MODIFIED (G'/G)-EXPANSION METHOD WITH GENERALIZED RICCATI EQUATION TO THE SIXTH-ORDER BOUSSINESQ EQUATION

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Abstract. In this article, abundant traveling wave solutions of the sixth-order Boussinesq equation have been obtained in a uniform way by using the alternative (G'/G)-expansion method wherein the generalized Riccati equation is used. It is shown that the alternative (G'/G)-expansion method together with the generalized Riccati equation provides advance mathematical tool for solving nonlinear partial differential equations. Numerical results coupled with the graphical representation explicitly reveal the complete reliability and high efficiency of the proposed algorithm.

Keyword: (G'/G)-expansion method, sixth-order Boussinesq equation, traveling wave solutions, nonlinear evolution equations.

1. Introduction

It is well known that nonlinear evolution equations (NLEEs) are widely used as models to describe many important complex physical phenomena in various fields of science, such as, plasma physics, nonlinear optics, solid state physics, chemical kinematics, chemistry, biology and many others [1]-[51]. For better understanding of nonlinear phenomena as well as further applications in practical life, it is more significant to establish exact traveling wave solutions. In the recent past, a wide range of methods have been developed to generate analytical solutions by a diverse group of scientists. For instance, the Backlund transformation method [1], the inverse scattering method [2], the truncated Painleve expansion method [3], the Hirota's bilinear transformation method [4], the Jacobi elliptic function expansion method [5], the generalized Riccati equation method [6], the tanh-coth method [7], [8], the F-expansion method [9,10], the variational iteration method [11], [12],

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the direct algebraic method [13], the homotopy perturbation method [14]-[16], the Exp-function method [17]-[21] and others [22]-[28].

Another important method was introduced by Wang et al. [29] introduced a direct and concise method, called the (G'/G)-expansion method to look for traveling wave solutions of nonlinear partial differential equations, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda\, G'(\xi) + \mu\, G(\xi) = 0$; λ and μ are arbitrary constants. The (G'/G)-expansion method is one of the most powerful methods to solve nonlinear problems and several scientists investigated different kind of NLEEs to construct traveling wave solutions via this method and can be found in the articles [30]-[34] for better understanding.

In order to establish the effectiveness and reliability of the (G'/G)-expansion method and to expand the possibility of its application, further research has been carried out by several researchers. For instance, Zhang et al. [35] presented an improved (G'/G)-expansion method to seek more general traveling wave solutions. Zayed [36] presented a new approach of the (G'/G)-expansion method where $G(\xi)$ satisfies the Jacobi elliptic equation, $[G'(\xi)]^2 = e_2\, G^4(\xi) + e_1\, G^2(\xi) + e_0$, e_2, e_1, e_0 are arbitrary constants, and obtained new exact solutions. Zayed [37] again presented an alternative approach of this method in which $G(\xi)$ satisfies the Riccati equation $G'(\xi) = AG + BG^2(\xi)$ where A and B are arbitrary constants. Consequently, several researchers studied various nonlinear PDEs to generate traveling wave solutions via the improved (G'/G)-expansion method and can be found [38]-[41].

It is significant to observe that there exist some fundamental relationships among numerous complex nonlinear partial differential equations and some basic and soluble nonlinear ordinary differential equations (ODEs), such as the sine-Gordon equation, the sinh-Gordon equation, the Riccati equation, the Weierstrass elliptic equation etc. Therefore, it is natural to use the solutions of these nonlinear ODEs to construct exact solutions of various intricate nonlinear partial differential equations. Based on the relationships of complex nonlinear partial differential equations and ODEs, a number of methods, such as, the sinh-Gordon equation expansion method [42], the generalized F-expansion method [43], [44], the projective Riccati equation method [45], [46], the algebraic method [47] etc. have been developed.

In the present article, we combine the generalized Riccati equation with the (G'/G)–expansion method, called alternative (G'/G)-expansion method introduced recently by Akbar et al. [48] to find the exact traveling wave solutions of the sixth-order Boussinesq equation.

2. The Alternative (G'/G)-Expansion Method

Suppose we have the following nonlinear partial differential equation,

$$P(u,\, u_t,\, u_x,\, u_{tt},\, u_{xx},\, u_{xt}, \cdots) = 0. \tag{1}$$

where $u = u(x,\ t)$ is an unknown function, P is a polynomial in $u = u(x,\ t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. The main steps of the alternative (G'/G)-expansion method are:

Step 1. The traveling wave variable,

$$u\,(x,\,t) = u\,(\xi), \xi = x - V\,t, \tag{2}$$

where V is the speed of the traveling wave, which converts the Eq. (1) into an ODE in the form,

$$Q\,(u,\,u', u'',\,u''',\,\cdots) = 0, \tag{3}$$

where prime denotes the derivative with respect toξ.

Step 2. If Eq. (3) is integrable, integrate it term by term one or more times, yields constant(s) of integration.

Step 3. Suppose that the solution of the Eq. (3) can be expressed by means of a polynomial in (G'/G) as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\tfrac{G'}{G}\right)^i, a_n \neq 0 \tag{4}$$

where$G = G(\xi)$ satisfies the generalized Riccati equation,

$$G' = r + p\,G + q\,G^2, \tag{5}$$

where a_i $(i = 1,\,2,\,3, \cdots, n)$, $p,$ q and r are random constants to be determined later.

The generalized Riccati Eq. (5) has the following twenty seven solutions [49].

Family 1.
When $p^2 - 4\,qr < 0$ and $p\,q \neq 0\,($ or $\quad qr \neq 0)$, the solutions of Eq. (5) are,

$G_1 = \frac{1}{2\,q}\left[-p + \sqrt{4qr - p^2}\tan\left(\frac{1}{2}\sqrt{4qr - p^2}\xi\right)\right]$,

$G_2 = -\frac{1}{2\,q}\left[p + \sqrt{4qr - p^2}\cot\left(\frac{1}{2}\sqrt{4qr - p^2}\xi\right)\right]$,

$G_3 = \frac{1}{2\,q}\left[p + \sqrt{4qr - p^2}\left(\tan\left(\sqrt{4qr - p^2}\xi\right) \pm \sec\left(\sqrt{4qr - p^2}\xi\right)\right)\right]$,

$G_4 = -\frac{1}{2\,q}\left[p + \sqrt{4qr - p^2}\left(\cot\left(\sqrt{4qr - p^2}\xi\right) \pm \csc\left(\sqrt{4qr - p^2}\xi\right)\right)\right]$,

$G_5 = \frac{1}{4\,q}\left[-2p + \sqrt{4qr - p^2}\left(\tan\left(\frac{1}{4}\sqrt{4qr - p^2}\xi\right) - \cot\left(\frac{1}{4}\sqrt{4qr - p^2}\xi\right)\right)\right]$,

$G_6 = \frac{1}{2q}\left[-p + \frac{\sqrt{(A^2-B^2)\,(4qr-p^2)} - A\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\xi\right)}{A\sin\left(\sqrt{4qr-p^2}\xi\right)+B}\right]$,

$G_7 = \frac{1}{2q}\left[-p + \frac{\sqrt{(A^2-B^2)\,(4qr-p^2)} + A\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\xi\right)}{A\sin\left(\sqrt{4qr-p^2}\xi\right)+B}\right]$,

where A and B are two non-zero real constants and satisfies the condition $A^2 - B^2 > 0$.

$$G_8 = \frac{-2\,r\,\cos\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)}{\sqrt{\ 4qr-p^2}\,\sin\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)+p\cos\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)}\ ,$$

$$G_9 = \frac{2\,r\,\sin\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)}{-p\,\sin\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)+\sqrt{(4qr-p^2)}\,\cos\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)},$$

$$G_{10} = \frac{-2\,r\,\cos\left(\sqrt{4qr-p^2}\xi\right)}{\sqrt{(4qr-p^2)}\,\sin\left(\sqrt{4qr-p^2}\xi\right)+p\,\cos\left(\sqrt{4qr-p^2}\xi\right)\pm\sqrt{(4qr-p^2)}}\ ,$$

$$G_{11} = \frac{2\,r\,\sin\left(\sqrt{4qr-p^2}\xi\right)}{-p\,\sin\left(\sqrt{4qr-p^2}\xi\right)+\sqrt{(4qr-p^2)}\,\cos\left(\sqrt{4qr-p^2}\xi\right)\pm\sqrt{(4qr-p^2)}}\ ,$$

$$G_{12} = \frac{4\,r\,\sin\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)}{-2p\,\sin\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)+2\sqrt{(4qr-p^2)}\,\cos^2\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)-\sqrt{(4qr-p^2)}}\ .$$

Family 2.

When $p^2 - 4\,qr > 0$ and $p\,q \neq 0$ (or $qr \neq 0$), the solutions of Eq. (5) are,

$$G_{13} = -\frac{1}{2\,q}\left[p+\sqrt{p^2-4qr}\tanh\left(\frac{1}{2}\sqrt{p^2-4qr}\,\xi\right)\right],$$

$$G_{14} = -\frac{1}{2\,q}\left[p+\sqrt{p^2-4qr}\coth\left(\frac{1}{2}\sqrt{p^2-4qr}\,\xi\right)\right],$$

$$G_{15} = -\frac{1}{2\,q}\left[p+\sqrt{p^2-4qr}\left(\tanh\left(\sqrt{p^2-4qr}\,\xi\right)\pm i\,\mathrm{sec}\,h\left(\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{16} = -\frac{1}{2\,q}\left[p+\sqrt{p^2-4qr}\left(\coth\left(\sqrt{p^2-4qr}\xi\right)\pm \mathrm{csc}\,h\left(\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{17} = -\frac{1}{4\,q}\left[2p+\sqrt{p^2-4qr}\left(\tanh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)+\coth\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{18} = \frac{1}{2q}\left[-p+\frac{\sqrt{(A^2+B^2)\,(p^2-4qr)}-A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)}{A\,\sinh\left(\sqrt{p^2-4qr}\,\xi\right)+B}\right],$$

$$G_{19} = \frac{1}{2q}\left[-p-\frac{\sqrt{(B^2-A^2)\,(p^2-4qr)}+A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)}{A\,\sinh\left(\sqrt{p^2-4qr}\,\xi\right)+B}\right],$$

where A and B are two non-zero real constants and satisfies the condition $B^2 - A^2 > 0$.

$$G_{20} = \frac{2\,r\,\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\,\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)-p\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}\ ,$$

$$G_{21} = \frac{2\,r\,\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\,r\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)-p\,\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}\ ,$$

$$G_{22} = \frac{2\,r\,\cosh\left(\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\,\sinh\left(\sqrt{p^2-4qr}\xi\right)-p\,\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm i\sqrt{p^2-4qr}}\ ,$$

$$G_{23} = \frac{2\,r\,\sinh\left(\sqrt{p^2-4qr}\xi\right)}{-p\,\sinh\left(\sqrt{p^2-4qr}\xi\right)+\sqrt{p^2-4qr}\,\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm\sqrt{p^2-4qr}}\ ,$$

$$G_{24} = \frac{4\,r\,\sinh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)}{-2p\,\sinh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)+2\sqrt{p^2-4qr}\,\cosh^2\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)-\sqrt{p^2-4qr}}\ .$$

Family 3.

When $r = 0$ and $p\,q \neq 0$,the solutions of Eq. (5) are,

$$G_{25} = \frac{-p\,d}{q\,[d+\cosh(p\,\xi)-\sinh(p\,\xi)]},$$

$$G_{26} = -\frac{p\,[\cosh(p\,\xi)+\sinh(p\,\xi)]}{q\,[d+\cosh(p\,\xi)+\sinh(p\,\xi)]},$$

where d is an arbitrary constant.

Family 4.
When $q \neq 0$ and $r = p = 0$, the solution of Eq. (5) is,
$G_{27} = -\frac{1}{q\,\xi + d_1}$,
where d_1 is an arbitrary constant.

Step 4. In Eq. (4), n is a positive integer which is usually obtained by balancing the highest order nonlinear term(s) with the linear term(s) of the highest order come out in Eq. (3).

Step 5. Substituting Eq. (4) into Eq. (3) and utilizing Eq. (5), we obtain polynomials in G^i and G^{-i} $(i = 0, 1, 2, 3, \cdots)$. Vanishing each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for a_n, p, q, r, V and constant(s) of integration, if applicable. Suppose with the aid of symbolic computation software such as Maple, the unknown constants a_n, p, q, r and V can be found by solving these set of algebraic equations and substituting these values into Eq. (4), we obtain new exact traveling wave solutions of the nonlinear partial differential equation (1).

3. Some new traveling wave solutions of the sixth-order Boussinesq equation

In this section, the alternating (G'/G)-expansion method together with the generalized Riccati equation is employed to construct some new traveling wave solutions for the sixth-order Boussinesq equation [50], [51]:

$$u_{tt} - u_{xx} - [15\,u\,u_{4x} + 30\,u_x u_{3x} + 15\;(u_{2x})^2 + 45\,u^2 u_{2x} + 90\,u\,u_x^2 + u_{6x}] = 0, \tag{1}$$

Now, using the traveling wave variable (2) in Eq. (1), we have

$$V^2 u'' - u'' - [15\,u\,u^{(4)} + 30\,u' u''' + 15\;(u'')^2 + 45\;u^2\,u'' + 90\,u\;(u')^2 + u^{(6)}] = 0, \tag{2}$$

where $u^{(4)}$ and $u^{(6)}$ denotes the fourth and sixth derivative of u with respect to ξ. Integrating Eq. (2) once, we obtain:

$$C - \left(1 + 45\,u^2 - V^2\right)\; u' - 15\,u'u'' - 15\,u\,u''' - u^{(5)} = 0. \tag{3}$$

where C is constant of integration. According to Step 3, the solution of Eq. (3) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = a_0 + a_1\,(\tfrac{G'}{G}) + a_2\,(\tfrac{G'}{G})^2 + ... + a_n\,(\tfrac{G'}{G})^n, \quad a_n \neq 0 \tag{4}$$

where $a_i, (i = 0, 1, 2, 3, \cdots n)$ are constant to be determined and $G = G(\xi)$ satisfies the generalized Riccati Eq. (5). Considering the homogeneous balance between the highest order derivative and the nonlinear terms in Eq. (3), we obtain $n = 2$.

Therefore, the solution of Eq. (4) takes the form:

$$u(\xi) = a_0 + a_1\,(\tfrac{G'}{G}) + a_2\,(\tfrac{G'}{G})^2, \quad a_2 \neq 0 \tag{5}$$

Using Eq. (5), Eq. (5) can be rewritten as:

$$u(\xi) = a_0 + a_1\,(p + r\,G^{-1} + q\,G) + a_2\,(p + r\,G^{-1} + q\,G)^2. \tag{6}$$

Substituting Eq. (6) into Eq. (3), we obtain the following polynomials inG^i and G^{-i}, $(i = 0,\, 1,\, 2,\, 3, \cdots)$. Setting each coefficient of these resulted polynomial to zero, we obtain a set of simultaneous algebraic equations for $a_0,\ a_1,\ a_2,\ p,\ q,\ r$ and V as follows:

$540\,a_2^2\,q^7 + 720\,a_2\,q^7 + 90\,a_2^3\,q^7 = 0,$

$600\,a_1\,a_2\,q^6 + 25\,a_1a_2^2\,q^6 + 540\,a_2^3\,p\,q^6 + 2490\,a_2^2\,p\,q^6 + 2640\,a_2\,p\,q^6$
$+120\,a_1\,q^6 = 0,$

$2220a_1a_2pq^5 + 4590a_2^2p^2q^5 + 1350\,a_2^3\,p^2q^5 + 120\,a_1^2q^5 + 360\,a_0a_2q^5$
$+1740\,a_2^2rq^6 + 180a_0a_2^2q^5 + 1125a_1a_2^2pq^5 + 180a_2a_1^2q^5 + 450a_2^3rq^6$
$+3720a_2p^2q^5 + 1680a_2rq^6 + 360a_1pq^5 = 0,$

$3105\,a_1a_2p^2q^4 + 345a_1^2\,pq^4 + 240a_1rq^5 + 6120\,a_2^2rpq^5 + 720\,a_0a_2^2pq^4$
$+2160\,a_2^3rpq^5 + 990a_0a_2pq^4 + 900a_1a_2^2rq^5 + 270a_0a_1a_2q^4 + 1800a_2^3p^3q^4$
$+1500a_1a_2rq^5 + 2250a_1a_2^2p^2q^4 + 4440a_2rpq^5 + 390a_1p^2q^4 + 4260a_2^2p^3q^4$
$+720a_2a_1^2pq^4 + 45a_1^3q^4 + 90a_0a_1q^4 + 2490a_2p^3q^4 = 0,$

$782\,a_2p^4q^3 + 180a_1p^3q^3 + 1232a_2r^2q^5 + 90a_0a_1^2q^3 + 90\,a_2a_0^2q^3$
$+135\,a_1^3pq^3 + 810a_2^3r^2q^5 + 1350a_2^3p^4q^3 + 240a_1^2rq^3 + 345a_1^2p^2q^3$
$+3960a_1a_2rpq^4 + 3375a_1a_2^2rpq^4 + 810a_0a_1a_2pq^3 + 1995a_1a_2p^3q^3$
$+540a_0a_2^2rq^4 + 1080a_0a_2^2p^2q^3 + 540a_1^2a_2rq^4 + 1080a_1^2a_2p^2q^3$
$+2250a_1a_2^2p^3q^3 + 4050a_2^3rp^2q^4 + 180a_0a_1pq^3 + 930a_0a_2p^2q^3$
$+600a_0a_2rq^4 + 8010a_2^2rp^2q^4 + 4064\,a_2rp^2q^4 + 480a_1rpq^4$
$+1980a_2^2r^2q^5 + 2040a_2^2p^4q^3 + 2a_2q^3 - 2V^2a_2q^3 = 0,$

$45\,a_1a_0^2q^2 + 136a_1r^2q^4 + 31a_1p^4q^2 + 94a_2p^5q^2 + 90\,a_1^3rq^3$
$+135\,a_1^3p^2q^2 + 540a_2^3p^5q^2 + 135a_1^2p^3q^2 + 3510a_1a_2rp^2q^3$
$+1020a_0a_2rpq^3 + 4500a_1a_2^2rp^2q^3 + 540a_0a_1a_2rq^3$
$+810a_0a_1a_2p^2q^2 + 1440a_0a_2^2rpq^3 + 1440a_2a_1^2rpq^3$
$+1200a_1a_2r^2q^4 + 450a_1^2rpq^3 + 555a_1a_2p^4q^2 + 180a_0^2a_2pq^2$
$+720a_0a_2^2p^3q^2 + 180a_0a_1^2pq^2 + 1125a_1a_2^2r^2q^4 - V^2a_1q^2$
$+720a_2a_1^2p^3q^2 + 1125a_1a_2^2p^4q^2 + 2700a_2^3r^2pq^4 + 3600\,a_2^3rp^3q^3$
$+120a_0a_1rq^3 + 105a_0a_1p^2q^2 + 330a_0a_2p^3q^2 + 4770a_2^2r^2pq^4$
$+4680a_2^2rp^3q^3 + 292a_1rp^2q^3 + 1984a_2r^2pq^4 + 1468a_2rp^3q^3$
$+450a_2^2p^5q^2 + a_1q^2 + 4a_2pq^2 - 4V^2a_2pq^2 = 0,$

$272a_2r^3q^4 + a_1p^5q + 810a_0a_1a_2rpq^2 + 2a_2p^6q + 45\,a_1^3p^3q$
$+90\,a_2^3p^6q + 1080a_1^2a_2rp^2q^2 + 2250a_2^2a_1p^3q^2 + 1740a_1a_2\,r^2pq^3$
$+1095a_1a_2\,rp^3q^2 + 120a_0a_1rpq^2 + 450a_0a_2rp^2q^2 + 15a_1^2p^4q$
$+30a_2^2p^6q + 780a_2^2r^3q^4 + 1080a_0a_2^2rp^2q^2 + 270a_0a_1a_2p^3q$
$+a_1pq - V^2a_1pq + 2a_2rq^2 + 450a_2^3r^3q^4 + 120a_1^2r^2q^3 + 2a_2p^2q$
$+2250a_1a_2^2r^2pq^3 + 90a_0a_1^2q - 2V^2a_2p^2q + 15a_0a_1p^3q + 30a_0a_2p^4q$
$+240a_0a_2r^2q^3 + 3420a_2^2r^2p^2q^3 + 1080a_2^2rp^4q^2 + 135a_1^3rpq^2 + 90a_0a_1^2rq^2$
$+360a_0a_2^2r^2q^3 + 2700a_2^3r^2p^2q^3 + 136a_1r^2pq^3 + 52a_1rp^3q^2 856a_2r^2p^2q^3$
$+45a_1a_2p^5q + 225a_1^2rp^2q^2 + 360a_2a_1^2r^2q^3 + 45a_1a_0^2pq + 90a_2a_0^2rq^2$
$+90a_2a_0^2p^2q + 180a_2a_1^2p^4q + 225a_1a_2^2p^5q + 1350a_2^3rp^4q^2 + 166a_2rp^4q^2$
$+180a_0a_2^2p^4q - 2V^2a_2rq^2 = 0,$

$$
\begin{aligned}
&94a_2r^2p^5+45a_1a_0^2r^2+31a_1r^2p^4+136a_1r^4q^2+135a_1^3r^2p^2\\
&+a_1r^2+540a_2^3r^2p^5+a_0a_1a_2r^3q+3510a_1a_2r^3p^2q\\
&+1020a_0a_2r^3pq+810a_0a_1a_2r^2p^2+90a_1^3r^3q+450a_2^2r^2p^5\\
&+1040a_0a_2^2r^3pq+1040a_2a_1^2r^3pq+4500a_2^2a_1r^3p^2q+135a_1^2r^2p^3\\
&+4a_2r^2p+120a_0a_1r^3q+105a_0a_1r^2p^2+330a_0a_2r^2p^3+3600a_2^3r^3p^3q\\
&+292a_1r^3p^2q+4680a_2^2r^3p^3q+4770a_2^2r^4pq^2+1200a_1r^4a_2q^2\\
&+450a_1^2r^3pq+555a_1r^2a_2p^4+180a_2a_0^2r^2p+180a_0a_1^2r^2p+1984a_2r^4pq^2\\
&+1468a_2r^3p^3q+1125a_2^2a_1r^4q^2+2700a_2^3r^4pq^2+720a_0a_2^2r^2p^3\\
&+720a_2a_1^2r^2p^3+1125a_2^2a_1r^2p^4-4V^2a_2r^2p-V^2a_1r^2=0,\\
&135\,a_1^3r^3p+180a_0a_1r^3p+1080\,a_0a_2^2r^3p^2+1350a_2^3r^3p^4+4064a_2r^4p^2q\\
&+3375a_1r^4a_2^2pq+8010r^4a_2^2p^2q+180a_1r^3p^3+930a_0a_2r^3p^2+810a_0a_1a_2r^3p\\
&+782a_2r^3p^4+3960a_1a_2r^4pq+90a_0a_1^2r^3+480a_1r^4pq+4050a_2^3r^4p^2q\\
&+2040a_2^2r^3p^4+600a_0a_2r^4q+240a_1^2r^4q+1080a_2a_1^2r^3p^2+1232a_2r^5q^2\\
&+810a_2^3r^5q^2+90a_2a_0^2r^3+2a_2r^3+1980a_2^2r^5q^2+345a_1^2r^3p^2+540a_0a_2^2r^4q\\
&+1995a_1a_2r^3p^3+2250a_1a_2^2r^3p^3+540a_2a_1^2r^4q-2V^2a_2r^3=0,\\
&720a_0a_2^2\,r^4p+4440a_2r^5pq+6120a_2^2r^5pq+240a_1r^5q+1800\,a_2^3r^4p^3\\
&+45\,a_1^3r^4+4260\,a_2^2r^4p^3+390a_1r^4p^2+720a_2a_1^2r^4p+345a_1^2r^4p\\
&+3105a_1a_2r^4p^2+2160a_2^3r^5pq+990a_0a_2r^4p+2250a_1a_2^2\,r^4p^2\\
&+270a_0a_1a_2\,r^4+90a_0a_1\,r^4+2490a_2\,r^4p^3+900a_1a_2^2\,r^5q+1500a_1a_2\,r^5q=0,\\
&4590a_2^2\,r^5p^2+360a_1r^5p+3720\,a_2r^5p^2+2220a_1a_2r^5p+1125\,a_1a_2^2r^5p\\
&+1350a_2^3r^5p^2+120a_1^2\,r^5+180a_1^2a_2r^5+180\,a_0a_2^2\,r^5+360a_0a_2r^5\\
&+1680a_2r^6q+1740a_2^2\,r^6q+450a_2^3r^6q=0,
\end{aligned}
$$

$$
\begin{aligned}
540a_2^3\,r^6p+2490a_2^2r^6p+2640a_2r^6p+600a_1a_2r^6+120a_1r^6\\
+225a_1a_2^2r^6=0,
\end{aligned}
\tag{7}
$$

$$540\,a_2^2r^7+90\,a_2^3r^7+720a_2r^7\;=0,$$

$C=0.$

Solving the above set of algebraic equations by using the symbolic computation software, such as, Maple, we obtain

$$
\begin{aligned}
a_0 &= -\frac{1}{3}p^2+\frac{4}{3}\,rq,\;\; a_1=2p,\;\; a_2=-2,\\
V &= \sqrt{p^4-8\,p^2q\,r+16\,q^2r^2+1},\;\; C=0,
\end{aligned}
\tag{8}
$$

where $p,\ q$ and r are arbitrary constants.

Now, on the basis of the solutions of Eq. (5), we obtain the following families of solutions of Eq. (1).

Family 1.
When $p^2-4\,qr<0$ and $p\,q\neq 0\,($ or $\quad qr\neq 0),$ the periodic form solutions of Eq. (1) are,

$$u_1=-\frac{1}{3}p^2+\frac{4}{3}rq+2p\left(\frac{2\Psi^2\sec^2(\Psi\xi)}{-p+2\Psi\tan(\Psi\xi)}\right)-2\left(\frac{2\Psi^2\sec^2(\Psi\xi)}{-p+2\Psi\tan(\Psi\xi)}\right)^2,$$

where $\Psi=\frac{1}{2}\sqrt{4qr-p^2}$, $\xi=x-\sqrt{p^4-8p^2qr+16q^2r^2+1}\;t$, p, q and r are arbitrary constants.

$$u_2=-\frac{1}{3}p^2+\frac{4}{3}rq-2p\left(\frac{2\Psi^2\csc^2(\Psi\xi)}{p+2\Psi\cot(\Psi\xi)}\right)-2\left(\frac{2\Psi^2\csc^2(\Psi\xi)}{p+2\Psi\cot(\Psi\xi)}\right)^2,$$

$$u_3=-\frac{1}{3}p^2+\frac{4}{3}rq+2p\left(\frac{4\Psi^2\sec(2\Psi\xi)(1\pm\sin(2\Psi\xi))}{-p\cos(2\Psi\xi)+2\Psi\sin(2\Psi\xi)\pm2\Psi}\right)$$
$$-2\left(\frac{4\Psi^2\sec(2\Psi\xi)(1\pm\sin(2\Psi\xi))}{-p\cos(2\Psi\xi)+2\Psi\sin(2\Psi\xi)\pm2\Psi}\right)^2,$$

$$u_4=-\tfrac{1}{3}p^2+\tfrac{4}{3}rq-2p\left(\frac{4\Psi^2\csc(2\Psi\xi)(1\pm\cos(2\Psi\xi))}{p\sin(2\Psi\xi)+2\Psi\cos(2\Psi\xi)\pm2\Psi}\right)-2\left(\frac{4\Psi^2\csc(2\Psi\xi)(1\pm\cos(2\Psi\xi))}{p\sin(2\Psi\xi)+2\Psi\cos(2\Psi\xi)\pm2\Psi}\right)^2,$$

$$u_5=-\tfrac{1}{3}p^2+\tfrac{4}{3}rq-2p\left(\frac{2\Psi^2\csc(\Psi\xi)}{p\sin(\Psi\xi)+2\Psi\cos(2\Psi\xi)}\right)-2\left(\frac{2\Psi^2\csc(\Psi\xi)}{p\sin(\Psi\xi)+2\Psi\cos(2\Psi\xi)}\right)^2,$$

$$u_6=-2p\left(\frac{4A\Psi^2\left\{\sqrt{A^2-B^2}\cos(2\Psi\xi)-B\sin(2\Psi\xi)-A\right\}\{A\sin(2\Psi\xi)+B\}}{\{A^2\cos^2(2\Psi\xi)-A^2-B^2-2AB\sin(2\Psi\xi)\}\left\{pA\sin(2\Psi\xi)+2A\Psi\cos(2\Psi\xi)+pB-2\Psi\sqrt{A^2-B^2}\right\}}\right)$$
$$-2\left(\frac{4A\Psi^2\left\{\sqrt{A^2-B^2}\cos(2\Psi\xi)-B\sin(2\Psi\xi)-A\right\}\{A\sin(2\Psi\xi)+B\}}{\{A^2\cos^2(2\Psi\xi)-A^2-B^2-2AB\sin(2\Psi\xi)\}\left\{pA\sin(2\Psi\xi)+2A\Psi\cos(2\Psi\xi)+pB-2\Psi\sqrt{A^2-B^2}\right\}}\right)^2$$
$$-\tfrac{1}{3}p^2+\tfrac{4}{3}rq,$$

$$u_7=-2p\left(\frac{4A\Psi^2\left\{\sqrt{A^2-B^2}\cos(2\Psi\xi)+B\sin(2\Psi\xi)+A\right\}\{A\sin(2\Psi\xi)+B\}}{\{A^2\cos^2(2\Psi\xi)-A^2-B^2-2AB\sin(2\Psi\xi)\}\left\{pA\sin(2\Psi\xi)-2A\Psi\cos(2\Psi\xi)+pB-2\Psi\sqrt{A^2-B^2}\right\}}\right)$$
$$-2\left(\frac{4A\Psi^2\left\{\sqrt{A^2-B^2}\cos(2\Psi\xi)+B\sin(2\Psi\xi)+A\right\}\{A\sin(2\Psi\xi)+B\}}{\{A^2\cos^2(2\Psi\xi)-A^2-B^2-2AB\sin(2\Psi\xi)\}\left\{pA\sin(2\Psi\xi)-2A\Psi\cos(2\Psi\xi)+pB-2\Psi\sqrt{A^2-B^2}\right\}}\right)^2$$
$$-\tfrac{1}{3}p^2+\tfrac{4}{3}rq,$$

where A and B are two non-zero real constants and satisfies the condition $A^2-B^2>0$.

$$u_8=-\tfrac{1}{3}p^2+\tfrac{4}{3}rq-2p\left(\frac{2\Psi^2\sec(\Psi\xi)\{p\cos(\Psi\xi)+2\Psi\sin(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)+4\Psi^2}\right)$$
$$-2\left(\frac{2\Psi^2\sec(\Psi\xi)\{p\cos(\Psi\xi)+2\Psi\sin(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)+4\Psi^2}\right)^2,$$

$$u_9=-\tfrac{1}{3}p^2+\tfrac{4}{3}rq+2p\left(\frac{2\Psi^2\csc(\Psi\xi)\{p\sin(\Psi\xi)-2\Psi\cos(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)-p^2}\right)$$
$$-2\left(\frac{2\Psi^2\csc(\Psi\xi)\{p\sin(\Psi\xi)-2\Psi\cos(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)-p^2}\right)^2,$$

$$u_{10}=-\tfrac{1}{3}p^2+\tfrac{4}{3}rq-2p\left(\frac{2\Psi^2\sec(2\Psi\xi)\{1\pm\sin(2\Psi\xi)\}\{p\cos(2\Psi\xi)+2\Psi\sin(2\Psi\xi)\pm2\Psi\}}{(p^2-2rq)\cos^2(2\Psi\xi)+2\Psi\{1\pm\sin(2\Psi\xi)\}\{2\Psi\pm p\cos(2\Psi\xi)\}}\right)$$
$$-2\left(\frac{2\Psi^2\sec(2\Psi\xi)\{1\pm\sin(2\Psi\xi)\}\{p\cos(2\Psi\xi)+2\Psi\sin(2\Psi\xi)\pm2\Psi\}}{(p^2-2rq)\cos^2(2\Psi\xi)+2\Psi\{1\pm\sin(2\Psi\xi)\}\{2\Psi\pm p\cos(2\Psi\xi)\}}\right)^2,$$

$$u_{11}=-\tfrac{1}{3}p^2+\tfrac{4}{3}rq\pm2p\left(\frac{2\Psi^2\csc(2\Psi\xi)\{-p\sin(2\Psi\xi)+2\Psi\cos(2\Psi\xi)\pm2\Psi\}}{(2rq-p^2)\cos(2\Psi\xi)-2p\{1\pm\sin(2\Psi\xi)\}\{2p\Psi\sin(2\Psi\xi)\pm2rq\}}\right)$$
$$-2\left(\frac{2\Psi^2\csc(2\Psi\xi)\{-p\sin(2\Psi\xi)+2\Psi\cos(2\Psi\xi)\pm2\Psi\}}{(2rq-p^2)\cos(2\Psi\xi)-2p\{1\pm\sin(2\Psi\xi)\}\{2p\Psi\sin(2\Psi\xi)\pm2rq\}}\right)^2,$$

$$u_{12}=-\tfrac{1}{3}p^2+\tfrac{4}{3}rq+2p\left(\frac{2\Psi^2\csc(\Psi\xi)\{p\sin(\Psi\xi)-2\Psi\cos(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)-p^2}\right)$$
$$-2\left(\frac{2\Psi^2\csc(\Psi\xi)\{p\sin(\Psi\xi)-2\Psi\cos(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)-p^2}\right)^2.$$

Family 2.

When $p^2-4qr>0$ and $pq\neq0$ (or $qr\neq0$), the soliton and soliton-like solutions of Eq. (1) are,

$$u_{13}=-\tfrac{1}{3}p^2+\tfrac{4}{3}rq+2p\left(\frac{2\Delta^2\sec h^2(\Delta\xi)}{p+2\Delta\tanh(\Delta\xi)}\right)-2\left(\frac{2\Delta^2\sec h^2(\Delta\xi)}{p+2\Delta\tanh(\Delta\xi)}\right)^2,$$

where $\Delta = \frac{1}{2}\sqrt{p^2-4qr}$, $\xi = x - \sqrt{p^4 - 8\,p^2qr + 16\,q^2r^2 + 1}\,t$, p, q and r are arbitrary constants.

$$u_{14} = -\tfrac{1}{3}p^2 + \tfrac{4}{3}\,rq - 2p\,\left(\tfrac{2\Delta^2 \csc h^2(\Delta\xi)}{p+2\Delta\coth(\Delta\xi)}\right) - 2\,\left(\tfrac{2\Delta^2 \csc h^2(\Delta\xi)}{p+2\Delta\coth(\Delta\xi)}\right)^2,$$

$$u_{15} = -\tfrac{1}{3}p^2+\tfrac{4}{3}\,rq+2p\,\left(\tfrac{4\Delta^2 \sec h\,(2\Delta\xi)\,(1\mp i\sinh(2\Delta\xi))}{p\cosh(2\Delta\xi)+2\Delta\sinh(2\Delta\xi)\pm i\,2\Delta}\right)-2\,\left(\tfrac{4\Delta^2 \sec h\,(2\Delta\xi)\,(1\mp i\sinh(2\Delta\xi))}{p\cosh(2\Delta\xi)+2\Delta\sinh(2\Delta\xi)\pm i\,2\Delta}\right)^2,$$

$$u_{16} = -\tfrac{1}{3}p^2+\tfrac{4}{3}\,rq-2p\,\left(\tfrac{4\Delta^2 \csc h\,(2\Delta\xi)\,(1\pm\cosh(2\Delta\xi))}{p\sinh(2\Delta\xi)+2\Delta\cosh(2\Delta\xi)\pm 2\Delta}\right)-2\,\left(\tfrac{4\Delta^2 \csc h\,(2\Delta\xi)\,(1\pm\cosh(2\Delta\xi))}{p\sinh(2\Delta\xi)+2\Delta\cosh(2\Delta\xi)\pm 2\Delta}\right)^2,$$

$$u_{17} = -\tfrac{1}{3}p^2 + \tfrac{4}{3}\,rq - 2p\,\left(\frac{\Delta^2 \sec h^2(\Delta\xi/2)}{2\left\{\cosh^2(\Delta\xi/2)-1\right\}\{p+\Delta(\tanh(\Delta\xi/2)+\coth(\Delta\xi/2))\}}\right)$$
$$-2\,\left(\frac{\Delta^2 \sec h^2(\Delta\xi/2)}{2\left\{\cosh^2(\Delta\xi/2)-1\right\}\{p+\Delta(\tanh(\Delta\xi/2)+\coth(\Delta\xi/2))\}}\right)^2,$$

$$u_{18} = -\tfrac{1}{3}p^2 + \tfrac{4}{3}\,rq - 2p\,\left(\frac{4A\Delta^2\left\{A-B\sinh(2\Delta\xi)-\sqrt{A^2+B^2}\cosh(2\Delta\xi)\right\}}{(A\sin(2\Delta\xi)+B)\left\{pA\sinh(2\Delta\xi)+pB-2\Delta\sqrt{A^2+B^2}+2A\Delta\cosh(2\Delta\xi)\right\}}\right)$$
$$-2\,\left(\frac{4A\Delta^2\left\{A-B\sinh(2\Delta\xi)-\sqrt{A^2+B^2}\cosh(2\Delta\xi)\right\}}{(A\sin(2\Delta\xi)+B)\left\{pA\sinh(2\Delta\xi)+pB-2\Delta\sqrt{A^2+B^2}+2A\Delta\cosh(2\Delta\xi)\right\}}\right)^2,$$

$$u_{19} = -\tfrac{1}{3}p^2 + \tfrac{4}{3}\,rq - 2p\,\left(\frac{4A\Delta^2\left\{A-B\sinh(2\Delta\xi)+\sqrt{A^2+B^2}\cosh(2\Delta\xi)\right\}}{(A\sin(2\Delta\xi)+B)\left\{pA\sinh(2\Delta\xi)+pB+2\Delta\sqrt{A^2+B^2}+2A\Delta\cosh(2\Delta\xi)\right\}}\right)$$
$$-2\,\left(\frac{4A\Delta^2\left\{A-B\sinh(2\Delta\xi)+\sqrt{A^2+B^2}\cosh(2\Delta\xi)\right\}}{(A\sin(2\Delta\xi)+B)\left\{pA\sinh(2\Delta\xi)+pB+2\Delta\sqrt{A^2+B^2}+2A\Delta\cosh(2\Delta\xi)\right\}}\right)^2,$$

where A and B are two non-zero real constants and satisfies the condition $A^2 - B^2 < 0$.

$$u_{20} = -\tfrac{1}{3}p^2 + \tfrac{4}{3}\,rq - 2p\,\left(\tfrac{2\Delta^2 \sec h\,(\Delta\xi)}{2\,\Delta\sinh(\Delta\xi)-p\cosh(\Delta\xi)}\right) - 2\,\left(\tfrac{2\Delta^2 \sec h\,(\Delta\xi)}{2\,\Delta\sinh(\Delta\xi)-p\cosh(\Delta\xi)}\right)^2,$$

$$u_{21} = -\tfrac{1}{3}p^2 + \tfrac{4}{3}\,rq + 2p\,\left(\tfrac{2\Delta^2 \csc h\,(\Delta\xi)}{2\,\Delta\cosh(\Delta\xi)-p\sinh(\Delta\xi)}\right) - 2\,\left(\tfrac{2\Delta^2 \csc h\,(\Delta\xi)}{2\,\Delta\cosh(\Delta\xi)-p\sinh(\Delta\xi)}\right)^2,$$

$$u_{22} = -\tfrac{1}{3}p^2+\tfrac{4}{3}\,rq+2p\,\left(\tfrac{4\Delta^2 \sec h\,(2\Delta\xi)\,(1\mp i\sinh(2\Delta\xi))}{p\cosh(2\Delta\xi)-2\Delta\sinh(2\Delta\xi)\mp i\,2\Delta}\right)-2\,\left(\tfrac{4\Delta^2 \sec h\,(2\Delta\xi)\,(1\mp i\sinh(2\Delta\xi))}{p\cosh(2\Delta\xi)-2\Delta\sinh(2\Delta\xi)\mp i\,2\Delta}\right)^2,$$

$$u_{23} = -\tfrac{1}{3}p^2+\tfrac{4}{3}\,rq+2p\,\left(\tfrac{4\Delta^2 \csc h\,(2\Delta\xi)\,(1\pm\cosh(2\Delta\xi))}{-p\sinh(2\Delta\xi)+2\Delta\cosh(2\Delta\xi)\pm 2\Delta}\right)-2\,\left(\tfrac{4\Delta^2 \csc h\,(2\Delta\xi)\,(1\pm\cosh(2\Delta\xi))}{-p\sinh(2\Delta\xi)+2\Delta\cosh(2\Delta\xi)\pm 2\Delta}\right)^2,$$

$$u_{24} = -\tfrac{1}{3}p^2 + \tfrac{4}{3}\,rq + 2p\,\left(\tfrac{2\Delta^2 \csc h\,(2\xi)}{2\Delta\cosh(\Delta\xi)-p\sinh(\Delta\xi)}\right) - 2\,\left(\tfrac{2\Delta^2 \csc h\,(2\xi)}{2\Delta\cosh(\Delta\xi)-p\sinh(\Delta\xi)}\right)^2.$$

Family 3.

When $r = 0$ and $p\,q \neq 0$,the solutions of Eq. (1) are,

$$u_{25} = -\frac{1}{3}p^2 + \frac{4}{3}\,rq + 2p\,\left(\tfrac{p\,(\cosh(p\xi)-\sinh(p\xi))}{d+\cosh(p\xi)-p\sinh(p\xi)}\right) - 2\,\left(\tfrac{p\,(\cosh(p\xi)-\sinh(p\xi))}{d+\cosh(p\xi)-p\sinh(p\xi)}\right)^2,$$

$$u_{26} = -\frac{1}{3}p^2 + \frac{4}{3}\,rq + 2p\,\left(\tfrac{p\,d}{d+\cosh(p\xi)+p\sinh(p\xi)}\right) - 2\,\left(\tfrac{p\,d}{d+\cosh(p\xi)+p\sinh(p\xi)}\right)^2.$$

where d is an arbitrary constant.

Family 4.

When $q \neq 0$ and $r = p = 0$, the solutions of Eq. (1) are

$$u_{27} = -\frac{1}{3}p^2 + \frac{4}{3}\,rq - 2p\,\left(\tfrac{q}{q\,\xi+d_1}\right) - 2\,\left(\tfrac{q}{q\,\xi+d_1}\right)^2.$$

where d_1 is an arbitrary constant.

4. Graphical presentation

Graph is an influential tool for communication and it illustrates clearly the solutions of the problems. Therefore, some graphs of the solutions are given below. The graphs readily have shown the periodic and solitary wave forms of the solutions.

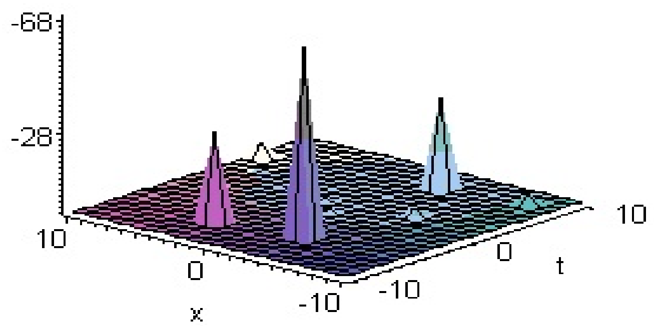

Figure 1: Solitons corresponding to solutions u_1 for $p = q = r = 1$.

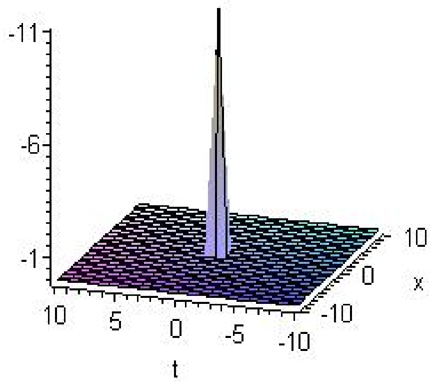

Figure 2: Solitons corresponding to solutions u_2 for $p = q = 1$, $r = 2$.

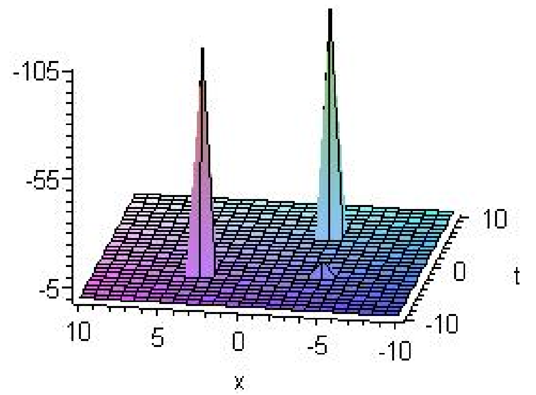

Figure 3: Solitons corresponding to solutions u_8
for $p = q = r = 2$.

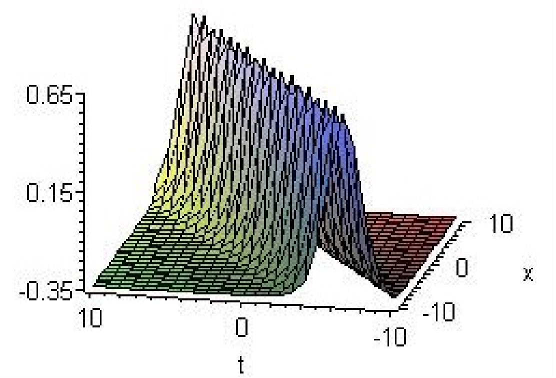

Figure 4: Solitons corresponding to solutions u_{13}
for $p = 3$, $q = 2$, $r = 1$.

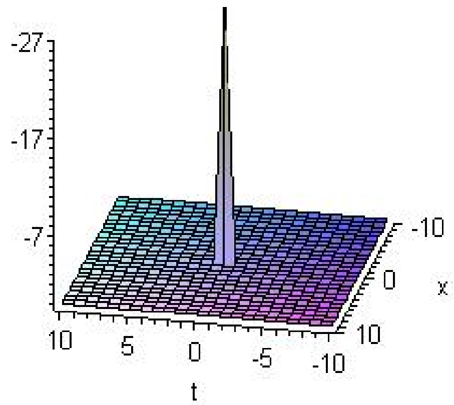

Figure 5: Solitons corresponding to solutions u_{14} for $p = 2,\ q = 1,\ r = 0.5$.

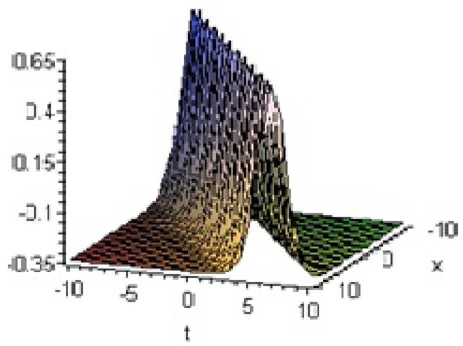

Figure 6: Solitons corresponding to solutions u_{20} for $p = 3,\ q = 1,\ r = 2$.

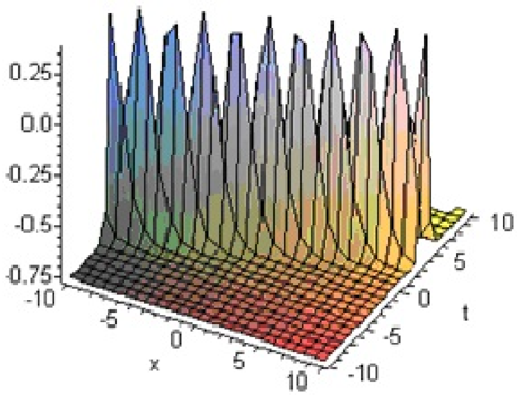

Figure 7: Solitons corresponding to solutions u_{26} for $p = 1.5,\ q = 1,\ r = 0.$

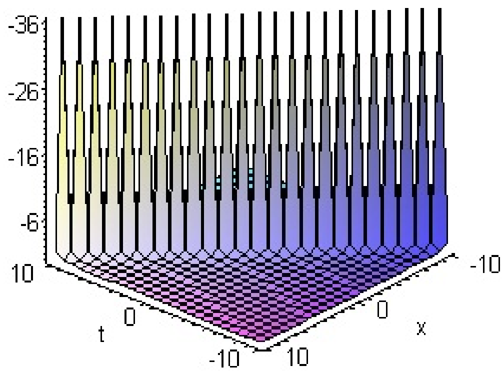

Figure 8: Solitons corresponding to solutions u_{27} for $p = 0,\ q = 1,\ ,r = 0.$

5. Conclusion

The (G'/G)-expansion method is an advance mathematical tool for investigating exact solutions of nonlinear partial differential equations associated with complex physical phenomena wherein, in general the second order linear ordinary differential equation is employed as an auxiliary equation. But, in this article, we utilize the generalized Riccati equation as an auxiliary equation; in consequence we obtain further new exact solutions of the sixth-order Boussinesq equation in a unified way. The obtained exact solutions may be important and significant to reveal the internal mechanism of some complicated physical phenomena. The algorithm presented in this article is effective and more powerful and it can be used for other kind of nonlinear evolution equations in mathematical physics.

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