NON-FACTORIZABLE GROUPS

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Abstract. A group G is called a factorizable group if there are proper subgroups A and B of G such that $G = AB$. If G is non-trivial and no such a factorization exists G is called a non-factorizable group. In this paper we will show that if G is a non-factorizable group with Frattini subgroup as a minimal normal subgroup of order p^n, $1 \leq n \leq 5$, where p is a prime number, then G is a quasi-simple group.

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1. Introduction and preliminaries

Let G be a group and A be a subgroup of G. If there is a subgroup B of G such that $G = AB$, then B is called a supplement of A in G. If A and B are non-trivial subgroups of G such that $G = AB$, then G is called a factorizable group. A non-factorizable group is a non-trivial group G such that for every subgroup A of G there is no subgroup B of G such that $G = AB$. The question of whether a group is factorizable or not is raised in [1] page 13. But finding all the factorizable groups is a difficult task, moreover there are groups which are not factorizable, for example finite cyclic groups of prime order or the Prüfer p-group $\mathbb{Z}(p^\infty)$. If $G = AB$ is a factorizable group with A and B maximal subgroup of G, then $G = AB$ is called a maximal factorization of G. If $G = AB$ is a finite factorizable group, then A and B are contained in maximal subgroups M and N of G respectively, hence $G = MN$ is a maximal factorization of G. Therefore, if a finite group does not have maximal factorization, then G is a non-factorizable group. In [8], the maximal factorizations of all the almost finite simple groups are found from which we can find all the non-factorizable finite simple groups. If $G = AB$ is a factorizable group, then the impact of A and B on the structure of G is studied extensively by some researcher. In [5] it is proved that if A and B are abelian, then G is metabelian. A generalization of Ito's result is given in [14] in which it is proved that if G is a finite group with A and B nilpotent and $(|A|, |B|) = 1$, then G is solvable. O. Kegel in [6] generalized this result of Wielandt without the assumption $(|A|, |B|) = 1$. Chernikov in [2] without the condition of finiteness on G proved that if A and B are nilpotent subgroups of G satisfying the minimum condition, then G is a solvable group.

Recently, factorizable groups which can be factored as product of two simple groups or an alternating group and a simple group have been studied. To mention a few result, one can see [13] and [3].

In [7] non-factorizable groups have been investigated. In particular the following result has been proved:

Result 1. Let G be a non-factorizable group, then G is one of the following three types of groups:

(i) a cyclic p-group,

(ii) a non-factorizable simple group,

(iii) a perfect group with $\Phi(G) \neq 1$ and $G/\Phi(G)$ a non-factorizable simple group.

In the above result $\Phi(G)$ denotes the Frattini subgroup of G which by definition is the intersection of all the maximal subgroups of G. And a perfect group is a group G with $G = G'$. Referring to Result 1, if G is a finite non-factorizable group, then $G \cong \mathbb{Z}_{p^n}$, $n \in \mathbb{N}$, or G is isomorphic to one of the simple groups listed in Table 4.1 of [7] or that a group satisfying condition (iii) of the Result 1, which are called type (iii) groups. In this paper our aim is to investigate type (iii) groups and obtain some properties of these groups.

2. Main result

In this section, our results are concerned with the type (iii) groups G as mentioned in part (iii) of Result 1. It is worth mentioning that if G is an arbitrary group and $G/\Phi(G)$ is a non-abelian simple group, then $G = G'$, i.e. G is a perfect group. Therefore to find type (iii) group G it is enough to investigate the possibility of the equality $G/\Phi(G)=S$ where S is a non-abelian non-factorizable simple group.

In [7] page 50, it is mentioned that the groups $SL_2(q)$ for $q\neq 7, 11, 19, 23, 29, 59$ or $q \equiv 1(mod\ 4)$ from a collection of type (iii) groups. But by [8] the group $PSL_2(q) = SL_2(q)/\Phi(SL_2(q))$ for all q's are factorizable, hence $SL_2(q)$ can not a type (iii) group. In Table I, using [8] we give the list of all the non-abelian non-factorizable simple groups:

Table I: non-abelian non-factorizable simple groups

Classical group	$U_{2m+1}(q)$ except $U_3(3)$, $U_3(5)$, $U_3(8)$, $U_9(2)$
Exceptional groups of Lie type	$E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$ (q *odd*), $G_2(q)$ ($q \neq 3^n$ or $q \neq 4$), ${}^3D_4(q^3)$, ${}^2E_6(q^2)$, ${}^2B_2(2^{2n+1})$, ${}^2F_4(2^{2n+1})$, ${}^2G_2(3^{2n+1})$
Sporadic groups	M_{22}, Mc, J_1, J_3, J_4, Ly, $O'N$, CO_2, CO_3, Fi_{23}, Fi'_{24}, HN, Th, BM, M

Let $G = SU_{2m+1}(q)$, $(m, q) \neq (1, 2)$. Then $G = G'$, $\Phi(G)$=$Z(G)$ is a cyclic group of order $(2m + 1, q - 1)$, and moreover $PSU_{2m+1}(q) = U_{2m+1}(q)$ =$G/Z(G)$ is an example of a type (iii) group. In this case G is called a quasi-simple group, or a covering group for the simple group $U_{2m+1}(q)$.

If G is a type (iii) group, then every proper normal subgroup N of G must be contained in every maximal subgroup of G, hence $N \leq\Phi(G)$ and $\Phi(G)$ is the unique maximal normal subgroup of G, in particular $Z(G) \leq\Phi(G)$. The case of equality, i.e. $Z(G) =\Phi(G)$, is interesting because in this case G would be the covering group of a simple group. In [7] and [9] the case that $|\Phi(G)|= p$ or p^2 are investigated and it is proved that $Z(G) =\Phi(G)$, although some part of the proof of Theorem 1 in [9] is in error.

Lemma 2.1 *Let H be the non-abelian group of order pq, where p, q are prime numbers, $p > q$, $q \mid p - 1$. Then $Aut(H)$ has order $p(p - 1)$ and is isomorphic to the group $\mathbb{Z}_p{}^* \rtimes \mathbb{Z}_p$ with the composition law*

$$(i, j)(k, l) = (ik, il + j)$$

where $i, k \in \mathbb{Z}_p{}^* = \mathbb{Z}_p - \{0\}$ and $j, l \in \mathbb{Z}_p$. Hence $Aut(H)$ is a metabelian group.

Proof. Any group of order pq, p and q prime, $p > q$, $q \mid p - 1$, is isomorphic to the group H with the following presentation, $H = \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^{\alpha}\rangle$, where $\alpha \equiv 1(mod\ p)$ and $\alpha^q \equiv 1(mod\ p)$. Since a Sylow p-subgroup $\langle x\rangle$ of H is normal in H it should be invariant under any automorphism ϕ of H. Let us set $\phi(x) = x^i$, $1 \leq i < p$. To define ϕ completely we must determine $\phi(y)$. Let $\phi(y) = x^j y^k$, where $0 \leq j < p$ and $0 \leq k < p$. We extend ϕ to the whole of G by being a homomorphism, so that $\phi(y^{-1}xy) = \phi(y)^{-1}\phi(x)\phi(y)$. Therefore $x^{i\alpha} = (x^j y^k)^{-1}x^i(x^j y^k) = y^{-k}x^i y^k = x^{i\alpha^k}$. Hence $x^{i\alpha(\alpha^{k-1}-1)} = 1$ which implies $p \mid i\alpha(\alpha^{k-1} - 1)$. Since p is prime to i and α, we obtain $p \mid \alpha^{k-1} - 1$, hence $\alpha^{k-1} \equiv 1(mod\ p)$. Using the condition $\alpha^q \equiv 1(mod\ p)$ we obtain $q \mid k - 1$, implying $y^k = y$. Finally we obtain $\phi(y) = x^j y$ where $0 \leq j < p$. With this definition the equality $\phi(y^q) = 1$ is also satisfied, hence: $Aut(H) = \{\phi_{i,j}|\phi_{i,j}(x) = x^i, \phi_{i,j}(y) = x^j y, 1 \leq i < p, 0 \leq j < p\}$. Now it is easy to verify that

$$\phi_{i,j}\phi_{k,l} = \phi_{ik,il+j}$$

and the isomorphism stated in the lemma holds. ∎

Theorem 2.2 *Let G be a type (iii) group, p and q distinct prime numbers. If $|\Phi(G)|= p^2$ or pq, then $\Phi(G)= Z(G)$, i.e. G is a quasi-simple group.*

Proof. In any case we have $G/C_G(\Phi(G))$ isomorphic to a subgroup of $Aut(\Phi(G))$. If $\Phi(G)$ is abelian, then $\Phi(G) \trianglelefteq C_G(\Phi(G)) \trianglelefteq G$, and because $\Phi(G)$ is a maximal normal subgroup of G we obtain $C_G(\Phi(G)) = \Phi(G)$or G.
If $C_G(\Phi(G)) = G$ then $\Phi(G) \leqslant Z(G)$ and then $\Phi(G)= Z(G)$, the theorem holds.

Next, we consider the case $C_G(\Phi(G)) = \Phi(G)$. Hence $G/\Phi(G)$ is a simple non-abelian non-factorizable subgroup of $Aut(\Phi(G))$. If $\Phi(G) \cong \mathbb{Z}_{p^2}$ is a cyclic group of order p^2, then $Aut(\Phi(G))$ is abelian and does not contain non-abelian simple group. If $\Phi(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $Aut(\Phi(G)) = GL_2(p)$ and it is easy to see that any simple subgroup of $GL_2(p)$ is isomorphic to a simple subgroup of $SL_2(p)$. But by [11] page 404, the group $SL_2(p)$ does not contain any simple subgroup. If $\Phi(G) \cong \mathbb{Z}_{pq}$ is a cyclic group of order pq, then $Aut(\Phi(G))$ is abelian, hence $G/C_G(\Phi(G))$ can not be isomorphic to a subgroup of $Aut(\Phi(G))$, a contradiction. Finally we consider the case that $\Phi(G)$is a non-abelian group of order pq, where p and q are distinct primes, $p > q$. In this case $G/C_G(\Phi(G))$ is isomorphic to a subgroup of $Aut(\Phi(G))$. Let us set $C = C_G(\Phi(G))$. By Lemma 2.1, $Aut(\Phi(G))$ is a metabelian group,and using the fact that $(G/C)'$ is abelian and $G' = G$, it follows that G/C is abelian. Hence $G' \leq C$ or $G \leq C$, i.e. $G = C_G(\Phi(G))$. It follows that $\Phi(G) \leq Z(G)$ which implies $\Phi(G)= Z(G)$ and again the theorem holds. ∎

Let G be a type (iii) group with $\mid G \mid$ as small as possible, i.e. if H is any type (iii) group with order less than $\mid G \mid$, then or equal to $H = G$. We know that $\Phi(G) \neq 1$ and $\Phi(G)$ is a nilpotent group. If $\Phi(G)$ contains a non-trivial subgroup N of G properly, then $\Phi(G/N) = \Phi(G)/N$ is non-trivial and $(G/N)' = G/N$, moreover

$$(G/N)/\Phi(G/N) = (G/N)/\Phi(G)/N \cong G/\Phi(G)$$

is a non-abelian non-factorizable simple group. Therefore G/N is a type (iii) group whose order is less than $\mid G \mid$. Hence we may assume that N is characteristically simple, so that it does not contain any normal subgroup of G. Thus $N = \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p = {\mathbb{Z}_p}^n$, the direct product of n copies of cyclic groups of order p. In this case $\Phi(G)$ is a minimal normal subgroup of G. Again we have $G/C_G(G)$ isomorphic to a subgroup of $Aut(\Phi(G)) \cong GL_n(p)$. Since $\Phi(G)$ is abelian we obtain $\Phi(G) \leq C_G(\Phi(G))$ hence $G = C_G(\Phi(G))$ or $\Phi(G)$. In the first case we obtain $\Phi(G)= Z(G)$, hence $\Phi(G)= Z(G)$ and G is a quasi-simple group. In the second case we obtain $G/\Phi(G) \leq GL_n(p)$. Therefore either $\Phi(G)= Z(G)$ and G is a quasi-simple group or $\Phi(G) \neq Z(G)$ and $G/\Phi(G)$ is isomorphic to a non-factorizable simple subgroups of $GL_n(p)$. In the next lemma we discuss such subgroups of $GL_n(p)$ in case $n = 3, 4$ and 5.

Lemma 2.3 *Let p be a prime number and $n \in \mathbb{N}, n \leq 5$. Then $GL_n(p)$ does not have a non-abelian non-factorizable simple group.*

Proof. It is obvious to see that any simple subgroup of $GL_n(p)$ should be contained in $SL_n(p)$. The case $n = 1$ is easy to see, and the case $n = 2$ follows from [11] page 404. If G is a simple non-abelian subgroup of $SL_n(p)$, then G is isomorphic to a subgroup of the simple group $PSL_n(p)$, $n \geq 2$. From the other hand since we assume that G is non-factorizable, hence G is one of groups listed in $TableI$. Now using the subgroup structure of the groups $PSL_n(p)$, $n \leq 5$, given in [12], [4] and [10] no possibilities for G exists. ∎

Theorem 2.4 *Let G be a type (iii) group. If $\mid \Phi(G) \mid = p^n$, $n \leq 5$, p a prime number, and $\Phi(G)$is an elementary abelian $p-$group, then G is a quasi-simple group.*

Proof. Since $\Phi(G)$is abelian we obtain $C_G(\Phi(G)) = \Phi(G)$ or G. If $C_G(\Phi(G)) = G$, then $\Phi(G) = Z(G)$. Hence G is a quasi–simple group. Next we will consider the case $C_G(\Phi(G)) = \Phi(G)$. We have $G/\Phi(G)$isomorphic to a non-factorizable non-abelian simple subgroup of $GL_n(p)$. But by *Lemma* 1.4 such subgroups don't exist in $GL_n(p)$ for $n \leq 5$, the theorem is proved. ∎

References

[1] Amberg,B. Franciosi, S. and Degiovanni, F., *Product of groups*, New York, Oxford University Press, 1992.

[2] Chernikov, N.S., *Infinite groups that are products of nilpotent groups*, Dokl. Akad. Nauk SSSR, 252 (1980), 57-60 (Soviet Math. Dokl.) 21, (1980).

[3] Darafsheh, M.R., —it Finite groups which factor as product of an alternating group and a symmytric group, Comm. Alg., vol 32, no. 2, (2004), 637-647.

[4] Dimartino, L. and Wager, A., *The irreducible sbgroups of $PSL(V_5, q)$, where q is odd*, Resultate Math., 2 (1979), 54-61.

[5] Ito, N., *Uber das product von zwei abelschen gruppen*, Math. Z., 62 (1955), 400-401.

[6] Kegel, O., *Product nilpotenter gruppen*, Arch. Math. (Basel), 12 (1961), 90-93.

[7] Kirtland, J., *Finding proper factorizations in finite groups*, Algebra and Discrete Mathematics, 2 (2009), 45-59.

[8] Liebeck, M.W., Praeger, C.E. and Saxl, J., *The maximal factorizations of the finite simple groups and their automorphism groups*, Mem. AMS, 86 (432), 1990.

[9] Mahmiani, A., *On factorizatoin of finite groups*, Journal of Algebra, Number Theory: Advances and Applications, vol. 3, no. 1 (2010), 1-4.

[10] Mitchell, H.H., *Determination of the ordinary and modular ternary linear groups*, Trans. Amer. Math. Soc., 12 (1911), 207-242.

[11] Suzuki, A., *Group Theory*. I, Springer-Verlag, 1982.

[12] Wagner, A., *The subgroups of* $PSL(5, 2^a)$, Resultate Math., 1 (2) (1978), 207-226.

[13] Walls, G., *Products of simple groups and symmetric groups*, Arch. Math., 58 (1992), 312-321.

[14] Wieldant, H., *Uber product nilpotenter gruppen*, Illinois J. Math., 2 (1958), 611-618.

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