ON THE SUBGROUPS OF TORSION-FREE GROUPS WHICH ARE SUBRINGS IN EVERY RING

A.M. Aghdam

Department Of Mathematics
University of Tabriz
Tabriz
Iran
e-mail: mehdizadeh@tabrizu.ac.ir

F. Karimi[1]
A. Najafizadeh

Mathematics Department
Payame Noor University
Tehran
Iran
e-mails: fatemehkarimi63@yahoo.com
ar_najafizadeh@yahoo.com

Abstract. In this paper we study the subgroups of a torsion-free Abelian group A which are subrings in every ring over A. In particular, we get a necessary and sufficient condition for the case of rank one and rank two torsion-free groups. Moreover, we introduce the notion of SR-group and obtain some related results.

Keywords: ring on a group, nil group, idempotent type, SR-group.

2000 Mathematics Subject Classification: 20K15.

1. Introduction

All groups considered in this paper are Abelian, with addition as the group operation. Given an Abelian group A, we call R a ring over A if the group A is isomorphic to the additive group of R. In this situation we write $R = (A, *)$, where $*$ denotes the ring multiplication. This multiplication is not assumed to be associative. Every group may be turned into a ring in a trivial way, by setting all products equal to zero; such a ring is called a zero-ring. If this is the only

[1]Corresponding author e-mail: fatemehkarimi63@yahoo.com

multiplication over A, then A is said to be a nil group. Stratton [9] studies the type set of a torsion-free Abelian group of rank two which supports a non-zero ring and determines all the cases for the type set of such a group. Aghdam [1] uses the cases determined by Stratton to find the rings which may be defined on rank two torsion-free groups. We use the type set and the rings over a torsion-free Abelian group A of rank two to give necessary and sufficient conditions for the subgroups of A which are subrings in every ring on A. Moreover, we give such necessary and sufficient conditions for the subgroups of completely decomposable torsion-free groups with the given typeset. Finally, we introduce the notion of an SR-group and obtain some related results.

2. Notations and preliminary results

All groups in this paper are torsion-free Abelian and in general, we will follow the notation and conventions of [8]. For a group A and $x \in A$, the $p-$height, height sequence and the type of x are respectively denoted by $h_p^A(x), \chi_A(x), t(x)$ and if $\chi = (k_1, k_2, \ldots, k_n, \ldots)$ and $\mu = (l_1, l_2, \ldots, l_n, \ldots)$ are two height sequences, then their product is defined as

$$\chi\mu = (k_1 + l_1, k_2 + l_2, \ldots, k_n + l_n, \ldots).$$

Moreover, we will write $\langle x \rangle_*$ for the pure subgroup of A generated by x and if $T(A) = \{t(a) \mid 0 \neq a \in A\}$, then for any $t \in T(A)$ we consider $A(t) = \{a \in A \mid t(a) \geq t\}$, which is a fully invariant pure subgroup of A.

Furthermore, following [5], suppose that $\{x, y\}$ is any independent set of a rank two torsion-free group A, then each element a of A has the unique representation $a = ux + vy$, where u, v are rational numbers. Consider

$$U_0 = \{u_0 \in \mathbb{Q} : u_0 x \in A\},\ \ U = \{u \in \mathbb{Q} : ux + vy \in A \text{ for some } v \in \mathbb{Q}\}$$

$$V_0 = \{v_0 \in \mathbb{Q} : v_0 y \in A\},\ \ V = \{v \in \mathbb{Q} : ux + vy \in A \text{ for some } u \in \mathbb{Q}\}.$$

Clearly, U_0 and V_0 are subgroups of U and V respectively and U, U_0, V, V_0 are called the groups of rank one *belonging* to the independent set $\{x, y\}$.

We generalize these notions for any subgroup C of A as follows:

$$U_0^C = \{u_0 \in \mathbb{Q} : u_0 x \in C\},\ \ U^C = \{u \in \mathbb{Q} : ux + vy \in C \text{ for some } v \in \mathbb{Q}\}$$

$$V_0^C = \{v_0 \in \mathbb{Q} : v_0 y \in C\},\ \ V^C = \{v \in \mathbb{Q} : ux + vy \in C \text{ for some } u \in \mathbb{Q}\},$$

and for any subgroups R, S of $\mathbb{Q}$ we define

$$Rx \dotplus Sy = \{rx + sy \in A \mid r \in R, \quad s \in S\}.$$

For any subgroup C of a group A, let

$$\mathrm{I(C)} = \langle \varphi(\mathrm{C}) \mid \varphi \in \mathrm{Hom(A, E(A))} \rangle,$$

where E(A) is the endomorphism group of A, i.e, I(C) is the subgroup of E(A) generated by all homomorphic images of C in E(A). Now by an argument similar to that in Theorem 117.2 in [8], it is straightforward to see that:

Proposition 2.1 *A subgroup C of an abelian group A is a subring of every ring on A exactly if* I(C).C $\leq$ C.

Finally, by Theorem 3.3 of [9], we know if A is a non-nil torsion-free group of rank two, then $T(A)$ contains a unique minimal member and at most three elements.

Using this fact we can obtain the next proposition which describes the possible rings on rank two completely decomposable non-homogeneous groups and will be needed in Theorem 5.9:

Proposition 2.2 *Let $A = A_1 \oplus A_2$ be a completely decomposable non-homogeneous group of rank two with $t(A_1) = t_1, t(A_2) = t_2$. Let x and y be non-zero elements of A_1 and A_2 respectively. If A is non-nil, then any ring on A satisfies one of the following cases:*

1. $T(A) = \{t_0, t_1, t_2\}$ *with* $t_0 < t_1, t_0 < t_2$.

   (a) t_1, t_2 *are incomparable and in general* $x^2 = ax, y^2 = by, xy = yx = 0$ *for some* $a \in U_0$ *and* $b \in V_0$.

   (b) *If* $t_1^2 = t_1, t_2^2 \neq t_2$, *then* $x^2 = ax, y^2 = xy = yx = 0$ *for some* $a \in U_0$.

   (c) *If* $t_1^2 = t_1, t_2^2 = t_2$, *then* $x^2 = ax, y^2 = by, xy = yx = 0$ *for some* $a \in U_0$ *and* $b \in V_0$.

2. $T(A) = \{t_1, t_2\}$ *with* $t_1 < t_2$.

   (d) *If* $t_1^2 \neq t_1, t_2^2 = t_2$, *then* $x^2 = ay, y^2 = by, xy = cy, yx = dy$ *for some* $a, b, c, d, f \in V_0$.

   (e) *If* $t_1^2 \neq t_1, t_2^2 \neq t_2$, *then* $x^2 = ay, y^2 = 0, xy = cy, yx = dy$, *for some* $a, c, d \in V_0$.

   (f) *If* $t_1^2 = t_1, t_2^2 = t_2$, *then* $x^2 = a'x + by, y^2 = cy, xy = dy, yx = fy$, *for some* $a' \in U_0$ *and* $b, c, d, f \in V_0$, *in which if* $b \neq 0$, *then* $a' \neq 0$.

   (g) *If* $t_1^2 = t_1, t_2^2 \neq t_2$, *then* $x^2 = a'x + by, y^2 = 0, xy = dy, yx = fy$, *for some* $a' \in U_0$ *and* $a, b, c, d, f \in V_0$, *in which if* $b \neq 0$, *then* $a' \neq 0$.

Proof. 1) See [4, Proposition 2.7, Lemma 3.1 and Lemma 3.3].

2) (d) Clearly, $t(x^2) \geqq t(x)^2 = t_1^2 > t_1$. Now the hypothesis that $T(A)$ contains two elements, implies that $t(x^2) = t_2$. This yields $x^2 = ay$ for some $a \in V_0$. By the same reasoning, the other parts are obtained. ∎

3. Rank one and indecomposable rank two groups

In this section, we give for rank one and rank two torsion-free groups a necessary and sufficient condition for subgroups to be subrings in every ring.

Let A be a torsion-free group and C be a subgroup of A. We define the nucleus of C by:

$$N(C) = \{\alpha \in \mathbb{Q} \mid \alpha.x \in C, \ \forall x \in C\}.$$

Proposition 3.1 *Let A be a torsion-free group of rank one and C be a subgroup of A. Then C is a subring of every ring on A exactly if* $\mathrm{I(C)} \subseteq \mathrm{N(C)}$.

Proof. $\Longrightarrow$) Suppose that C is a subring of every ring on A, then by Proposition 2.1 we have $\mathrm{I(C).C} \leq \mathrm{C}$. But C, being a group of rank 1, is isomorphic to a subgroup of $\mathbb{Q}$, which means every generator of I(C) acts on C as multiplication by a rational number. Therefore $\mathrm{I(C).C} \leq \mathrm{C}$ implies that $r.c \in C$ for all $c \in C$ and $r \in \mathrm{I(C)}$. Consequently, $\mathrm{I(C)} \subseteq \mathrm{N(C)}$.
$\Longleftarrow$) Obvious. In fact, if $\mathrm{I(C)} \subseteq \mathrm{N(C)}$, then $\mathrm{I(C).C} \subseteq \mathrm{N(C).C} \subseteq \mathrm{C}$. ∎

Now, we consider rank two groups. Our arguments are based on the following cases for the typeset of a non-nil rank two group A which is a consequence of Theorem 3.3 in [9]:

(a) $|T(A)| = 1$; in this case the type must be idempotent.

(b) $|T(A)| = 2$; one of the types is minimal and the other is maximal.

(c) $|T(A)| = 3$; one of the types is minimal and the other two types are maximal. In this case, at least one of the maximal types is idempotent.

Theorem 3.2 *Let A is a torsion-free group of rank two and* $\mathrm{T(A)} = \{\mathrm{t_0, t_1, t_2}\}$ *such that $t_0 < t_1$ and $t_0 < t_2$. Let $x, y \in A$, such that $t(x) = t_1$ and $t(y) = t_2$. If $t_1^2 = t_1$, $t_2^2 \neq t_2$, then $x^2 = ax$, $y^2 = xy = yx = 0$ for some rational number a.*

Proof. See [4, Lemma 3]. ∎

Now, consider a subgroup T of U_0 defined as follows:

$$T = \{r \in \mathbb{Q} \mid x^2 = rx,\ y^2 = xy = yx = 0 \text{ yields a ring on } A\},$$

then we have:

Theorem 3.3 *Let A be a torsion-free group of rank two and* $\mathrm{T(A)} = \{\mathrm{t_0, t_1, t_2}\}$ *such that $t_0 < t_1, t_2$ and $t_1^2 = t_1$, $t_2^2 \neq t_2$. Let $x, y \in A$ such that $t(x) = t_1$ and $t(y) = t_2$. Then a subgroup C of A is a subring of every ring on A exactly if $TU^CU^C \subseteq U_0^C$. Moreover, if $r(C) = 1$, then C is a subring of every ring on A exactly if $C \leq V_0 y$ or $C = U_0^C(nx)$ for some integer n.*

Proof. Suppose that C is a subgroup which is a subring of any ring on A and let $c_1 = \alpha x + \beta y$, $c_2 = \gamma x + \nu y$ be two arbitrary elements of C. We know that every $\varphi \in \mathrm{Hom(A, End(A))}$ gives rise to a ring multiplication by defining the product of $a, a' \in A$ as $a.a' = (\varphi(a))(a')$. Therefore, in view of Theorem 3.2, $(\varphi(c_1))(c_2) = c_1.c_2 = \alpha\gamma r x$ for some $r \in T$. This implies that $\mathrm{I(C).C} = \mathrm{TU^C.U^Cx}$. Now, the first part is obtained from Proposition 2.1. But, if $r(C) = 1$, then suppose R is an arbitrary non-zero ring over A. So there exists a non-zero $a \in T$ such that $x^2 = ax$, $y^2 = xy = yx = 0$. Now, if $c = \alpha x + \beta y \in C$ be such that $\alpha \neq 0$ and $\beta \neq 0$, then $c^2 = \alpha^2 ax \neq 0$. This implies that a non-zero multiple of x is in C. Therefore, a non-zero multiple of y is in C. Hence C must be of rank two, which is a contradiction. Therefore, $\alpha = 0$ or $\beta = 0$. Consequently, $C \leq V_0 y$ or $C = U_0^C(nx)$ in which n is the smallest positive integer such that $nx \in C$. ∎

Theorem 3.4 *Let A be a torsion-free group of rank two and* $\mathrm{T(A)} = \{\mathrm{t}_0, \mathrm{t}_1, \mathrm{t}_2\}$ *such that* $t_0 < t_1$ *and* $t_0 < t_2$. *Let* $x, y \in A$ *such that* $t(x) = t_1$ *and* $t(y) = t_2$. *If* $t_1^2 = t_1$, $t_2^2 = t_2$, *then* $x^2 = ax$, $y^2 = by$, $xy = yx = 0$ *for some rational numbers* a, b *which are not both zero.*

Proof. See [1, Proposition 9]. ∎

In this case, consider the following subgroups of U_0 and V_0 respectively:

$T' = \{r \in \mathbb{Q} \mid x^2 = rx,\ y^2 = sy,\ xy = yx = 0$ yields a ring on A for some $s \in \mathbb{Q}\}$,

$S = \{s \in \mathbb{Q} \mid x^2 = rx,\ y^2 = sy,\ xy = yx = 0$ yields a ring on A for some $r \in \mathbb{Q}\}$,

then we have:

Theorem 3.5 *Let A be a torsion-free group of rank two and* $\mathrm{T(A)} = \{\mathrm{t}_0, \mathrm{t}_1, \mathrm{t}_2\}$ *such that* $\mathrm{t}_0 < \mathrm{t}_1, \mathrm{t}_2$ *and* $\mathrm{t}_1^2 = \mathrm{t}_1$, $\mathrm{t}_2^2 = \mathrm{t}_2$. *If* $x, y \in A$ *such that* $t(x) = \mathrm{t}_1$, $t(y) = \mathrm{t}_2$, *then any subgroup C of A is a subring of every ring on A exactly if* $T'U^CU^Cx \dotplus SV^CV^Cy \subseteq C$. *Moreover, if C is a rank one subgroup of A which is a subring of every ring on A, then* $C = U_0^C(nx)$ *or* $C = V_0^C(my)$ *for some integers m and n.*

Proof. The proof is similar to the proof of Theorem 3.3. In fact, if $c_1 = \alpha x + \beta y$ and $c_2 = ux + vy$ be two arbitrary elements of C and $\varphi \in \mathrm{Hom(A, End(A))}$, then

$$(\varphi c_1)(c_2) = c_1.c_2 = \alpha u a x + \beta v s y,$$

for some $a \in T'$ and $s \in S$. This implies $\mathrm{I(C).C} = \mathrm{T'U^CU^C} \dotplus \mathrm{SV^CV^C}$. Therefore, the first assertion follows from Proposition 2.1. Now, let C be a rank one subgroup of A which is a subring of every ring on A. Suppose that $c = \alpha x + \beta y \in C$, with $0 \neq \alpha, 0 \neq \beta$ and R be a ring on A with $x^2 = rx, y^2 = sy, xy = yx = 0$ for some $r \neq 0, s \neq 0$. Then $c^2 = \alpha^2 rx + \beta^2 sy \in C$. This yields a non-zero multiple of y and therefore a non-zero multiple of x lies in C. We conclude that C is of rank two, that is a contradiction. Hence $\alpha = 0$ or $\beta = 0$ and a similar argument to the used in Theorem 3.3 yields the result. ∎

Theorem 3.6 *Let A be an indecomposable torsion-free group of rank two and* $\mathrm{T(A)} = \{\mathrm{t}_1, \mathrm{t}_2\}$ *such that* $t_1 < t_2$. *If* $\{x, y\}$ *be an independent set such that* $t(x) = t_1$, $t(y) = t_2$, *then all non-trivial rings on A satisfy* $x^2 = by$, $xy = yx = y^2 = 0$, *for some rational number b.*

Proof. See [1, Lemma 3]. ∎

In this situation, we define the following subgroup of V_0 as follows

$$W = \{r \in \mathbb{Q} \mid x^2 = ry,\ y^2 = xy = yx = 0 \text{ yields a ring on A}\}.$$

Theorem 3.7 *Let A be an indecomposable torsion-free group of rank two and* $\mathrm{T(A)} = \{\mathrm{t}_1, \mathrm{t}_2\}$ *such that* $t_1 < t_2$. *Let* $\{x, y\}$ *be an independent set such that* $t(x) = t_1, t(y) = t_2$. *Then any subgroup C of A is a subring of every ring on A exactly if* $WU^CU^C \subseteq V_0^C$. *Moreover, any rank one subgroup of A which is a subring of every ring on A, is a subgroup of* $V_0 y$.

Proof. Let C be a subgroup of A. Then, in view of Theorem 3.6, for any arbitrary $\varphi \in \mathrm{Hom}(\mathrm{A}, \mathrm{End}(\mathrm{A}))$, $c_1 = ux + vy$, $c_2 = \alpha x + \beta y$ in C we have $(\varphi c_1)(c_2) = \alpha ury$ for some $r \in W$. This means $\mathrm{I}(A).C = WU^C U^C y$, thus the first assertion follows from Proposition 2.1. Now let C be a rank one subgroup of A which is a subring of every ring on A. Let $\alpha x + \beta y \in C$ be a non-zero element of C. We claim that $\alpha = 0$. By the way of contradiction, suppose that $\alpha \neq 0$. If R is any non-zero ring on A then there exists a non-zero $r \in W$ such that $x^2 = ry, xy = yx = y^2 = 0$. Hence $0 \neq \alpha^2 ry = c^2 \in C$. Therefore non-zero multiples of x and y are in C which means C is of rank two, a contradiction. ∎

4. Completely decomposable groups

By [8], any element a of a torsion-free group A with maximal independent set $\{x_1, x_2, ...\}$, has a unique representation $a = u_1x_1 + u_2x_2 + \cdots + u_nx_n$, for some positive integer n and $u_1, \cdots, u_n \in \mathbb{Q}$. Let

$$U_i = \{u_i \in \mathbb{Q} \mid \sum_{j=1}^{k} u_j x_j \in A, \ \ for \ \ some \ \ u_1, \cdots, u_{i-1}, u_{i+1}, ...u_k \in \mathbb{Q}\} \setminus \{0\}$$

and

$$U_{0i} = \{u_i \in \mathbb{Q} \mid u_i x_i \in A\}.$$

So, if $A = \bigoplus_{i=1}^{\infty} A_i$ is a completely decomposable group and $x_i \in A_i, i = 1, 2, ...,$ then $U_i = U_{0i}$ and it is straightforward to see that $t(U_{0i}) = t(A_i)$.

Moreover, since the typeset of a completely decomposable group has at most $r(A)$ maximal element, we can reduce our consideration of such groups to three cases:

(1) Homogeneous completely decomposable groups.

(2) Completely decomposable groups where the types of all elements in a maximal independent set of a group are maximal and incomparable.

(3) Completely decomposable groups where some types of elements in a maximal independent set are equal or are not maximal.

Theorem 4.1 *Let $A = \oplus_{i \in I} A_i$ be a homogeneous completely decomposable group. If A is non-nil, then A contains no non-trivial subgroup of rank less than $r(A)$, which is a subring of every ring on A.*

Proof. Let $t(A) = t$. If $x_i \in A_i$ and $\{x_1, x_2, ...\}$ is a maximal independent set of A, then $t(U_i) = t = t(U_iU_j)$ for all $i, j \in \{1, 2, ...\}$, because A is non-nil and hence $t^2 = t$. By the way of contradiction, suppose that C is a non-trivial subgroup of A with $r(C) < r(A)$ such that C is a subring of every ring on A. Let $0 \neq c = \sum_{i=1}^{n} \alpha_i x_i$ be an element of C. Then there exists $i \in \{1, 2, \cdots, n\}$ such that $\alpha_i \neq 0$. Without loss of generality suppose that $\alpha_1 \neq 0$. Since $t(U_1^2) = t(U_1) = t(U_2) = \cdots = t(U_n)$, there exist some non-zero integer numbers $m_1, m_2, ..., m_n, k_1, k_2, ..., k_n$ such that: $m_1U_1^2 = k_1U_1, m_2U_1^2 = k_2U_2, ..., m_nU_1^2 = k_nU_n$. Now we define $*_1$ as follows

$$x_i *_1 x_j = \begin{cases} m_1x_1 & \text{if } i = j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $u_1 = \beta_1x_1 + \cdots + \beta_nx_n$ and $u_2 = \gamma_1x_1 + \cdots + \gamma_nx_n$ are two arbitrary elements of A, then $u_1 *_1 u_2 = m_1\beta_1\gamma_1x_1$. But $m_1\beta_1\gamma_1 \in m_1U_1^2 = k_1U_1 \subseteq U_1$, hence $*_1$ is actually a ring on A. Now we have $c *_1 c = m_1.\alpha_1^2x_1 \in C$. Similarly we may define multiplications $*_2, *_3, ...$ on A such that for all $l = 2, 3, ...$ we have

$$0 \neq c *_l c = m_l\alpha_1x_l \in C.$$

This implies that $r(C) = r(A)$, a contradiction. ■

Now, if A is a homogeneous separable group of infinite rank and $\{x_1, x_2, ...\}$ be a maximal independent set of A, then, by Proposition 87.2 of [8], for all $i = 1, 2, ..., \langle x_i \rangle_*$ is a direct summand of A and hence $U_{0i} = U_i$. So, by a similar proof to the previous theorem we will have:

Proposition 4.2 *If A be a non-nil homogeneous separable group (of rank $\aleph_1$), then A contains no non-trivial subgroup of rank less than $r(A)$, which is a subring of every ring on A.*

Theorem 4.3 *Let $A = \oplus_{i\in I}A_i$ be a completely decomposable group and $S = \{x_i \mid x_i \in A_i,\ i \in I\}$ a maximal independent set of A such that $t(x_i) = t_i s$ are maximal and incomparable in $T(A)$ for all $i \in I$. Then*

(i) *Any rank one subgroup C which is a non-zero subring of every ring on A, is of the form $C = U_i^C(mx_i)$ with $t_i^2 = t_i, m \in \mathbb{Z} \setminus \{0\}$.*

(ii) *Any subgroup C of rank k which is a subring of every ring on A, is generated by $l(\leq k)$ rational multiples of some elements in S with idempotent types and $k - l$ combinations with rational coefficients of some elements in S with non-idempotent types. Moreover, if $l \neq 0$ then C is a nonzero subring.*

Proof. (i) Let C be any rank one subgroup of A which is a subring of every ring on A and let $c = \sum_{i=1}^{n} \alpha_ix_i$ be a non-zero element of C. We consider two cases. First suppose that $\alpha_i \neq 0$, for some $i \in \{1, 2, ..., n\}$ with $t^2(x_i) = t(x_i)$. For example let $\alpha_1 \neq 0$ and $t^2(x_1) = t(x_1)$. This implies $t(U_1^2) = t(U_1)$. Hence, similar to the proof of Theorem 4.1, there exists a non-zero integer m such that

$$x_i * x_j = \begin{cases} mx_1 & \text{if } i = j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

is a ring on A. Clearly, $0 \neq c * c = \alpha_1mx_1 \in C$. Now if $\alpha_j \neq 0$ for some $j \neq 1$, then $r(C) \geq 2$, which is a contradiction. Consequently, $C = U_1^C(mx_1)$ for some $m \in \mathbb{Z} \setminus \{0\}$.

If $c = \sum_{i=1}^{n} \beta_i x_i, t^2(x_i) \neq t(x_i)$, for all $i = 1, 2, ..., n$, then we know $t(x_i)$ is maximal and non-idempotent, hence for any ring on A we must have: $x_i x_i = x_i x_j = 0$ which yields C is zero subring.

(ii) Let C be a rank $k > 1$ subgroup of A which is a subring of every ring on A and let $\{c_1 = \alpha_{11}x_1 + \cdots + \alpha_{1n}x_n, c_2 = \alpha_{21}x_1 + \cdots + \alpha_{2n}x_n, ..., c_k = \alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n\}$ be a maximal independent set of C. If there exist $i \in \{1, 2, ..., k\}$ and $j \in \{1, 2, ..., n\}$ such that $\alpha_{ij} \neq 0$ and $t^2(x_j) = t(x_j)$, then as in case (i) there exist a non-zero integer m and a ring on A with $0 \neq c_i^2 = \alpha_{ij}^2 m x_j \in C$. Let $\alpha_{ij}^2 m = \beta_j$, hence there exist $c'_2, ..., c'_k \in C$ such that $\{\beta_j x_j, c'_2, ..., c'_k\}$ is an independent set in C and for all $i = 1, 2, ..., k$,

$$c'_i = \alpha'_{i1}x_1 + \cdots + \alpha'_{ij-1}x_{j-1} + \alpha'_{ij+1}x_{j+1} + \cdots + \alpha'_{in}x_n.$$

Repeating this procedure we get a maximal independent set in C

$$\{\beta_{j_1}x_{j_1}, ..., \beta_{j_l}x_{j_l}, c''_1, ..., c''_{k-l}\},$$

such that $t^2(x_{j_1}) = t(x_{j_1}), ..., t^2(x_{j_l}) = t(x_{j_l})$ and $c''_1, ..., c''_{k_l}$ are rational combinations of some elements in S with non-idempotent types.
In fact if C is a rank k subgroup of A which is a non-zero subring of every ring on A, then there exists an element $c = \sum_{i=1}^{n} \alpha_i x_i$ with $\alpha_i \neq 0$ and $t^2(x_i) = t(x_i)$. For otherwise, the maximality of types implies that $x_i x_j = x_j x_i = 0$ for all $i, j = 1, 2, ..., n$. This implies that C is a zero subring, a contradiction.

Now, our final assertion is obvious. ■

Theorem 4.4 *Let $A = \oplus_{i \in I} A_i$ be a completely decomposable group and $S = \{x_i \mid x_i \in A, i \in I\}$ a maximal independent set of A such that some of $t(x_i)$s are equal or are not maximal, Then any rank one subgroup C which is a non-zero subring of every ring on A, is of the form $C = U_i^C(lx_i)$ for some non-zero integer l with $t(x_i)$ idempotent.*

Proof. Let C be any rank one subgroup of A which is a subring of every ring over A and $c = \sum_{i=1}^{n} \alpha_i x_i \in C$. If $\alpha_i \neq 0$ for some i in which $t^2(x_i) = t(x_i)$, then by the proof of Theorem 4.1, there exists a non-zero integer m such that

$$x_r * x_s = \begin{cases} mx_i & \text{if } r = s = i, \\ 0 & \text{otherwise,} \end{cases}$$

yields a ring on A such that $c^2 = m\alpha_i^2 x_i \in C$. Moreover, by $r(C) = 1$ we obtain $\alpha_j = 0$ for all $j \neq i$. Consequently, $C = U_i^C(lx_i)$, for some non-zero integer l.

Now, suppose that C is a subring and any arbitrary element of C is of the form αc for some $\alpha \in \mathbb{Q}$ and $c = \sum_{j=1}^{n} \alpha_j x_j$ such that $t(x_j)$s are not idempotent.

Suppose there exists exactly one index $i \in J$ such that $\alpha_i \neq 0$. If there is a ring on A with $x_i^2 \neq 0$ then $t(x_i) < t^2(x_i) \leq t(x_i^2)$ and so $x_i^2 = \Sigma_k \beta_k x_k$, for some $\beta_k \in \mathbb{Q}$ and $x_k \in S$ with $t(x_k) > t(x_i)$. Hence $c^2 = \alpha_i^2 x_i^2 = \Sigma_k \beta_k \alpha_i^2 x_k \notin C$ which means C is not a subring of every ring over A.

This yields that if $C = \langle \alpha_i x_i \rangle$, with $t(x_i)$ non-idempotent, then C is a subring of any ring over A exactly if C be the zero subring. Moreover, if $C = \langle \alpha x_i + \beta x_j \rangle$ and α, β are both non-zero, then for any ring on A we must have

$$c^2 = \gamma c = \gamma(\alpha x_i + \beta x_j). \tag{$*$}$$

But

$$c^2 = \alpha^2 x_i^2 + \alpha\beta x_i x_j + \beta\alpha x_j x_i + \beta^2 x_j^2. \tag{$**$}$$

Now if $c^2 \neq 0$, for a ring on A, then we must have $(*) = (**)$. Hence if $x_i^2 \neq 0$, it must be equal to a rational multiple of x_j (because $t^2(x_i) \neq t(x_i)$) which yields

$$t(x_i) < t(x_j). \tag{1}$$

By the same reason, if $x_j^2 \neq 0$, then it must be a rational multiple of x_i, which yields

$$t(x_j) < t(x_i). \tag{2}$$

Moreover, if $x_i x_j \neq 0$ or $x_j x_i \neq 0$, then similarly they must be non-zero multiples of x_i or x_j. But in this case if, for example, $x_i x_j$ be a non-zero rational multiples of x_i, then $t(x_i) = t(x_i x_j) \geq t(x_i)t(x_j)$. Now if $t(x_i) = t(x_j)$, then $t(x_i)t(x_j) > t(x_i)$; because $t(x_i)$ is not idempotent, and if $t(x_i) \neq t(x_j)$ then it is clear that $t(x_i)t(x_j) > t(x_i)$. This yields $t(x_i) = t(x_i x_j) \geq t(x_i)t(x_j) > t(x_i)$ which is a contradiction. Thus, $x_i x_j = 0$. Similarly, $x_j x_i = 0$; and so $c^2 = \alpha x_i^2 + \beta x_j^2$, which never could be of the form $(*)$.

Thus, $C = \langle \alpha x_i + \beta x_j \rangle$ with $\alpha, \beta \neq 0$ and $t(x_i), t(x_j)$ non-idempotent, is a subring of A iff $x_i^2 = x_j^2 = 0$ for every ring on A and in this case C is a zero subring. [For example if $t(x_i)$ and $t(x_j)$ are maximal and non-idempotent, then for every ring on A we have $x_i^2 = x_j^2 = 0$ and C will be a zero subring of any ring over A.]

So, by induction we could see that A doesn't contain any subgroup of the form $C = \left\langle \sum_{j=1}^{n} \alpha_j x_j \right\rangle$ with the $t(x_j)$s non-idempotent and C is a non-zero subring of any ring on A. ■

5. $SR-$Groups

We begin with the definition:

A group A is called an SR-group if every subgroup of A is a subring in every ring over A.

The class of SR-groups is extensive since every nil group is an SR-group and a wide range of countable groups are nil; in fact it is even possible to exhibit a proper class (i.e., not a set) of nil groups. This is achieved by utilizing the following:

Proposition 5.1

(i) *If R is any countable reduced torsion-free ring with identity which has no zero divisors, then there is a countable reduced torsion-free group A which is a nil group and* $\mathrm{End(A)} \cong \mathrm{R}$.

(ii) *If R is a cotorsion-free ring (i.e. R^+ is torsion-free and reduced and contains no non-trivial pure-injective subgroup) with identity which has no zero divisors, then there exist arbitrarily large torsion-free groups A which are nil and* $\mathrm{End(A)} \cong \mathrm{R}$.

Proof. For part (i), we utilize Corner's famous realization theorem [[2], Theorem A] to produce a group A with $R \leq_* A \leq_* \hat{R}$, where $A = \langle R, e_r R\rangle_*$ for suitable elements $e_r = \alpha_r 1 + \beta_r r$, where α_r, β_r are algebraically independent over a certain subring of $\hat{\mathbb{Z}}$ containing $\mathbb{Z}$. To see that A is nil we compute $\mathrm{Hom(A, End(A))}$. Since $\mathrm{End(A)} \cong \mathrm{R}$ and $R \leq A$, we can construe any homomorphism $\varphi \in \mathrm{Hom(A, End(A))}$ as an endomorphism of A and hence it is equivalent to scalar multiplication on A by an element $s \in R$. Consequently $e_r s \in R$ for each $e_r \in A$ and so $(\alpha_r 1 + \beta_r r)s \in R$, giving $\beta_r rs \in R$. The algebraic independence of the β_r force $rs = 0$ for all r, and this in turn, by the assumption on R, forces $s = 0$, so that $\varphi = 0$. Since $\varphi \in \mathrm{Hom(A, End(A))}$ was arbitrary, we conclude that $\mathrm{Hom(A, End(A))} = 0$ and A is nil, as required.

The proof of (ii) is similar except that one uses a more powerful "Black Box"-type realization theorem (see e.g., [3]) to obtain the group A which is now sandwiched between a large free R-module B and its natural completion $\hat{B}$. Specifically, if $\lambda = \lambda^{\aleph_0}$ is a given cardinal, we chose B to be the direct sum of λ copies of R so that $|A| \geq \lambda$. The assumption that R has no zero divisors means that B and $\hat{B}$ are both torsion-free R-modules and hence so also is A. Thus every endomorphism of A is monic. That A is nil now follows from Proposition 121.2 of [8] or alternatively the argument in part(i) may be replaced by a support argument on the "branch" elements used in the construction of A. Finally, since there exist arbitrary large cardinals λ with $\lambda^{\aleph_0} = \lambda$, we can construct the desired group A to have cardinality exceeding an arbitrarily large cardinal; clearly this results in a proper class of such groups. ∎

Moreover, we have criteria to realize some groups which could not be SR-group.

Proposition 5.2 *Let M be a module over a commutative ring R. Suppose that $R^+ = A$, and $M^+ = B$. If there exist $m \in M$ and $r \in R$ such that $r^2 m \neq 0$, then $A \oplus B$ is not an SR-group.*

Proof. For $r_1, r_2 \in R$, and $m_1, m_2 \in M$ define $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1 + r_1r_2m)$. This multiplication induces a ring structure S on $A \oplus B$. However the subgroup $T = \{(r, 0) \mid r \in R\}$ is not a subring of S, because $(r, 0)(r, 0) = (r^2, r^2m) \notin T$. ∎

Corollary 5.3 *For every torsion-free group $B \neq 0$, the group $\mathbb{Z} \oplus B$ is not an SR-group.*

Proposition 5.4 *A direct summand of an SR-group is an SR-group.*

Proof. Let $A = B \oplus C$ is a SR-group and suppose that $S = (B, *)$ is a ring on B such that there exists a subgroup H of B which is not a subring of S. Now define $R = (A, .)$ by a multiplication $(b_1, c_1).(b_2, c_2) = (b_1 * b_2, 0)$. Then $\{(b, 0) \mid b \in H\}$ is a subgroup of A which is not a subring of R. This contradiction yields the result. ∎

Corollary 5.5 *If $\mathbb{Q}$ is a subgroup of an abelian group A, then A is not an SR-group.*

Proof. Follows from Proposition 5.4 and the fact that $\mathbb{Q}$ is not a SR-group. ∎

Corollary 5.6 *A torsion-free SR-group is reduced.*

Proposition 5.7 *A non-nil homogeneous completely decomposable group of rank greater than one is not an SR-group.*

Proof. By Theorem 4.1, every subgroup of rank one is not a subring of every ring on this group. ∎

Lemma 5.8 *Let $A = \bigoplus_{i \in I} A_i$ be a completely decomposable group with $r(A_i) = 1$. Then A is nil if and only if $t(A_i)t(A_j) \nleq t(A_k)$ for all $i, j, k \in I$.*

Proof. See [6, Corollary 2.1.3]. ∎

Now, we want to show that a non-nil completely decomposable group with rank greater than one is not SR-group.

So, let $A = \bigoplus_{i \in I} A_i$ be a completely decomposable group with $r(A_i) = 1$ and $|I| > 1$. By Lemma 5.8 we know that, if A is non-nil, then there exist $i, j, k \in I$ such that $t(A_i)t(A_j) \leq t(A_k)$ and so there exists $0 \neq \varphi \in Hom(A_i \otimes A_j, A_k)$. Therefore, we will have one of the following cases:

(I) $i \neq j \neq k$;

(II) $(i = j) \neq k$;

(III) $i = j = k$;

(IV) $i \neq j, j = k$.

Now, in each case we will find a direct summand of A which is not a SR-group and then by Proposition 5.4, A is not a SR-group.

(I) For convenience let $i = 1, j = 2, k = 3$. Then, by Proposition 5.4, it suffices to show that $A' = A_1 \oplus A_2 \oplus A_3$ is not a SR-group. As we saw there exist $0 \neq \varphi \in Hom(A_1 \otimes A_2, A_3)$. Let $a_i \in A_i$, $i = 1, 2$, be such that $\varphi(a_1 \otimes a_2) \neq 0$. Now suppose R is a ring on A' which is defined by the multiplication $(b_1, b_2, b_3)(c_1, c_2, c_3) = (0, 0, \varphi(b_1 \otimes c_2 + c_1 \otimes b_2))$ for all $b_i, c_i \in A_i$, $i = 1, 2, 3$.

Let B be the subgroup of A' generated by $\{(a_1, 0, 0), (0, a_2, 0) \mid a_1 \in A_1, a_2 \in A_2\}$. Then $(a_1, 0, 0)(0, a_2, 0) = (0, 0, \varphi(a_1 \otimes a_2)) \notin B$. Therefore B is a subgroup of A' which is not a subring of R. So A', and therefore A, is not an SR-group.

(II) $(i = j) \neq k$; in this case we have $t^2(A_i) \leq t(A_k)$ and hence there exists $0 \neq \varphi \in Hom(A_i \otimes A_i, A_k)$. Let $a_1, a_2 \in A_i$ be such that $\varphi(a_1 \otimes a_2) \neq 0$. Let R be a ring on $A' = A_i \oplus A_k$ which is defined by $(a_i, a_k)(a_i', a_k') = (0, \varphi(a_i \otimes a_i'))$. Then $R \neq 0$ and $(a_i, 0)(a_i', 0) = (0, \varphi(a_i \otimes a_i')) \notin A_i$. Hence A_i is a subgroup of A' which is not subring of A' and this completes this part.

(III) Let $i = j = k = 1$. Then $t(A_1)t(A_1) \leq t(A_1)$, because $t(A_i)t(A_j) \leq t(A_k)$, hence $t^2(A_1) = t(A_1)$.
If $t(A_1) = t(\mathbb{Z})$, then we have two cases:

(a) there exists some $l \in I$ such that $l \neq 1$ and $t(A_l) = t(\mathbb{Z})$. Then $B = A_1 \oplus A_l$ is a direct summand of A and by Proposition 5.7, B is not SR-group and so A is not SR-group too.

(b) if there exists some $l \neq 1$ such that $t(A_1) < t(A_l)$, then $t^2(A_1) = t(A_1)t(A_1) < t(A_1)t(A_l) = t(A_l)$, because $t(A_1) = (0, 0, ...)$. So, $t^2(A_1) \leq t(A_l)$ and by (II), $B = A_1 \oplus A_l$ is not SR-group and A will not be SR-group, too.

If at least one component of $t(A_1)$ is ∞, for example suppose the $p-$component of $t(A_1)$ is ∞. In this case, by Theorem 121.1, if R be a ring on A_1, then there exists an integer m such that $(m, p) = 1$, $R^2 = mR$ and $R \cong m\mathbb{Z}(p^{-1})$. Now consider the subgroup $C = \langle m/p \rangle$ of $(m\mathbb{Z}(p^{-1}))^+$. Since C is not a subring of $m\mathbb{Z}(p^{-1})$, the additive group of $m\mathbb{Z}(p^{-1})$ which is isomorphic to A_1 is not an SR-group and this implies that A is not an SR-group.

(IV) Let $i \neq (j = k)$ and so $t(A_i)t(A_k) \leq t(A_k)$ which means $t(A_i) \leq t(A_k)$ and therefore if $A' = A_i \oplus A_k$, then $|T(A')| = 2$.
If $t(A_i) = t(A_k)$ then A' is homogeneous and is not an SR-group. So consider $t(A_i) < t(A_k)$, which implies $t(A_k) > t(\mathbb{Z})$.
Now if at least one of the $t(A_i)$ or $t(A_k)$ is idempotent:

(a) $t(A_k)$ is idempotent. By $t(A_k) > t(\mathbb{Z})$, at least one of its component is ∞ and (III) yields the result.

(b) $t(A_i)$ is idempotent;
If $t(A_i) = t(\mathbb{Z})$, then $t^2(A_i) = t(A_i)t(A_i) \leq t(A_i)t(A_k) = t(A_k)$ and (II) yields the result.
If $t(A_i) \neq t(\mathbb{Z})$, then $t(A_i)$ has infinity as a component and (III) yields the result.

(c) If $t(A_i)$ and $t(A_k)$ are not idempotent then by Proposition 2.2, $B = \langle x + y \rangle; x \in A_i, y \in A_k$ is a subgroup of A' which is not a subring. This means A', and therefore A is not an SR-group.

As a result of $(I) - (IV)$ we are in a position to prove:

Theorem 5.9 *Let $A = \bigoplus_{i \in I} A_i$ be a completely decomposable torsion-free group with $r(A_i) = 1$. Then A is an SR-group exactly if $A \cong \mathbb{Z}$ or A is nil.*

Proof. It is clear that if $A \cong \mathbb{Z}$ or A is nil, then A is an SR-group.

Now, suppose that A is an SR-group. If A is a rank 1 group then A is nil or $A \cong \mathbb{Z}$; in fact, if A is a rank one non-nil group, and $t(A) = (k_1, k_2, \ldots, k_n, \ldots)$ then $k_i = \infty$ or $k_i = 0$ for all i. If there exists an i such that $k_i = \infty$, then A contains a subgroup which is isomorphic with a subgroup of $\mathbb{Q}^{(p_i)}$ which is clearly not a subring of A. Consequently, $t(A) = (0, 0, \ldots, 0, 0, \ldots)$ hence $A \cong \mathbb{Z}$.

Moreover, if $r(A) > 1$, then by $(I) - (IV)$ it couldn't be a non-nil SR-group. ∎

It would be interesting to know if there exists a non-nil SR-group other than the group of integers $\mathbb{Z}$.

Acknowlegment. This work was completed while the corresponding author was a visitor at the School of Mathematical Sciences, Dublin Institute of Technology. The author would like to thank Prof. B. Goldsmith and his colleagues for the opportunity of presenting her work at their seminar.

References

[1] Aghdam, A.M., *On the strong nilstufe of rank two torsion-free groups,* Acta. Sci. Math. (Szeged), 49 (1985), 53-61.

[2] Corner, A.L.S., *Every countable reduced torsion-free ring is an endomorphism ring,* Proc. London Math. Soc., 13 (1963), 687–710.

[3] Corner, A.L.S., R. Göbel, R., *Prescribing endomorphism algebras – A unified treatment,* Proc. London Math. Soc., 50 (1985), 447 -479.

[4] Aghdam, A.M., Najafizadeh, A., *Square subgroups of rank two abelian groups,* Colloq. Math., 117 (1) (2009), 19–28.

[5] Beaumont, R.A., Wisner, R.J., *Rings with additive group which is a torsion-free group of rank two,* Acta. Sci. Math. (Szeged), 20(1959), 105-116.

[6] FEIGELSTOCK, S., *The additive group of rings,* Pitman Research Notes, Boston-London, 1983.

[7] FEIGELSTOCK, S., *Additive groups of rings whose subrings are ideal,* Bull. Austral. Math. Soc., 55 (1997), 477–481.

[8] FUCHS, L., *Infinite abelian groups,* vol. 1, 2, Academic Press, New York, London, 1973.

[9] STRATTON, A.E., *The type set of torsion-free rings of finite rank,* Comment. Math. Univ. Sancti Pauli, 27 (1978), 199–211.

Accepted: 24.06.2012