0, 1]$. Define $x \wedge y = \min\{x, y\}$, $x \otimes y = xy$ and $x \sim y = [x \wedge y, 1]$, for all $x, y \in H$. Then it is not difficult to check that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra.

Proposition 3.3. *Let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra such that $x \to y = (x \wedge y) \sim x$ and $\bar{x} = x \sim 1$. Then the following conditions hold, for all $x, y, z \in H$ and $A, B, C \subseteq H$:*

(i) $x \ll x \otimes 1$, $\quad A \ll A \otimes 1$;

(ii) $1 \in x \sim x$, $1 \ll x \to x$ *and* $1 \in A \sim A$;

(iii) $z \otimes (x \wedge y) \ll (z \otimes x) \wedge (z \otimes y)$;

(iv) *if* $A \ll B$ *and* $B \ll C$, *then* $A \ll C$;

(v) $x \sim y \ll y \sim x$, $A \sim B \ll B \sim A$;
(vi) $x \sim y \ll x \to y$;
(vii) *if* $x \leq y$, *then* $1 \in x \to y$;
(viii) $x \ll \bar{x}$;
(*ix*) *if* $A \ll B$, *then* $A \otimes C \ll B \otimes C$;
(x) $(x \sim y) \otimes (y \sim z) \ll x \sim z$ *and* $(x \to y) \otimes (y \to z) \ll x \to z$;
(xi) $(x \to y) \otimes (y \to x) \ll x \sim y$;
(xii) $\bar{x} \otimes \bar{y} \ll x \sim y$;
(xiii) $x \otimes (x \sim y) \ll \bar{y}$;
(xiv) *if* $x \leq y$, *then* $\bar{x} \ll \bar{y}$, $x \sim y = y \to x$, $z \to x \ll z \to y$, $y \to z \ll x \to z$;
(xv) *if* $A \ll B$, *then* $C \to A \ll C \to B$;
(xvi) $y, \bar{y} \ll x \to y$;
(xvii) $(z \to (x \wedge y)) \otimes (x \sim t) \ll z \to (t \wedge y)$;
(xviii) *if* $x \ll y \to z$, *then* $x \otimes y \ll \bar{z}$;
(xix) *if* $x \leq y \leq z$, *then* $z \sim x \ll z \sim y$ *and* $x \sim z \ll x \sim y$;
(xx) $(x \sim y) \otimes (z \sim t) \ll (x \wedge z) \sim (y \wedge t)$;
(xxi) $x \sim y \ll (x \wedge z) \sim (y \wedge z)$;
(xxii) $x \sim y \ll ((x \wedge t) \sim z) \sim ((y \wedge t) \sim z)$ *and* $x \sim y \ll (z \sim (x \wedge t)) \sim (z \sim (y \wedge t))$;
(xxiii) $x \to y = x \to (x \wedge y)$;
(xxiv) $x \sim y \ll (x \sim z) \sim (y \sim z)$ *and* $x \sim y \ll (z \sim x) \sim (z \sim y)$;
(xxv) $x \sim y \ll (z \to x) \sim (z \to y)$;
(xxvi) $x \to y \ll (z \to x) \to (z \to y)$.

Proof. (i): By definition of identity, $x \in x \otimes 1$. Since $x \leq x$, then $x \ll x \otimes 1$. The second part is clear.
(ii): By (HEQ7), $1 \in 1 \otimes 1 \leq 1 \sim 1$. Since 1 is a top element, then $1 \in 1 \sim 1$. Now, by (HEQ6), $1 \sim 1 = ((1 \wedge 1) \sim 1) \ll (1 \wedge 1 \wedge x) \sim (1 \wedge x) = x \sim x$. Since 1 is a top element, then $1 \in x \sim x$. Hence $1 \ll x \sim x$. The rest is clear.
(iii): Since $x \wedge y \ll x, y$ and $\otimes$ is isotone, then $z \otimes (x \wedge y) \ll (z \otimes x)$ and $(z \otimes y)$. Hence $z \otimes (x \wedge y) \ll (z \otimes x) \wedge (z \otimes y)$.
(iv): The proof is straightforward.
(v): By (i), (ii), (iv) and (HEQ3),

$$x \sim y \ll (x \sim y) \otimes 1 = ((x \wedge 1) \sim y) \otimes 1 \ll ((x \wedge 1) \sim y) \otimes (x \sim x) \\ \ll y \sim (x \wedge 1) = y \sim x$$

The second part is clear.
(vi): By (i), (ii), (HEQ3), (iv) and (v),

$$x \sim y \ll 1 \otimes (x \sim y) \ll ((x \wedge x) \sim x) \otimes (x \sim y) \ll ((x \wedge x) \sim x) \otimes (y \sim x) \\ \ll x \sim (x \wedge y) \ll (x \wedge y) \sim x = x \to y$$

(vii): If $x \leq y$, then $x \to y = (x \wedge y) \sim x = x \sim x$. Hence, by (ii), $1 \in x \to y$.
(viii): By (i), (iv) and (HEQ7), $x \ll x \otimes 1 \ll x \sim 1 = \bar{x}$.

(ix): The proof is straightforward.
(x): By (HEQ3), (HEQ2), (iv) and (v),

$$\begin{aligned}(x \sim y) \otimes (y \sim z) &= (y \sim z) \otimes (x \sim y) = ((y \wedge 1) \sim z) \otimes (x \sim y) \ll (z \sim (x \wedge 1)) \\ &= (z \sim x) \ll x \sim z\end{aligned}$$

Now, by (HEQ5), (HEQ1), (iv) and (v),

$$\begin{aligned}(x \to y) \otimes (y \to z) &= ((x \wedge y) \sim x) \otimes ((y \wedge z) \sim y) \ll (x \sim (x \wedge (y \wedge z))) \\ &\ll ((x \wedge (y \wedge z)) \sim x) = (((x \wedge z) \wedge y) \sim x) \\ &\ll (x \wedge z) \sim x = x \to z.\end{aligned}$$

(xi): By (iv), (v), (x) and (HEQ3),

$$\begin{aligned}(x \to y) \otimes (y \to x) &= ((x \wedge y) \sim x) \otimes ((x \wedge y) \sim y) \\ &\ll (x \sim (x \wedge y)) \otimes ((x \wedge y) \sim y) \ll x \sim y\end{aligned}$$

(xii): By (iv), (v) and (x), $\bar{x} \otimes \bar{y} = (x \sim 1) \otimes (y \sim 1) \ll (x \sim 1) \otimes (1 \sim y) \ll x \sim y$.
(xiii): By (viii), (iv), (v), (ix) and (x),

$$x \otimes (x \sim y) \ll (x \sim 1) \otimes (x \sim y) \ll (1 \sim x) \otimes (x \sim y) \ll 1 \sim y \ll y \sim 1 = \bar{y}$$

(xiv): Let $x \leq y$. By (HEQ5), $(x \wedge y \wedge 1) \sim 1 \ll (y \wedge 1) \sim 1$. Then $x \sim 1 \ll y \sim 1$ i.e. $\bar{x} \ll \bar{y}$. Again, by (HEQ5),

$$z \to x = (z \wedge x) \sim z = (z \wedge x \wedge y) \sim z \ll (z \wedge y) \sim z = z \to y$$

The proof of other part is similar.
(xv): The proof is straightforward.
(xvi): Since $x \leq 1$, then by (xiv), $1 \to y \ll x \to y$. Hence, $\bar{y} = y \sim 1 = 1 \to y \ll x \to y$. Moreover by (viii) and (iv), $y \ll x \to y$.
(xvii): By (HEQ3), (iv) and (v),

$$\begin{aligned}(z \to (x \wedge y)) \otimes (x \sim t) &\ll (z \to (x \wedge y)) \otimes (t \sim x) \\ &= ((z \wedge (x \wedge y)) \sim z) \otimes (t \sim x) \\ &\ll z \sim (z \wedge (t \wedge y)) \ll z \to (t \wedge y)\end{aligned}$$

(xviii): Let $x \ll y \to z$. By (viii), (iv), (ix), (x) and (HEQ5),

$$\begin{aligned}x \otimes y \ll (y \to z) \otimes \bar{y} &= ((y \wedge z) \sim y) \otimes (y \sim 1) \ll (y \wedge z) \sim 1 \\ &= (y \wedge z \wedge 1) \sim 1 \ll (z \wedge 1) \sim 1 = z \sim 1 = \bar{z}\end{aligned}$$

(xix): If $x \leq y \leq z$, then by (HEQ5) and (iv),

$$z \sim x \ll (x \wedge z) \sim z = (x \wedge y \wedge z) \sim z \ll (y \wedge z) \sim z = y \sim z$$

For the other case, by (HEQ6),

$$x \sim z = (x \wedge z) \sim z \ll (x \wedge y \wedge z) \sim (z \wedge y) = x \sim y$$

(xx): By (HEQ6), $x \sim y = (x \wedge 1) \sim y \ll (x \wedge 1 \wedge z) \sim (y \wedge z) = (x \wedge z) \sim (y \wedge z)$ and in the same way $z \sim t \ll (y \wedge z) \sim (y \wedge t)$. Now, by (x), (ix) and (iv),

$$(x \sim y) \otimes (z \sim t) \ll ((x \wedge z) \sim (y \wedge z)) \otimes ((y \wedge z) \sim (y \wedge t)) \ll (x \wedge z) \sim (y \wedge t)$$

(xxi): By (i), (ii), (iv), (v) and (HEQ3),

$$x \sim y \ll y \sim x \ll 1 \otimes (y \sim x) \ll ((x \wedge z) \sim (x \wedge z)) \otimes (y \sim x) \ll (x \wedge z) \sim (y \wedge z)$$

(xxii): By (i), (ii), (iv), (xxi) and (HEQ4),

$$\begin{aligned} x \sim y \ll (x \wedge t) \sim (y \wedge t) &\ll ((x \wedge t) \sim (y \wedge t)) \otimes 1 \\ &\ll ((x \wedge t) \sim (y \wedge t)) \otimes (z \sim z) \\ &\ll ((x \wedge t) \sim z) \sim ((y \wedge t) \sim z) \end{aligned}$$

The proof of second part is similar.
(xxiii): The proof is clear.
(xxiv): By (xxii), it is enough to take $t = 1$.
(xxv): Let $t = z$. Then by (xxii), the proof is clear.
(xxvi): By (xxv), (iv), (v), (vi) and (xiv),

$$\begin{aligned} x \to y = (x \wedge y) \sim x &\ll (z \to (x \wedge y)) \sim (z \to x) \ll (z \to x) \sim (z \to (x \wedge y)) \\ &\ll (z \to x) \to (z \to (x \wedge y)) \ll (z \to x) \to (z \to y). \end{aligned}$$ ∎

Proposition 3.4. *Let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra with bottom element "0". If we set $\neg x = 0 \sim x$, then, for all $x, y, t \in H$;*
(i) $1 \in \neg 0$;
(ii) *if* $x \leq y$, *then* $\neg y \ll \neg x$;
(iii) $x \otimes \neg x \ll \bar{0}$, $\bar{x} \otimes \bar{0} \ll \neg x$, $\neg x \otimes \bar{0} \ll \bar{x}$;
(iv) $\neg x \otimes \neg y \ll x \to y$;
(v) $\bar{0} \ll \neg x$;
(vi) $x \sim y \ll \neg x \sim \neg y$;
(vii) $(x \sim y) \otimes \neg y \ll \neg x, (x \to y) \otimes \neg y \ll \neg x$;
(viii) $\neg y \ll y \to z$;
(ix) $x \sim y \ll \neg(x \wedge t) \sim \neg(y \wedge t)$.

Proof. The proof of (i) and (ii) are clear.
(iii): Straightforward.
(iv): By Proposition 3.3 (iv), (v), (vi) and (x),

$$\neg x \otimes \neg y = (0 \sim x) \otimes (0 \sim y) \ll (x \sim 0) \otimes (0 \sim y) \ll x \sim y \ll x \to y.$$

(v): It follows from Proposition 3.3 (xix) and $0 \leq x \leq 1$.
(vi): By Proposition 3.3 (xxiv), $x \sim y \ll (0 \sim x) \sim (0 \sim y) = \neg x \sim \neg y$.
(vii): By Proposition 3.3 (iv), (v) and (x),

$$(x \sim y) \otimes \neg y = (x \sim y) \otimes (0 \sim y) \ll (x \sim y) \otimes (y \sim 0) \ll x \sim 0 \ll \neg x$$

The proof of other case is similar.
(viii): Since $0 \leq z$, then by Proposition 3.3 (xiv), $\neg y = 0 \sim y \ll y \to 0 \ll y \to z$.
(ix): By Proposition 3.3(xxii), $x \sim y \ll (0 \sim (x \wedge t)) \sim (0 \sim (y \wedge t)) = \neg(x \wedge t) \sim \neg(y \wedge t)$. ■

Theorem 3.5. *Each hyper EQ-algebra of order n, can be extend to a hyper EQ-algebra of order $n+1$, for any $2 < n \in \mathbb{N}$.*

Proof. Let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra of order n, e be an element such that $e \notin H$, and $\mathcal{H}_1 = H \cup \{e\}$. Then we define a binary operations $\wedge_1$ and two hyper operation $\sim_1$ and $\otimes_1$ on $\mathcal{H}_1$ by:

$$x \wedge_1 y = \begin{cases} x \wedge y & , \ x, y \in H \\ x & , \ x \in H, y = e \\ y & , \ y \in H, x = e \\ e & , \ x = e, y = e \end{cases},$$

$$x \otimes_1 y = \begin{cases} x \otimes y & , \ x, y \in H \\ \{x, e\} & , \ x \in H, y = e \\ \{y, e\} & , \ y \in H, x = e \\ e & , \ x = e, y = e \end{cases},$$

$$x \sim_1 y = x \sim y) \cup \{e\}$$

We define $x \leq y$ if and only if $x \wedge_1 y = x$. It is clear that for any $x \in H$, $x \leq e$. Now, by some modification we can show that $\mathcal{H}_1 = (H_1, \wedge_1, \otimes_1, \sim_1, e)$ is a hyper EQ-algebra. ■

Example 3.6. Let $H = \{1\}$. Then $1 \wedge 1 := 1, 1 \sim 1 := \{1\}, 1 \sim 1 := \{1\}$. Hence it is clear that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra.

Corollary 3.7. *There exists at least one hyper EQ-algebra of order n, for any $n \in \mathbb{N}$.*

Proof. By Examples $3.6, 3.2$(ii) and Proposition 3.5, the proof is clear. ■

Definition 3.8. Let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra. Then H is called,
(i) *separated* if $1 \in x \sim y$, then $x = y$, for all $x, y \in H$, (in other words $1 \in x \sim y$ if and only if $x = y$);
(ii) *good* if $x \sim 1 = x = 1 \sim x$, for all $x \in H$.

Example 3.9. (i) Let $H = \{0, a, 1\}$, which $0 < a < 1$, and $\wedge, \otimes$ and $\sim$ are defined on H as follows,

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\otimes$	0	a	1
0	{0}	{0}	{0}
a	{0}	{0,a}	{0,a}
1	{0}	{0,a}	{1}

$\sim$	0	a	1
0	{1}	{0,a}	{a}
a	{0,a}	{1}	{a}
1	{0,a}	{0,a}	{1}

Routine calculation shows that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a separated hyper EQ-algebra but it is not good.
(ii) Let $H = \{0, a, 1\}$, which $0 < a < 1$, and $\wedge, \otimes$ and $\sim$ are defined on H as follows,

$x \wedge y = \min\{x, y\}$,

$\otimes$	0	a	1
0	{0}	{0}	{0}
a	{0}	{0}	{0,a}
1	{0}	{0,a}	{1}

$\sim$	0	a	1
0	{1}	{0,a}	{0}
a	{0,a}	{1}	{a}
1	{0}	{a}	{1}

Then $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a good and separated hyper EQ-algebra.

Proposition 3.10. *Let $\mathcal{H}$ be a good hyper EQ-algebra. Then the following conditions hold, for all $x, y \in H$,*
(i) $x \ll (x \sim y) \sim y$, $x \ll (y \sim x) \sim y$;
(ii) $x \ll (x \to y) \to y$;
(iii) $x \otimes (x \sim y) \ll y$;
(iv) $x \otimes (x \to y) \ll y$;
(v) *if* $x \ll y \to z$, *then* $x \otimes y \ll z$;

Proof. (i): By Proposition 3.3 (i), (ii), (iv), (HEQ4) and goodness,

$$x \ll x \otimes 1 \ll (x \sim 1) \otimes (y \sim y) \ll (x \sim y) \sim (1 \sim y) = (x \sim y) \sim y.$$

The proof of second part is similar.
(ii): By (i) and Proposition 3.3(iv), (vi) and (xiv),

$$x \ll ((x \wedge y) \sim x) \sim (x \wedge y) = (x \to y) \sim (x \wedge y) \ll (x \to y) \to (x \wedge y) \ll (x \to y) \to y.$$

(iii), (iv): By Proposition 3.3 (xiii) and (iv), the proof are straightforward.
(v): Let $x \ll y \to z$, for some $x, y, z \in H$. Then by (iv), $x \otimes y \ll (y \to z) \otimes y = y \otimes (y \to z) \ll z$. Hence $x \otimes y \ll z$. ■

Proposition 3.11. *Every good hyper EQ-algebra is separated.*

Proof. Let $1 \in x \sim y$. Then by Proposition 3.3 (i), (ii), (ix) and Proposition 3.10 (iii), $x \ll x \otimes 1 \ll x \otimes (x \sim y) \ll y$. Therefore, $x \leq y$. By the similar way, $y \leq x$. Thus $x = y$. Therefore, $\mathcal{H}$ is separated. ■

4. Filters in hyper EQ-algebra

In this section we introduce the concept of prefilters and filters in hyper EQ-algebras and we give some related results.

In what follows, let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ denote a hyper EQ-algebra, unless otherwise state.

Definition 4.1. Let F be a subset of H such that $1 \in F$. Then F is called a
(i) *prefilter* of H, if $x \to y \subseteq F$ and $x \in F$, imply $y \in F$, for some $x, y \in H$ and $(x \otimes y) \subseteq F$, for all $x, y \in F$.
(ii) *filter* of H, if F is a prefilter and $x \to y \subseteq F$, imply $(x \otimes z) \to (y \otimes z) \subseteq F$, for all $x, y, z \in H$.

Example 4.2. (i) Let $H = \{0, a, b, 1\}$, such that $0 < a < b < 1$. Define $\wedge, \otimes$ and $\sim$ on H as follows:

$\otimes$	0	a	b	1
0	{0}	{0}	{0}	{0}
a	{0}	{0,a}	{0,a}	{0,a}
b	{0}	{0,a}	{0,b}	{0,b}
1	{0}	{0,a}	{0,b}	{1}

,

$\sim$	0	a	b	1
0	{1}	{1}	{1}	{0,1}
a	{1}	{1}	{a,b,1}	{a,1}
b	{1}	{a,b,1}	{1}	{b,1}
1	{1}	{a,1}	{b,1}	{1}

$$x \wedge y = \min\{x, y\}$$

It is easy to see that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra. It is clear that $F_1 = \{1\}$ is (pre)filter. Moreover, since $0 \to a = \{1\} \subseteq F$ but $a \notin F$, then $F_2 = \{0, b, 1\}$ is not a prefilter.

(ii) According to Example 3.9(i), $F = \{1, a\}$ is prefilter. Since $1 \to a = \{a\} \subseteq F$ and $1 \otimes a \to a \otimes a = \{0, a, 1\} \nsubseteq F$, then F is not a filter.

(iii) Let $H = \{0, a, b, 1\}$ such that $0 < a < b < 1$. Define $\wedge, \otimes$ and $\sim$ on H as follows;

$x \wedge y = \min\{x, y\}$,

$\sim$	0	a	b	1
0	{1}	{a,b,1}	{a,1}	{1}
a	{a,b,1}	{a,1}	{a,b,1}	{1}
b	{a,1}	{a,b,1}	{b,1}	{1}
1	{1}	{1}	{1}	{1}

, $x \otimes y = x \wedge y$

It is easy to see that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra. Let $F = \{1\}$. Since $1 \to a = a \sim 1 = \{1\} \subseteq F$ and $1 \in F$, but $a \notin F$, then F is not a (pre)filter.

Definition 4.3. A non-empty subset D of H is said to be $S_{\to}reflexive$ ($S_{\sim}reflexive$) if $(x \to y) \cap D \neq \emptyset((x \sim y) \cap D \neq \emptyset)$, then $x \to y \subseteq D(x \sim y \subseteq D)$, for all $x, y \in H$.

Example 4.4. According of Example 3.2(iii), $F = \{1, a\}$ is $S_{\to}reflexive$ and $S_{\sim}reflexive$.

Remark 4.5. Let D be $S_{\sim}reflexive$ and $(x \to y) \cap D \neq \emptyset$, for $x, y \in D$. Then $((x \wedge y) \sim x) \cap D \neq \emptyset$. Since D is $S_{\sim}reflexive$, then $((x \wedge y) \sim x) \subseteq D$ or $(x \to y) \subseteq D$. Therefore, D is $S_{\to}reflexive$.

Lemma 4.6. Let F be $S_{\sim}reflexive$ and (pre)filter of H. Then the following conditions hold, for all $x, y, z \in H$,

(i) if $x \in F$ and $x \leq y$, then $y \in F$, and if $A \subseteq F$ and $A \ll B$, then $B \cap F \neq \emptyset$;
(ii) if $x \in F$ and $(x \sim y) \cap F \neq \emptyset$, then $y \in F$;
(iii) if $x, y \in F$, then $(x \wedge y) \in F$;
(iv) if $A \subseteq F$ and $(A \sim B) \cap F \neq \emptyset$, then $B \cap F \neq \emptyset$;
(v) if $(x \sim y) \subseteq F$ and $(y \sim z) \subseteq F$, then $(x \sim z) \cap F \neq \emptyset$;
(vi) if $(x \to y) \subseteq F$ and $(y \to z) \subseteq F$, then $(x \to z) \cap F \neq \emptyset$;
(vii) if $x \ll y \to z$ and $x, y \in F$, then $z \in F$.

Proof. (i); Let $x \leq y$ and $x \in F$, for some $x, y \in H$. By Proposition 3.3 (vii), $1 \in x \to y$, then $(x \to y) \cap F \neq \emptyset$. Since F is $S_{\sim}reflexive$, then by Remark 4.5, F is $S_{\to}reflexive$, and so $x \to y \subseteq F$. Now since F is a (pre)filter and $x \in F$, then $y \in F$. The proof of second part is clear.
(ii): Since $(x \sim y) \cap F \neq \emptyset$, then $(x \sim y) \subseteq F$ and since $x \sim y \ll x \to y$ then by (i), $(x \to y) \cap F \neq \emptyset$. By Remark 4.5, F is $S_{\to}reflexive$, then $x \to y \subseteq F$. Now, since F is a (pre)filter and $x \in F$, thus $y \in F$.
(iii): Since $y \ll x \to y$ and $y \in F$, then by (i), $(x \to y) \cap F \neq \emptyset$. Since F is $S_{\sim}reflexive$, then by Remark 4.5, $x \to y \subseteq F$. By Proposition 3.3(xxiii), $x \to y = x \to (x \wedge y)$ and since $x \in F$, then $x \wedge y \in F$.
(iv): The proof is straightforward.
(v): From Proposition 3.3(xxiv), $x \sim y \ll (y \sim z) \sim (x \sim z)$. Then by (i), $((y \sim z) \sim (x \sim z)) \cap F \neq \emptyset$ and $y \sim z \subseteq F$. Hence, by (iv), $(x \sim z) \cap F \neq \emptyset$.
(vi): Let $x \to y \subseteq F$ and $(y \to z) \subseteq F$, then by Definition 4.1, $(x \to y) \otimes (y \to z) \subseteq F$. By Proposition 3.3 (x), $(x \to y) \otimes (y \to z) \ll x \to z$. Then by (i), $(x \to z) \cap F \neq \emptyset$.
(vii): Let $x \ll y \to z$. Since $x \in F$, then by (i), $y \to z \cap F \neq \emptyset$. Thus by Remark 4.5, $(y \to z) \subseteq F$ and $y \in F$. Hence $z \in F$. ■

Proposition 4.7. *Let F be an $S_{\sim}reflexive$ filter of H. If $(x \sim y) \cap F \neq \emptyset$ and $(z \sim t) \cap F \neq \emptyset$, for all $x, y, z, t \in H$. Then the following conditions hold:*
(i) $((x \wedge z) \sim (y \wedge t)) \cap F \neq \emptyset$;
(ii) $((x \sim z) \sim (y \sim t)) \cap F \neq \emptyset$;
(iii) $((x \to z) \sim (y \to t)) \cap F \neq \emptyset$;
(iv) $((x \otimes z) \sim (y \otimes t)) \cap F \neq \emptyset$.

Proof. (i): By Proposition 3.3 (xxi), $(x \sim y) \ll ((x \wedge z) \sim (y \wedge z))$. Since $(x \sim y) \cap F \neq \emptyset$ and F is $S_{\sim}reflexive$, then $(x \sim y) \subseteq F$. Hence by Lemma 4.6 (i), $((x \wedge z) \sim (y \wedge z)) \cap F \neq \emptyset$. By the similar way, since $(z \sim t) \ll ((z \wedge y) \sim (t \wedge y))$, we have $((z \wedge y) \sim (t \wedge y)) \cap F \neq \emptyset$. Since F is $S_{\sim}reflexive$ then, $((x \wedge z) \sim (y \wedge z)) \subseteq F$ and $((z \wedge y) \sim (t \wedge y)) \subseteq F$. Now, by Lemma 4.6 (v), $((x \wedge z) \sim (y \wedge t)) \cap F \neq \emptyset$.
(ii), (iii): The proof are similar to the proof of (i).
(iv): Let $(x \sim y) \cap F \neq \emptyset$. Since F is a $S_{\sim}reflexive$ filter and $x \sim y \ll x \to y$, then by Remark 4.5, $(x \to y) \subseteq F$ and by Definition 4.1, $((x \otimes z) \to (y \otimes z)) \subseteq F$. By the similar way from $(z \sim t) \cap F \neq \emptyset$, we have $((z \otimes y) \to (t \otimes y)) \subseteq F$. Since F is a filter, then $((x \otimes z) \to (y \otimes z)) \otimes ((z \otimes y) \to (t \otimes y)) \subseteq F$. Hence by Proposition 3.3(x) and Lemma 4.6(i), $((x \otimes z) \sim (y \otimes t)) \cap F \neq \emptyset$. ■

Now, let F be an $S_{\sim}reflexive$ filter of H. We define, for all $x, y \in H$:

$$x \equiv_F y \quad iff \quad (x \sim y) \cap F \neq \emptyset$$

By Proposition 3.3 (ii), $1 \in x \sim x$. Since F is a filter, then $1 \in F$. Hence $(x \sim x) \cap F \neq \emptyset$, and so $x \equiv_F x$. If $x \equiv_F y$, then $(x \sim y) \cap F \neq \emptyset$. By Proposition 3.3 (v) and Lemma 4.6 (i), $(y \sim x) \cap F \neq \emptyset$, Thus $y \equiv_F x$. Now, let $x \equiv_F y$ and $y \equiv_F z$. Then $(x \sim y) \cap F \neq \emptyset$ and $(y \sim z) \cap F \neq \emptyset$. Since F is $S_{\sim}reflexive$, then $(x \sim y) \subseteq F$ and $(y \sim z) \subseteq F$. Hence by Lemma 4.6 (v), $(x \sim z) \cap F \neq \emptyset$, and so $x \equiv_F z$. Therefore, " $\equiv_F$ " is an equivalent relation on H.

Let $x \equiv_F y$. Then $(x \sim y) \cap F \neq \emptyset$. Since F is $S_{\sim}reflexive$, then $(x \sim y) \subseteq F$. By Proposition 3.4(vi), $x \sim y \ll \neg x \sim \neg y$. Then by Lemma 4.6 (i), $(\neg x \sim \neg y) \cap F \neq \emptyset$, and so $\neg x \equiv_F \neg y$. Also, if $x \equiv_F y$ and $z \equiv_F t$, then $(x \sim y) \cap F \neq \emptyset$ and $(z \sim t) \cap F \neq \emptyset$, and so by Proposition 4.7, $((x \wedge z) \sim (y \wedge t)) \cap F \neq \emptyset$, $((x \sim z) \sim (y \sim t)) \cap F \neq \emptyset$, $((x \to z) \sim (y \to t)) \cap F \neq \emptyset$ and $((x \otimes z) \sim (y \otimes t)) \cap F \neq \emptyset$. Hence, $x \wedge z \equiv_F y \wedge t$, $x \sim z \equiv_F y \sim t$, $x \to z \equiv_F y \to t$ and $x \otimes z \equiv_F y \otimes t$. In the other words, " $\equiv_F$ " is a congruence relation on H. Now, let $\dfrac{H}{\equiv_F} = \{[x] | x \in H\}$. Define operation $\bar{\wedge}$ and hyper operations $\bar{\otimes}$ and $\bar{\sim}$ on $\dfrac{H}{\equiv_F}$ as follows:

$$\begin{aligned} [x] \bar{\wedge} [y] &= [x \wedge y] \\ [x] \bar{\otimes} [y] &= \{[t] | t \in x \otimes y\} \\ [x] \bar{\sim} [y] &= \{[t] | t \in x \sim y\} \end{aligned}$$

According to the above mention, it is clear $\bar{\wedge}$, $\bar{\otimes}$ and $\bar{\sim}$ are well defined.

Now, we define $[x] \leq [y]$ if and only if $[x] \bar{\wedge} [y] = [x]$ if and only if $[x \wedge y] = [x]$ if and only if $x \wedge y \equiv_F x$ if and only if $(x \wedge y) \sim x \cap F \neq \emptyset$ if and only if $x)y \cap F \neq \emptyset$ and $A \ll B$ is defined as before. Then we have the following theorem.

Theorem 4.8. *Let F be an $S_{\sim}reflexive$ filter of $\mathcal{H}$. Then* $\dfrac{\mathcal{H}}{\equiv_F} = \left(\dfrac{H}{\equiv_F}, \bar{\wedge}, \bar{\otimes}, \bar{\sim}, [1] \right)$ *is a hyper EQ-algebra which is separated.*

Proof. (HEQ1), (HEQ2): It is easy to see that $\left(\dfrac{H}{\equiv_F}, \bar{\wedge} \right)$ is an idempotent commutative monoid, and $(\frac{H}{\equiv_F}, \bar{\otimes})$ is a commutative semihypergroup with identity [1] and $\bar{\otimes}$ is isotone.

(HEQ3): We show that $(([x] \bar{\wedge} [y]) \bar{\sim} [z]) \bar{\otimes} ([t] \bar{\sim} [x]) \ll ([z] \bar{\sim} ([t] \bar{\wedge} [y]))$ or, equivalently, $([x \wedge y] \bar{\sim} [z]) \bar{\otimes} ([t] \bar{\sim} [x]) \ll ([z] \bar{\sim} [t \wedge y])$. Let $[t] \in ([u] \bar{\otimes} [v]) \subseteq (([x \wedge y] \bar{\sim} [z]) \bar{\otimes} ([t] \bar{\sim} [x]))$, such that $[u] \in [x \wedge y] \bar{\sim} [z]$ and $[v] \in [t] \bar{\sim} [x]$. Then $u \in (x \wedge y) \sim z$ and $v \in (t \sim x)$. So by (HEQ3), $t \in (u \otimes v) \ll ((x \wedge y) \sim z) \otimes (t \sim x) \ll z \sim (t \wedge y)$. Thus there exists $w \in z \sim (t \wedge y)$ such that $t \in (u \otimes v) \ll w$ or $t \leq w$. Hence $1 \in t \to w$, or $1 \in (u \otimes v) \to w$, i.e. $((u \otimes v) \to w) \cap F \neq \emptyset$. Then $[u] \bar{\otimes} [v] \ll [w]$. Since $[w] \in [z \sim (t \wedge y)] = [z] \bar{\sim} ([t] \bar{\wedge} [y])$, Therefore, (HEQ3) holds. By the same way, other conditions hold.

If $[1] \in ([x]\tilde{\sim}[y])$, then $[1] \in \{[t] | t \in x \sim y\}$. Hence there exists $t \in x \sim y$ such that $[t] = [1]$, and so $t \sim 1 \cap F \neq \emptyset$. Then $(x \sim y) \sim 1 \cap F \neq \emptyset$. By Proposition 3.3 (ii), $((x \sim y) \sim 1) \ll (1 \sim (x \sim y))$. Then by Lemma 4.6 (i), $(1 \sim (x \sim y)) \cap F \neq \emptyset$. Since F is $S_{\sim}reflexive$, then $(1 \sim (x \sim y)) \subseteq F$. Now, since $1 \in F$, and F is a filter, then by Lemma 4.6 (iv), $(x \sim y) \cap F \neq \emptyset$. Hence $x \equiv_F y$, i.e. $[x] = [y]$. Therefore, $\frac{\mathcal{H}}{\equiv_F}$ is separated. ∎

5. Relation between hyper EQ-algebras and some other hyper structures

Definition 5.1. [4] Let H be a non-empty set and "$\circ$" be a hyper operation on H. Then H is called a *hyper BCK-algebra* if it contains a constant "0" and satisfies the following axioms, for all $x, y, z \in H$:

$(HK1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$;
$(HK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$;
$(HK3)$ $x \circ H \ll \{x\}$ (or $x \circ y \ll \{x\}$);
$(HK4)$ $x \ll y$ and $y \ll x$ implies $x = y$;

where, $x \ll y$ is defined by $0 \in x \circ y$, and for every $A, B \subseteq H$, $A \ll B$ means, for all $a \in A$ there exists $b \in B$ such that $a \ll b$. We say H is *bounded hyper BCK-algebra* if there exists an element $e \in H$ such that, for all $x \in H$, $x \ll e$ and such e is called the *unit* of H. If $x \wedge y = y \wedge x$, where $x \wedge y = y \circ (y \circ x)$, for all $x, y \in H$, then H is called a *commutative hyper BCK-algebra.*

Proposition 5.2. [4] *Let $\mathcal{H} = (H, \circ, 0)$ be a hyper BCK-algebra. Then the following conditions hold, for all $x, y, z \in H$ and $A, B, C \subseteq H$:*

(i) $x \ll x$;
(ii) $A \ll A$;
(iii) $x \in x \circ 0$;
(iv) $A \circ 0 = A$;
(v) $y \ll z$ *imply* $x \circ z \ll x \circ y$;
(vi) $0 \circ x = 0$.

Theorem 5.3. *Let $\mathcal{H}$ be a separated hyper EQ-algebra such that $z \to (y \to x) = y \to (z \to x)$, for all $x, y, z \in H$. Then $\mathcal{H}$ is a hyper BCK-algebra.*

Proof. By considering $x \circ y := y \to x$, $0 := 1$, $x \ll_1 y$ if and only if $1 \in x \circ y = (y \to x)$ and $A \ll_1 B$ means $B \ll A$, we have:

(HK1): Since by Proposition 3.3(xxvi), $y \to x \ll (z \to y) \to (z \to x)$, then $(x \circ z) \circ (y \circ z) \ll_1 x \circ y$.
(HK2): Since by assumption, $z \to (y \to x) = y \to (z \to x)$, then $(x \circ y) \circ z = (x \circ z) \circ y$.
(HK3): Since by Proposition 3.3(xvi), $x \ll y \to x$, then $x \circ y \ll_1 \{x\}$, for all $x, y \in H$.
(HK4): If $x \ll_1 y$ and $y \ll_1 x$, then $1 \in y \to x = ((x \wedge y) \sim y)$ and $1 \in x \to y =$

$((x \wedge y) \sim x)$. Since H is separated, then $x \wedge y = y$ and $x \wedge y = x$. Thus $x \leq y$ and $y \leq x$, i. e. $x = y$.
Therefore, $(H, \circ, 0)$ is a hyper BCK-algebra. ∎

Example 5.4. Since in Example 3.9(ii), $\mathcal{H}$ is a separated hyper EQ-algebra such that, $x \to (y \to z) = y \to (x \to z)$, for all $x, y, z \in H$, then $\mathcal{H}$ is a hyper BCK-algebra.

Definition 5.5. [9] A *super lattice* is a partially ordered set $(S, \leq)$ endowed with two binary hyperoperations $\vee$ and $\wedge$ satisfying the following properties, for all $x, y, z \in S$,
(SL1) $x \in (x \vee x) \cap (x \wedge x)$;
(SL2) $x \vee y = y \vee x$, $x \wedge y = y \wedge x$;
(SL3) $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
(SL4) $x \in ((x \vee y) \wedge x) \cap ((x \wedge y) \vee x)$;
(SL5) $x \leq y$ implies $y \in x \vee y$ and $x \in x \wedge y$;
(SL6) if $x \in x \wedge y$ or $y \in x \vee y$ then $x \leq y$.

Definition 5.6. [9] By a *hyper residuated lattice* we mean a non-empty set L endowed with four binary hyperoperations $\vee$, $\wedge$, $\otimes$, $)$ and two constants 0 and 1 satisfying the following conditions:
(HRL1) $(L, \leq, \vee, \wedge, 0, 1)$ is a bounded super lattice;
(HRL2) $(L, \otimes, 1)$ is a commutative semihypergroup with 1 as the identity;
(HRL3) $x \otimes z \ll y$ if and only if $z \ll x)y$;
where $A \ll B$ means that $x \leq y$, for some $x \in A$ and $y \in B$.

Note. Let $\mathcal{H}$ be a hyper BCK-algebra with unite e. If $e \circ (e \circ x) = x$, for any $x \in H$, then $e \circ x$ is singleton for any $x \in H$. Since, let $u, v \in e \circ x$. Then $e \circ u \subseteq e \circ (e \circ x) = x$, and so $e \circ u = x$. Since $v \in e \circ x \subseteq e \circ (e \circ u) = u$, then $u = v$. Therefore $e \circ x$ is singleton.

Theorem 5.7. *Let $\mathcal{H} = (H, \circ, 0)$ be a commutative and bounded hyper BCK-algebra with unit e and $e \circ (e \circ x) = x$, for all $x \in H$. Then $\mathcal{H}$ is a hyper residuated lattice.*

Proof. Let $\mathcal{H} = (H, \circ, 0)$ be a commutative and bounded hyper BCK-algebra with unit e and $e \circ (e \circ x) = x$, for all $x \in H$. We define $x' := e \circ x$, $x \to y := y \circ x$, $1 := 0$, $0 := e$, $x \otimes y := (x \to y')'$, $x \vee y := (y \to x) \to x = (x \to y) \to y$, $x \wedge y := (x' \vee y')'$. Moreover, we define $x \leq y$ if and only if $y \in x \vee y$ and $x \ll_1 y$ if and only if $1 \in y \circ x = x \to y$. By these definitions and assumption we have $x'' = x$ and by Proposition 5.2 (v), if $x \ll_1 y$, then $y' \ll_1 x'$ and by (HK2),

$$\begin{aligned} y' \to x' &= e \circ y \to e \circ x = (y \to e) \to (x \to e) = x \to ((y \to e) \to e) \\ &= x \to y'' = x \to y. \end{aligned}$$

Now, we prove that $\mathcal{H} = (H, \wedge, \vee, \otimes, 0, 1)$ is a hyper residuated lattice.
(HRL1): We show that $(H, \leq, \vee, \wedge, 0, 1)$ is a bounded super lattice. By Proposition 5.2 (i) and (iii), $x \in 1 \to x \subseteq (x \to x) \to x = x \vee x$, then $x \leq x$.

Hence $\leq$ is reflexive. Now, let $x \leq y$ and $y \leq x$. Then by assumption $x, y \in x \vee y = (x \to y) \to y = (y \to x) \to x$. By (HK3), $(x \to y) \to y \ll y$ and $(y \to x) \to x) \ll x$. Thus $x \ll y$ and $y \ll x$, and so by (HK4), $x = y$. Then $\leq$ is antisymmetric. Let $x \leq y$ and $y \leq z$. Since $y \in x \vee y = (y \to x) \to x$, then $z \in y \vee z = (y \to z) \to z \subseteq ((((y \to x) \to x) \to z) \to z)$. By (HK3), $(y \to x) \to x \ll x$. Thus by Proposition 5.2(v), $x \to z \ll (((y \to x) \to x) \to z)$, and so $((((y \to x) \to x) \to z) \to z) \ll ((x \to z) \to z) = ((z \to x) \to x \ll x$. Thus $1 \in x \to z$ and so $z \in (1 \to z) \subseteq ((x \to z) \to z) = x \vee z$. Hence $z \in x \vee z$, i.e. $x \leq z$. Then $\leq$, is transitive. Therefore, $(H, \leq)$ is a partially ordered set.
(SL1): According to the above mention $x \in x \vee x$. Since $x \wedge x = (x' \vee x')' = ((x' \to x') \to x')'$ and x', x'' are singleton and $1 \in x \to x = x' \to x'$, then $(1 \to x')' \subseteq ((x' \to x') \to x')'$. By Proposition 5.2(iv), $(1 \to x')' = (x')' = x$. Hence $x \in ((x' \to x') \to x')' = x \wedge x$, Therefore, $x \in (x \vee x) \cap (x \wedge x)$.
(SL2): Since H is commutative, then $x \vee y = (x \to y) \to y = (y \to x) \to x = y \vee x$ and $x \wedge y = (x' \vee y')' = ((x' \to y') \to y')' = ((y' \to x') \to x')' = y \wedge x$.
(SL3): By (HK2) and commutativity,

$$\begin{aligned}
x \vee (y \vee z) &= (x \to ((z \to y) \to y)) \to ((z \to y) \to y) \\
&= ((z \to y) \to (x \to y)) \to ((z \to y) \to y) \\
&= (z \to y) \to (((z \to y) \to (x \to y)) \to y) \\
&= (z \to y) \to (y' \to ((z \to y) \to (x \to y))') \\
&= y' \to ((z \to y) \to ((z \to y) \to (x \to y))') \\
&= y' \to (((z \to y) \to (x \to y)) \to (z \to y)') \\
&= y' \to (((x \to y)' \to (z \to y)') \to (z \to y)') \\
&= y' \to (((z \to y)' \to (x \to y)') \to (x \to y)') \\
&= y' \to (((x \to y) \to (z \to y)) \to (x \to y)') \\
&= ((x \to y) \to (z \to y)) \to (y' \to (x \to y)') \\
&= ((x \to y) \to (z \to y)) \to ((x \to y) \to y) \\
&= (z \to ((x \to y) \to y)) \to (x \vee y) \\
&= (z \to (x \vee y)) \to (x \vee y) \\
&= (x \vee y) \vee z
\end{aligned}$$

By the similar way we can prove the second part.
(SL4): By (HK2) and commutativity,

$$\begin{aligned}
(x \vee y) \wedge x &= ((x \to y) \to y)) \wedge x = (((x \to y) \to y)' \vee x')' \\
&= ((((x \to y) \to y)' \to x') \to x')' \\
&= ((x \to ((x \to y) \to y)) \to x')' = (((x \to y) \to (x \to y)) \to x')'.
\end{aligned}$$

By Proposition 5.2(ii), $1 \in (x \to y) \to (x \to y)$, and so $(1 \to x')' \subseteq (((x \to y) \to (x \to y)) \to x')'$. By Proposition 5.2(iv), $(1 \to x')' = (x')' = x$. Then $x \in (((x \to y) \to (x \to y)) \to x')'$, i.e. $x \in (x \vee y) \wedge x$. By the similar way we can prove the second part.

(SL5): Let $x \leq y$. Then $y \in x \vee y = (y \to x) \to x$. By (HK3), $(y \to x) \to x \ll x$ and so $y \ll x$. Hence $1 \in x \to y$, and so by Proposition 5.2(iii), $y \in (1 \to y) \subseteq (x \to y) \to y = x \vee y$. By the definition,

$$x \wedge y = (x' \vee y')' = ((x' \to y') \to y')' = ((y' \to x') \to x')' = ((x \to y) \to x')'$$

Since $1 \in x \to y$, we have $(1 \to x')' \subseteq ((x \to y) \to x')'$. By Proposition 5.2(iv), $(1 \to x')' = (x')' = x$. Then $x \in ((x \to y) \to x')' = x \wedge y$.
(SL6): Let $y \in x \vee y$. Then $x \leq y$. Now, let $x \in x \wedge y = (x' \vee y')'$. Since x' is singleton, then $x' \in ((x' \vee y')')' = x' \vee y' = (x' \to y') \to y'$. By (HK3), $(x' \to y') \to y' \ll y'$, then $x' \ll y'$, and so by Proposition 5.2(v), $(y')' \ll (x')'$, i.e. $y \ll x$. Thus $1 \in x \to y$. By Proposition 5.2(vi), $y \in 1 \to y \subseteq (x \to y) \to y = x \vee y$. Hence $x \leq y$. Therefore, (HRL1) is hold.
(HRL2): By definition and assumption, $x \otimes y = (x \to y')' = ((y')' \to x')' = (y \to x')' = y \otimes x$ and by (HK2),

$$\begin{aligned} x \otimes (y \otimes z) &= (x \to ((y \to z')')')' = (x \to (y \to z'))' \\ &= (y \to (x \to z'))' = (y \to (z \to x'))' \\ &= (z \to (y \to x'))' = ((y \to x')' \to z')' = (x \otimes y) \otimes z. \end{aligned}$$

By Proposition 5.2(iii) and since x' is singleton, then $x' \in 1 \to x'$. Hence $x \otimes 1 = (x \to 1')' = ((1')' \to x')' = (1 \to x')'$. Thus $(x')' \in (1 \to x')'$ or $x \in 1 \otimes x$. Therefore, $(H, \otimes, 1)$ is a commutative semihypergroup with 1 as a identity.
(HRL3): Let $x \otimes z \ll_1 y$. Then $y \ll (x \to z')'$. Hence there exists $t \in (x \to z')'$ such that $y \ll t$, and so $1 \in t \to y$. Thus $1 \in (x \to z')' \to y$. By (HK2),

$$1 \in (x \to z')' \to y = y' \to (x \to z') = y' \to (z \to x') = z \to (y' \to x') = z \to (x \to y)$$

Then there exists $u \in x \to y$ such that $1 \in z \to u$, and so $u \ll z$. Hence $z \ll_1 x \to y$. Now let $z \ll_1 x \to y$, then $x \to y \ll z$ and so, for all $t \in x \to y$, $t \ll z$. Hence

$$\begin{aligned} 1 \in z \to t \subseteq z \to (x \to y) &= x \to (z \to y) = x \to (y' \to z') \\ &= y' \to (x \to z') = (x \to z')' \to y = (x \otimes z) \to y \end{aligned}$$

Then there exists $v \in x \otimes z$ such that $1 \in v \to y$, and so $x \otimes z \ll_1 y$.
Therefore, $\mathcal{H} = (H, \vee, \wedge, \otimes, \to, 0, 1)$ is a hyper residuated lattice. ∎

Definition 5.8. [2] By a *weak hyper residuated lattice*, we mean a non-empty set L endowed with two binary operations $\vee, \wedge$, and two binary hyperoperations $\otimes, \to$ and two constants 0 and 1 satisfying the following conditions:
(WHRL1) $(L, \leq, \vee, \wedge, 0, 1)$ is a bounded lattice;
(WHRL2) $(L, \otimes, 1)$ is a commutative semihypergroup with 1 as the identity;
(WHRL3) $x \otimes z \ll y$ if and only if $z \ll x \to y$;
where $A \ll B$ means that $x \leq y$, for some $x \in A$ and $y \in B$. Also, $A \leq B$ means that for any $x \in A$, there exists $y \in B$ such that $x \leq y$.

Proposition 5.9. [2] *Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a weak hyper residuated lattice. Then the following conditions hold, for all $x, y, z \in L$ and $A, B, C \subseteq L$:*
(i) $x \leq y$ *implies* $1 \in x \to y$ *and* $A \ll B$ *implies* $1 \in A \to B$;
(ii) $A \ll B \to C$ *if and only if* $A \otimes B \ll C$ *if and only if* $B \ll A \to C$;
(iii) $x \otimes y \ll x, y$ *and* $A \otimes B \ll A, B$;
(iv) $x \ll y \to x, A \ll B \to A$ *and* $1 \in x \to (y \to x)$;
(v) $x \otimes (x \to y) \ll x, y$;
(vi) $x \leq y$ *implies* $x \otimes z \ll y \otimes z, z \to x \leq z \to y,\ y \to z \leq x \to z$;
(vii) $(x \to y) \otimes (y \to z) \ll x \to z$ *and* $y \to z \ll (x \to y) \to (x \to z)$.

Proposition 5.10. *Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a weak hyper residuated lattice such that $A \ll B$ implies $A \leq B$ and " $\leftrightarrow$ " is biresiduation operation, which is defined by*

$$x \leftrightarrow y = (x \to y) \wedge (y \to x)$$

Then the following conditions holds, for all $x, y, z \in L$;
(i) $x \leftrightarrow y \ll (z \leftrightarrow x) \to (z \leftrightarrow y)$;
(ii) $x \leftrightarrow y \ll (z \leftrightarrow x) \leftrightarrow (z \leftrightarrow y)$;
(iii) $(x \leftrightarrow y) \otimes (y \leftrightarrow z) \ll x \leftrightarrow z$;
(iv) $(x \leftrightarrow y) \otimes (z \leftrightarrow t) \ll (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow t)$;
(v) $((x \wedge y) \leftrightarrow z) \otimes (t \leftrightarrow x) \ll (z \leftrightarrow (t \wedge y))$.

Proof. At first notice that by Proposition 5.9(vi) and assumption, we have $z \otimes (x \wedge y) \ll (z \otimes x) \wedge (z \otimes y)$ and if $A \ll B$ and $B \ll C$, then $A \ll C$.
(i): By Proposition 5.9(iii) and (vii),

$$\begin{aligned}
& (x \leftrightarrow y) \otimes (z \leftrightarrow x) \\
= \ & ((x \to y) \wedge (y \to x)) \otimes ((z \to x) \wedge (x \to z)) \\
\ll \ & ((x \to y) \otimes (z \to x)) \wedge ((x \to y) \otimes (x \to z)) \wedge ((y \to x) \\
& \otimes (z \to x)) \wedge ((y \to x) \otimes (x \to z)) \\
\ll \ & ((z \to y) \wedge ((x \to y) \otimes (x \to z)) \wedge ((y \to x) \otimes (z \to x)) \wedge (y \to z) \\
\ll \ & (z \to y) \wedge (y \to z) = y \leftrightarrow z
\end{aligned}$$

Then by (WHRL3), $(x \leftrightarrow y) \ll (z \leftrightarrow x) \to (z \leftrightarrow y)$.
(ii): By (i), $x \leftrightarrow y \ll (z \leftrightarrow x) \to (z \leftrightarrow y)$ and $y \leftrightarrow x \ll (z \leftrightarrow y) \to (z \leftrightarrow x)$. Since $x \leftrightarrow y = y \leftrightarrow x$, then $x \leftrightarrow y \ll ((z \leftrightarrow x) \to (z \leftrightarrow y)) \wedge ((z \leftrightarrow y) \to (z \leftrightarrow x)) = ((z \leftrightarrow x) \leftrightarrow (z \leftrightarrow y))$.
(iii): By Proposition 5.9 (iii) and (vii),

$$\begin{aligned}
& (x \leftrightarrow y) \otimes (y \leftrightarrow z) \\
& = ((x \to y) \wedge (y \to x)) \otimes ((y \to z) \wedge (z \to y)) \\
& \ll ((x \to y) \otimes (y \to z)) \wedge ((x \to y) \otimes (z \to y)) \wedge ((y \to x) \\
& \otimes (y \to z)) \wedge ((y \to x) \otimes (z \to y)) \\
& \ll (x \to z) \wedge ((x \to y) \otimes (z \to y)) \wedge ((y \to x) \otimes (y \to z)) \wedge (z \to x) \\
& \ll (x \to z) \wedge (z \to x) = (x \leftrightarrow z).
\end{aligned}$$

(iv): By (iii),

$$(x \leftrightarrow y) \otimes (z \leftrightarrow t) \otimes (x \leftrightarrow z) \ll (x \leftrightarrow y) \otimes (x \leftrightarrow t) = (t \leftrightarrow x) \otimes (x \leftrightarrow y) \ll t \leftrightarrow y = y \leftrightarrow t$$

Then by (WHRL3), $(x \leftrightarrow y) \otimes (z \leftrightarrow t) \ll (x \leftrightarrow z) \to (y \leftrightarrow t)$. By the same way we have, $(x \leftrightarrow y) \otimes (z \leftrightarrow t) \ll (y \leftrightarrow t) \to (x \leftrightarrow z)$. Hence $(x \leftrightarrow y) \otimes (z \leftrightarrow t) \ll (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow t)$.

(v): First, we show that:

$$((x \wedge y) \to z) \otimes (t \to x) \ll (t \wedge y) \to z \qquad (1)$$

$$(z \to (x \wedge y)) \otimes (x \to t) \ll z \to (t \wedge y) \qquad (2)$$

(1): By Proposition 5.9(iii) and (v),

$$(t \wedge y) \otimes (t \to x) \ll (t \otimes (t \to x)) \wedge (y \otimes (t \to x)) \ll x \wedge y.$$

Then by Proposition 5.9(v) and (vi),

$$(t \wedge y) \otimes ((x \wedge y) \to z) \otimes (t \to x) \ll (x \wedge y) \otimes (x \wedge y \to z) \ll z,$$

and so, $((x \wedge y) \to z) \otimes (t \to x) \ll (t \wedge y) \to z$.

By the similar way, we can prove (2).

Now the same as (iii) and (iv), we can prove (v). ∎

Theorem 5.11. *Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a weak hyper residuated lattice such that $A \ll B$ means $A \leq B$. Then $\mathcal{L} = (L, \wedge, \otimes, \leftrightarrow, 1)$ is a separated hyper EQ-algebra.*

Proof. (HEQ1): By Definition 5.8, it is clear that $(L, \wedge, 1)$ is an idempotent commutative monoid ($x \leq y$ if and only if $x \wedge y = x$).

(HEQ2): By Definition 5.8, $(L, \otimes, 1)$ is a commutative semihypergroup with identity "1" and by Proposition 5.9(vi), $\otimes$ is isotone.

Let $x \sim y := x \leftrightarrow y$, for all $x, y \in H$. Then

(HEQ3): Since $((x \wedge y) \sim z) \otimes (t \sim x) = ((x \wedge y) \leftrightarrow z) \otimes (t \leftrightarrow x)$. Then by Proposition 5.10(v), $((x \wedge y) \leftrightarrow z) \otimes (t \leftrightarrow x) \ll (z \leftrightarrow (t \wedge y))$. Then

$$((x \wedge y) \sim z) \otimes (t \sim x) \ll (z \leftrightarrow (t \wedge y)) = (z \sim (t \sim y)).$$

(HEQ4): By Proposition 5.10(iv),

$$(x \sim y) \otimes (z \sim t) = (x \leftrightarrow y) \otimes (z \leftrightarrow t) \ll (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow t) = (x \sim z) \sim (y \sim t)$$

(HEQ5): Since $x \wedge y \leq x$, then by Proposition 5.9(i), $1 \in ((x \wedge y) \to x)$. Hence by definition $\ll$, we have $(x \wedge y \wedge z) \to x \ll (x \wedge y) \to x$. By Proposition 5.9(vi), $x \to (x \wedge y \wedge z) \ll x \to (x \wedge y)$. Hence,

$$\begin{aligned}(x \wedge y \wedge z) \sim x &= (x \wedge y \wedge z) \leftrightarrow x = ((x \wedge y \wedge z) \to x) \wedge (x \to (x \wedge y \wedge z)) \\ &\ll ((x \wedge y) \to x) \wedge (x \to (x \wedge y)) = (x \wedge y) \leftrightarrow x \\ &= (x \wedge y) \sim x\end{aligned}$$

(HEQ6): Since $x \wedge y \wedge z \leq x \wedge z$, then by Proposition 5.9(i), $1 \in (x \wedge y \wedge z) \to x \wedge z$. Hence $((x \wedge y) \to x) \ll ((x \wedge y \wedge z) \to (x \wedge z))$. By Proposition 5.9 (iii) and (v),

$$(x \to (x \wedge y)) \otimes (x \wedge z) \ll (x \otimes (x \to (x \wedge y))) \wedge (z \otimes (x \to (x \wedge y))) \ll x \wedge y \wedge z$$

Then by (WHRL3), $(x \to (x \wedge y)) \ll (x \wedge z) \to (x \wedge y \wedge z)$. Hence

$$\begin{aligned}(x \wedge y) \sim x &= (x \wedge y) \leftrightarrow x = (((x \wedge y) \to x)) \wedge (x \to (x \wedge y)) \\ &\ll (((x \wedge y \wedge z) \to (x \wedge z)) \wedge ((x \wedge z) \to (x \wedge y \wedge z))) \\ &= (x \wedge y \wedge z) \leftrightarrow (x \wedge z) \\ &= (x \wedge y \wedge z) \sim (x \wedge z)\end{aligned}$$

(HEQ7): By Proposition 5.9(iii) and (iv),

$$x \otimes y \leq x \wedge y \ll (x \to y) \wedge (y \to x) = x \leftrightarrow y = x \sim y.$$

Therefore, $\mathcal{L}$ is a hyper EQ- algebra.

Let $1 \in x \sim y = x \leftrightarrow y = (x \to y) \wedge (y \to x)$, then $1 \in x \to y$ and $1 \in y \to x$. By (WHRL2) and Proposition 5.9(v), $x \in x \otimes 1 \ll x \otimes (x \to y) \ll y$, then $x \leq y$. By the same way we have $y \leq x$, thus $x = y$. Therefore, $\mathcal{L}$ is separated hyper EQ-algebra. ∎

6. Conclusions and future works

In this paper, we introduce the concept of hyper EQ-algebras, which is a generalization of the concept of EQ-algebras and we give some properties and related results and relation between hyper EQ-algebra and hyper BCK-algebra and hyper (weak)residuated lattice. The good hyper EQ-algebra, some result on quotient structure and filter theory, could be topics for my next task.

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