SOME CLASSES OF p-VALENT MEROMORPHIC FUNCTIONS DEFINED BY A NEW OPERATOR

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Abstract. In this paper, we introduce some classes of p-valent meromorphic functions associated with a new operator and to investigate various properties for these subclasses.

Keywords and phrases: p-Valent meromorphic functions, Hadamard product, linear operator.

2000 Mathematics Subject Classification: 30C45.

1. Introduction

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U \backslash \{0\}$. Let $P_k(\gamma, p)$ $(k \geq 2, 0 \leq \gamma < p, p \in \mathbb{N})$ denote the class of functions

$$g(z) = p + \sum_{k=1}^{\infty} c_k z^k \tag{1.2}$$

which are analytic in U and satisfy for every $r<1$ $(z=re^{i\theta}\in U)$ the conditions

$$(1.3)\qquad \begin{array}{ll} (1) & g(0)=p, \\ (2) & \displaystyle\int_0^{2\pi} \frac{|\text{Re}\{g(z)\}-\gamma|}{(p-\gamma)}\, d\theta \le k\pi. \end{array}$$

The class $P_k(\gamma,p)$ was introduced and studied by Aouf [1].

We note that:

(1) $P_k(\gamma,1)=P_k(\gamma)$ $(k\ge 2, 0\le\gamma<1)$ (see Padmanabhan and Parvatham [8]);

(2) $P_k(0,1)=P_k$ $(k\ge 2)$ (see Pinchuk [9] and Robertson [10]);

(3) $P_2(\gamma,p)=P(\gamma,p)$ $(0\le\gamma<p,\ p\in\mathbb{N})$, where $P(\gamma,p)$ is the class of functions g of the form (1.2) and satisfy the conditions $g(0)=p$ and $\text{Re}\{g(z)\}>\gamma$ $(0\le\gamma<p)$ in U;

(4) $P_2(0,1)=P$, where P is the class of functions with positive real part in U;

(5) $P_2(\gamma,1)=P(\gamma)$ $(0\le\gamma<1)$, where $h(z)=(1-\gamma)p(z)+\gamma$, $h(z)\in P(\gamma)$ and $p(z)\in P$.

From (1.2), we have $g(z)\in P_k(\gamma,p)$ if and only if there exists $g_i\in P(\gamma,p)$, $i=1,2$ such that (see [1])

$$(1.4)\qquad g(z)=\left(\frac{k}{4}+\frac{1}{2}\right)g_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)g_2(z)\quad (z\in U).$$

For analytic functions $f(z)\in\sum_p$, given by (1.1) and $\phi(z)\in\sum_p$ given by $\phi(z)=z^{-p}+\sum\limits_{n=1}^{\infty}b_{n-p}z^{n-p}$ $(p\in\mathbb{N})$, the Hadamard product (or convolution) of $f(z)$ and $\phi(z)$, is defined by

$$(1.5)\qquad (f*\phi)(z)=z^{-p}+\sum_{n=1}^{\infty}a_{n-p}b_{n-p}z^{n-p}=(\phi*f)(z).$$

Aqlan et al. [4] defined the operator $Q^{\alpha}_{\beta,p}:\sum_p\to\sum_p$ by:

$$(1.6)\qquad Q^{\alpha}_{\beta,p}f(z)=\begin{cases} z^{-p}+\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\sum\limits_{n=1}^{\infty}\frac{\Gamma(n+\beta)}{\Gamma(n+\beta+\alpha)}a_{n-p}z^{n-p} & \left(\begin{array}{l}\alpha>0;\ \ \beta>-1;\\ p\in\mathbb{N};\ \ f\in\Sigma_p\end{array}\right) \\ f(z) & (\alpha=0\ \beta>-1; p\in\mathbb{N}; f\in\Sigma_p). \end{cases}$$

Mostafa [7] used Aqlan et al. operator and defined the following linear operator $H^{\alpha}_{p,\beta,\mu}:\Sigma_p\to\Sigma_p$ as follows:
First put

$$(1.7)\qquad G^{\alpha}_{\beta,p}(z)=z^{-p}+\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\sum_{n=1}^{\infty}\frac{\Gamma(n+\beta)}{\Gamma(n+\beta+\alpha)}z^{n-p}\quad (p\in\mathbb{N})$$

and let $G_{\beta,p,\mu}^{\alpha *}$ be defined by

$$G_{\beta,p}^{\alpha}(z) * G_{\beta,p,\mu}^{\alpha *}(z) = \frac{1}{z^p(1-z)^\mu} \quad (\mu > 0; p \in \mathbb{N})\,. \tag{1.8}$$

Then

$$H_{p,\beta,\mu}^{\alpha} f(z) = G_{\beta,p}^{\alpha *}(z) * f(z) \quad (f \in \Sigma_p)\,. \tag{1.9}$$

Using (1.7) and (1.9), we have

$$H_{p,\beta,\mu}^{\alpha} f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta+\alpha)(\mu)_n}{\Gamma(n+\beta)(1)_n} a_{n-p} z^{n-p}\,, \tag{1.10}$$

where $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)...(\nu+n-1) & (n \in \mathbb{N})\,. \end{cases}$$

It is readily verified from (1.10) that (see [7])

$$z(H_{p,\beta,\mu}^{\alpha} f(z))' = (\alpha+\beta) H_{p,\beta,\mu}^{\alpha+1} f(z) - (\alpha+\beta+p) H_{p,\beta,\mu}^{\alpha} f(z) \tag{1.11}$$

and

$$z(H_{p,\beta,\mu}^{\alpha} f(z))' = \mu H_{p,\beta,\mu+1}^{\alpha} f(z) - (\mu+p) H_{p,\beta,\mu}^{\alpha} f(z). \tag{1.12}$$

It is noticed that, putting $\mu = 1$ in (1.10), we obtain the operator

$$H_{p,\beta,1}^{\alpha} f(z) = H_{p,\beta}^{\alpha} f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha+\beta)}{\Gamma(n+\beta)} a_{n-p} z^{n-p}. \tag{1.13}$$

Now we define the following subclasses of the class $\sum_p$ for $0 \le \gamma, \beta < p, p \in \mathbb{N}$ and $k \ge 2$:

$$\sum S_k(p,\gamma) = \left\{ f(z) \in \sum\nolimits_p : -\frac{zf'(z)}{f(z)} \in P_k(p,\gamma)\,, z \in U \right\}, \tag{1.14}$$

$$\sum C_k(p,\gamma) = \left\{ f(z) \in \sum\nolimits_p : -\frac{\left(zf'(z)\right)'}{f'(z)} \in P_k(p,\gamma)\,, z \in U \right\}, \tag{1.15}$$

$$\begin{aligned} &\sum V_k(p,\gamma,\zeta) \\ &= \left\{ f(z) \in \sum\nolimits_p, g(z) \in \sum S_2(p,\gamma) : -\frac{zf'(z)}{g(z)} \in P_k(p,\zeta)\,, z \in U \right\} \end{aligned} \tag{1.16}$$

and

$$\begin{aligned}&\sum V_k^*(p,\gamma,\zeta)\\ &=\left\{f(z)\in\sum\nolimits_p, g(z)\in\sum C_2(p,\gamma): -\frac{\left(zf'(z)\right)'}{g'(z)}\in P_k(p,\zeta), z\in U\right\}.\end{aligned} \tag{1.17}$$

We can easily see that:

$$f(z)\in\sum C_k(p,\gamma) \iff -\frac{zf'(z)}{p}\in\sum S_k(p,\gamma) \tag{1.18}$$

and

$$f(z)\in\sum V_k^*(p,\gamma,\zeta) \iff -\frac{zf'(z)}{p}\in\sum V_k(p,\gamma,\xi). \tag{1.19}$$

We note that, for special choices for the parameters k and γ involved in the above classes, we can obtain well-known subclasses

$$\begin{aligned}\sum S_2(p,\gamma) &= \sum S_p^*(\gamma), \sum C_2(p,\gamma)=\sum C_p(\gamma),\\ \sum V_2(p,\gamma,\zeta) &= \sum V_p(\gamma,\zeta) \text{ and } \sum V_2^*(p,\gamma,\zeta)=\sum V_p^*(\gamma,\zeta).\end{aligned}$$

The classes $\sum S_p^*(\gamma)$, $\sum C_p(\gamma)$, $\sum V_p(\gamma,\zeta)$ and $\sum V_p^*(\gamma,\zeta)$ denote the meromorphic p-valent starlike of order γ, meromorphic p-valent convex of order γ, meromorphic p-valent close-to-convex of order γ and type $\zeta(0\le\gamma,\ \zeta<p, p\in\mathbb{N})$ and meromorphic p-valent quasi-convex of order γ and type $\zeta(0\le\gamma,\ \zeta<p,\ p\in\mathbb{N})$. The classes $\sum S_p^*(\gamma)$ and $\sum C_p(\gamma)$ were studied by Kumar and Shukla [5] and the classes $\sum V_p(\gamma,\zeta)$ and $\sum V_p^*(\gamma,\zeta)$ were introduced by Aouf et al. [2] and Aouf and Xu [3].

Next, by using the linear operator $H^{\alpha}_{p,\beta,\mu}f(z)$, we introduce the following classes of analytic functions for $0\le\gamma,\zeta<p$ and $k\ge2$

$$\sum S_{k,p}(\alpha,\beta,\mu;\gamma)=\left\{f(z)\in\sum\nolimits_p: H^{\alpha}_{p,\beta,\mu}f(z)\in\sum S_k(p,\gamma), z\in U\right\}, \tag{1.20}$$

$$\sum C_{k,p}(\alpha,\beta,\mu;\gamma)=\left\{f(z)\in\sum\nolimits_p: H^{\alpha}_{p,\beta,\mu}f(z)\in\sum C_k(p,\gamma), z\in U\right\}, \tag{1.21}$$

$$\sum V_{k,p}(\alpha,\beta,\mu;\gamma,\zeta)=\left\{f(z)\in\sum\nolimits_p: H^{\alpha}_{p,\beta,\mu}f(z)\in\sum V_k(p,\zeta), z\in U\right\} \tag{1.22}$$

and

$$\sum V^*_{k,p}(\alpha,\beta,\mu;\gamma,\zeta)=\left\{f(z)\in\sum\nolimits_p: H^{\alpha}_{p,\beta,\mu}f(z)\in\sum V_k^*(p,\zeta), z\in U\right\}. \tag{1.23}$$

We also note that

$$f(z) \in \sum C_{k,p}(\alpha,\beta,\mu;\gamma) \Leftrightarrow -\frac{zf'(z)}{p} \in \sum S_{k,p}(\alpha,\beta,\mu;\gamma) \tag{1.24}$$

and

$$f(z) \in \sum V_{k,p}(\alpha,\beta,\mu;\gamma,\zeta) \Leftrightarrow -\frac{zf'(z)}{p} \in \sum V^{*}_{k,p}(\alpha,\beta,\mu;\gamma,\zeta). \tag{1.25}$$

2. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $\alpha \geq 0,\ \mu > 0,\ \beta > -1,\ 0 \leq \gamma,\ \zeta < p,\ k \geq 2$ and $z \in U$.

In order to prove our results, we need the following lemma.

Lemma 1. [6] *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and $\Phi(u,v)$ be a complex-valued function satisfying the conditions:*

1. $\Phi(u,v)$ *is continuous in a domain* $D \in \mathbb{C}^2$.
2. $(0,1) \in D$ *and* $Re\Phi(1,0) > 0$.
3. $\Re e\{\Phi(iu_2,v_1)\} > 0$ *where* $(iu_2,v_1) \in D$ *and* $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + c_1 z + c_2 z^2 + ...$ is analytic in U such that $\left(h(z), zh'(z)\right) \in D$ and $Re\left\{\Phi\left(h(z), zh'(z)\right)\right\} > 0$ for $z \in U$, then $Re\{h(z)\} > 0$ in U.

Theorem 1. *Let* $0 \leq \eta \leq \gamma < p$ and $\eta < \alpha + \beta + p$, *then*

$$\sum S_{k,p}(\alpha+1,\beta,\mu;\gamma) \subset \sum S_{k,p}(\alpha,\beta,\mu;\eta), \tag{2.1}$$

where

$$\eta = \frac{2[2\gamma(\alpha+\beta+p)+p]}{1+2\alpha+2\beta+2p+2\gamma-\sqrt{[2(\alpha+\beta+\gamma+p)+1]^2-8[2\gamma(\alpha+\beta+p)+p]}}. \tag{2.2}$$

Proof. Let $f(z) \in \sum S_{k,p}(\alpha+1,\beta,\mu;\gamma)$ and

$$-\frac{z\left(H^{\alpha}_{p,\beta,\mu}f(z)\right)'}{H^{\alpha}_{p,\beta,\mu}f(z)} = H(z) = (p-\eta)h(z) + \eta, \tag{2.3}$$

where

$$h(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z), \tag{2.4}$$

where $h_i(z)$ $(i = 1, 2)$ are analytic in U and $h_i(0) = 1$ $(i = 1, 2)$. Using (1.11) in (2.3) and differentiating the resulting equation with respect to z, we have

$$-\frac{z\left(H^{\alpha+1}_{p,\beta,\mu}f(z)\right)'}{H^{\alpha+1}_{p,\beta,\mu}f(z)} - \gamma = \eta - \gamma + (p-\eta)\,h\,(z) - \frac{(p-\eta)\,zh'\,(z)}{(p-\eta)\,h\,(z) + \eta - (\alpha+\beta+p)}. \tag{2.5}$$

Now, we will show that $H\,(z) \in P_k\,(\gamma, p)$ or $h_i(z) \in P$. From (2.4) and (2.5) we have

$$-\frac{z\left(H^{\alpha+1}_{p,\beta,\mu}f(z)\right)'}{H^{\alpha+1}_{p,\beta,\mu}f(z)} - \gamma = \left(\frac{k}{4}+\frac{1}{2}\right)\left\{\eta - \gamma + (p-\eta)\,h_1\,(z) - \tfrac{(p-\eta)zh_1'(z)}{(p-\eta)h_1(z)+\eta-(\alpha+\beta+p)}\right\}$$
$$-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{\eta - \gamma + (p-\eta)\,h_2\,(z) - \tfrac{(p-\eta)zh_2'(z)}{(p-\eta)h_2(z)+\eta-(\alpha+\beta+p)}\right\},$$

this implies that

$$\operatorname{Re}\left\{\eta - \gamma + (p-\eta)\,h_i\,(z) - \frac{(p-\eta)\,zh_i'\,(z)}{(p-\eta)\,h_i\,(z) + \eta - (\alpha+\beta+p)}\right\} > 0 \quad (i = 1, 2).$$

We form the functional $\Phi\,(u, v)$ by taking $u = h_i(z)$, $v = zh_i'\,(z)$,

$$\Phi\,(u, v) = \eta - \gamma + (p-\eta)\,u - \frac{(p-\eta)\,v}{(p-\eta)\,u + \eta - (\alpha+\beta+p)}. \tag{2.6}$$

Clearly, the first two conditions of Lemma 1 are satisfied in the domain $D \subseteq \mathbb{C}\backslash\frac{\eta-\alpha+\beta+p}{\eta-p} \times \mathbb{C}$. Now, we verify condition (iii) as follows:

$$\begin{aligned}
\operatorname{Re}\{\Phi\,(iu_2, v_1)\} &= (\eta-\gamma) + \operatorname{Re}\left\{-\frac{(p-\eta)\,v_1}{(p-\eta)\,iu_2 + \eta - (\alpha+\beta+p)}\right\} \\
&\le (\eta-\gamma) - \frac{(p-\eta)\,(\alpha+\beta+p-\eta)\,(1+u_2^2)}{2\left[(p-\eta)^2 u_2^2 + (\eta-\alpha-\beta-p)^2\right]} \\
&= \frac{A + Bu_2^2}{2C},
\end{aligned}$$

where

$$\begin{aligned}
A &= 2(\eta-\gamma)\,(\eta-\alpha-\beta-p)^2 - (p-\eta)\,(\alpha+\beta+p-\eta)\,, \\
B &= 2(\eta-\gamma)\,(p-\eta)^2 - (p-\eta)\,(\alpha+\beta+p-\eta)\,, \\
C &= (p-\eta)^2\,u_2^2 + (\eta-\alpha-\beta-p)^2\,.
\end{aligned}$$

We note that $\operatorname{Re}\{\Phi\,(iu_2, v_1)\} < 0$ if and only if $A \le 0$ and $B < 0$. From η as given by (2.2), we obtain $A \le 0$ and from $0 \le \eta \le \gamma < p$ we have $B < 0$. Therefore applying Lemma 1, $h_i\,(z) \in P\,(i = 1, 2)$ and consequently $f \in \sum S_{k,p}\,(\alpha, \beta, \mu; \eta)$.

This completes the proof of Theorem 1.

Theorem 2. *Let* $0 \leq \eta \leq \gamma < p, \eta < \alpha + \beta + p$ *and* $k \geq 2$, *then*

$$\sum C_{k,p}(\alpha+1,\beta,\mu;\gamma) \subset \sum C_{k,p}(\alpha,\beta,\mu;\eta). \tag{2.7}$$

Proof. Applying (1.24) and Theorem 1, we observe that

$$\begin{aligned} f(z) &\in \sum C_{k,p}(\alpha+1,\beta,\mu;\gamma) \\ &\Longleftrightarrow -\frac{zf'(z)}{p} \in \sum S_{k,p}(\alpha,\beta,\mu;\gamma) \\ &\Longrightarrow -\frac{zf'(z)}{p} \in \sum S_{k,p}(\alpha,\beta,\mu;\eta) \\ &\Longleftrightarrow f(z) \in \sum C_{k,p}(\alpha,\beta,\mu;\eta), \end{aligned}$$

which evidently proves Theorem 2.

Theorem 3. *Let* $0 \leq \gamma, \zeta < p, \gamma < \alpha + \beta + p$ *and* $k \geq 2$, *then*

$$\sum V_{k,p}(\alpha+1,\beta,\mu;\gamma,\zeta) \subset \sum V_{k,p}(\alpha,\beta,\mu;\gamma,\zeta). \tag{2.8}$$

Proof. Let $f(z) \in \sum V_{k,p}(\alpha+1,\beta,\mu;\gamma,\zeta)$. Then, in view of the definition of the class $\sum V_{k,p}(\alpha+1,\beta,\mu;\gamma,\zeta)$, there exists a function $g(z) \in \sum S_{2,p}(\alpha+1,\beta,\mu;\gamma)$ such that

$$-\frac{z\left(H^{\alpha+1}_{p,\beta,\mu}f(z)\right)'}{H^{\alpha+1}_{p,\beta,\mu}g(z)} \in P_k(\zeta,p) \quad (z \in U).$$

Now let

$$-\frac{z\left(H^{\alpha}_{p,\beta,\mu}f(z)\right)'}{H^{\alpha}_{p,\beta,\mu}g(z)} = G(z) = (p-\zeta)\,h(z) + \zeta, \tag{2.9}$$

where $h(z)$ is given by (2.4). Using (1.11) in (2.9), we have

$$\begin{aligned} &(\alpha+\beta)\,H^{\alpha+1}_{p,\beta,\mu}f(z) - (\alpha+\beta+p)H^{\alpha}_{p,\beta,\mu}f(z) \\ &\qquad = -[(p-\zeta)\,h(z)+\zeta]H^{\alpha}_{p,\beta,\mu}g(z). \end{aligned} \tag{2.10}$$

Differentiating (2.10) with respect to z and multiplying by z, we obtain

$$\begin{aligned} &(\alpha+\beta)\,z(H^{\alpha+1}_{p,\beta,\mu}f(z))' - (\alpha+\beta+p)z(H^{\alpha}_{p,\beta,\mu}f(z))' \\ &\quad = -(p-\zeta)\,zh'(z)\,H^{\alpha}_{p,\beta,\mu}g(z) - [(p-\zeta)\,h(z)+\zeta]z(H^{\alpha}_{p,\beta,\mu}g(z))'. \end{aligned} \tag{2.11}$$

Since $g(z) \in \sum S_{2,p}(\alpha+1,\beta,\mu;\gamma)$, by Theorem 1, $g(z) \in \sum S_{2,p}(\alpha,\beta,\mu;\gamma)$, then we have

$$-\frac{z\left(H^{\alpha}_{p,\beta,\mu}g(z)\right)'}{H^{\alpha}_{p,\beta,\mu}g(z)} = (p-\gamma)\,q(z) + \gamma,$$

where $q(z) = 1 + c_1 z + c_2 z^2 + ...$ is analytic in U with $q(0) = 1$. Then by using (1.11), we have

$$-(\alpha+\beta)\frac{H^{\alpha+1}_{p,\beta,\mu}g(z)}{H^{\alpha}_{p,\beta,\mu}g(z)} = (p-\gamma)\,q(z) + \gamma - (\alpha+\beta+p). \tag{2.12}$$

From (2.11) and (2.12), we obtain

$$-\frac{z\left(H^{\alpha+1}_{p,\beta,\mu}f(z)\right)'}{H^{\alpha+1}_{p,\beta,\mu}g(z)} - \zeta = (p-\zeta)\,h(z) - \frac{(p-\zeta)\,zh'(z)}{(p-\gamma)\,q(z)+\gamma-(\alpha+\beta+p)}. \tag{2.13}$$

Now, we will show that $G(z) \in P_k(\zeta, p)$ or $h_i(z) \in P$, $i = 1, 2$. From (2.4) and (2.13) we have

$$\begin{aligned} &-\frac{z\left(H^{\alpha+1}_{p,\beta,\mu}f(z)\right)'}{H^{\alpha+1}_{p,\beta,\mu}g(z)} - \zeta \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{(p-\zeta)\,h_1(z) - \frac{(p-\zeta)\,zh_1'(z)}{(p-\gamma)\,q(z)+\gamma-(\alpha+\beta+p)}\right\} \\ &- \left(\frac{k}{4}-\frac{1}{2}\right)\left\{(p-\zeta)\,h_2(z) - \frac{(p-\zeta)\,zh_2'(z)}{(p-\gamma)\,q(z)+\gamma-(\alpha+\beta+p)}\right\}, \end{aligned}$$

this implies that

$$\operatorname{Re}\left\{(p-\zeta)\,h_i(z) - \frac{(p-\zeta)\,zh_i'(z)}{(p-\gamma)\,q(z)+\gamma-(\alpha+\beta+p)}\right\} > 0 \quad (z \in U; i = 1, 2).$$

We form the functional $\Phi(u, v)$ by choosing $u = h_i(z)$, $v = zh_i^{'}(z)$,

$$\Phi(u,v) = (p-\zeta)\,u - \frac{(p-\zeta)\,v}{(p-\gamma)\,q(z)+\gamma-(\alpha+\beta+p)}.$$

Clearly, the first two conditions of Lemma 1 are satisfied in the domain $D \subseteq \mathbb{C}\backslash Q^* \times \mathbb{C}$, where $Q^* = \left\{z \in \mathbb{C} \text{ and } \operatorname{Re}(q(z)) = q_1 > \dfrac{\gamma-(\alpha+\beta+p)}{\gamma-p}\right\}$ and $q(z) = q_1 + iq_2$.

Now, we verify the condition (iii) as follows:

$$\begin{aligned} \operatorname{Re}\{\Phi(iu_2, v_1)\} &= \operatorname{Re}\left\{-\frac{(p-\zeta)\,v_1}{(p-\gamma)\,(q_1+iq_2)+\gamma-(\alpha+\beta+p)}\right\} \\ &\le -\frac{[(\alpha+\beta+p-\gamma)-(p-\gamma)\,q_1]\,(p-\zeta)\,(1+u_2^2)}{2\left\{[(p-\gamma)\,q_1+\gamma-\alpha-\beta-p]^2+[(p-\gamma)\,q_2]^2\right\}.} \\ &< 0. \end{aligned}$$

By applying Lemma 1, $h_i(z) \in P$ $(i = 1, 2)$ and, consequently,

$$f(z) \in \sum V_{k,p}(\alpha+1, \beta, \mu; \gamma, \zeta).$$

This completes the proof of Theorem 3.

Theorem 4. *Let* $0 \le \gamma, \lambda < p, \gamma < \alpha + \beta + p$ *and* $k \ge 2$, *then*

$$\sum V^{*}_{k,p}(\alpha + 1, \beta, \mu; \gamma, \zeta) \subset \sum V^{*}_{k,p}(\alpha, \beta, \mu; \gamma, \zeta). \tag{2.14}$$

Proof. Applying (1.25) and Theorem 3, we observe that

$$\begin{aligned} f(z) \in \sum V^{*}_{k,p}(\alpha + 1, \beta, \mu; \gamma, \zeta) \\ \Longleftrightarrow -\frac{zf'(z)}{p} \in \sum V_{k,p}(\alpha + 1, \beta, \mu; \gamma, \zeta) \\ \Longrightarrow -\frac{zf'(z)}{p} \in \sum V_{k,p}(\alpha, \beta, \mu; \gamma, \zeta) \\ \Longleftrightarrow f(z) \in \sum V^{*}_{k,p}(\alpha, \beta, \mu; \gamma, \zeta), \end{aligned}$$

which, evidently, proves Theorem 4.

In [5], Kumar and Shukla defined the familiar integral operator $F_{\nu,p}(f)(z)$ as follows:

$$\begin{aligned} F_{\nu,p}(f)(z) &= \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} f(t)dt \qquad (\nu > 0) \\ &= z^{-p} + \sum_{n=1}^{\infty} \frac{\nu}{\nu + n} a_{n-p} z^{n-p}. \end{aligned} \tag{2.15}$$

It follows that:

$$z(H^{\alpha}_{p,\beta,\mu} F_{\nu,p}(f)(z))' = \nu H^{\alpha}_{p,\beta,\mu} f(z) - (\nu + p) H^{\alpha}_{p,\beta,\mu} F_{\nu,p}(f)(z) \ \ (\nu > 0). \tag{2.16}$$

Theorem 5. *If* $0 \le \gamma < p$, $k \ge 2$ *and* $f \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma)$, *then*

$$F_{\nu,p}(f)(z) \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma) \ \ (\nu > 0).$$

Proof. Let $f \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma)$ and set

$$-\frac{z\left(H^{\alpha}_{p,\beta,\mu} F_{\nu,p}(f)(z)\right)'}{H^{\alpha}_{p,\beta,\mu} F_{\nu,p}(f)(z)} = M(z) = (p - \gamma)\, h(z) + \gamma, \tag{2.17}$$

where $h(z)$ is given by (2.4). Using (2.16) and (2.17), we have

$$\nu \frac{H^{\alpha}_{p,\beta,\mu} f(z)}{H^{\alpha}_{p,\beta,\mu} F_{\nu,p}(f)(z)} = -(p - \gamma)\, h(z) - \gamma + \nu + p. \tag{2.18}$$

Taking the logarithmic differentiation on both sides of (2.18) with respect to z and multiplying by z, we have

$$-\frac{z\left(H^{\alpha}_{p,\beta,\mu}f\left(z\right)\right)^{'}}{H^{\alpha}_{p,\beta,\mu}f\left(z\right)}-\gamma=\left(p-\gamma\right)h\left(z\right)+\frac{\left(p-\gamma\right)zh^{'}\left(z\right)}{\left(p-\gamma\right)h\left(z\right)+\gamma-\left(\nu+p\right)}. \tag{2.19}$$

Now, we will show that $M\left(z\right)\in P_k\left(\gamma,p\right)$ or $h_i(z)\in P$. From (2.4) and (2.19) we have

$$\begin{aligned}-\frac{z\left(H^{\alpha}_{p,\beta,\mu}f(z)\right)^{'}}{H^{\alpha}_{p,\beta,\mu}f(z)}-\gamma &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{\left(p-\gamma\right)h_1\left(z\right)-\frac{(p-\gamma)zh_1^{'}(z)}{(p-\gamma)h_1(z)+\gamma-(\nu+p)}\right\}\\ &\quad-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{\left(p-\gamma\right)h_2\left(z\right)-\frac{(p-\gamma)zh_2^{'}(z)}{(p-\gamma)h_2(z)+\gamma-(\nu+p)}\right\},\end{aligned}$$

this implies that

$$\operatorname{Re}\left\{\left(p-\gamma\right)h_i\left(z\right)-\frac{\left(p-\gamma\right)zh_i^{'}\left(z\right)}{\left(p-\gamma\right)h_i\left(z\right)+\gamma-\left(\nu+p\right)}\right\}>0\quad\left(z\in U;i=1,2\right). \tag{2.20}$$

We form the functional $\Phi\left(u,v\right)$ by choosing $u=h_i(z)$, $v=zh_i^{'}\left(z\right)$,

$$\Phi\left(u,v\right)=\left(p-\gamma\right)u-\frac{\left(p-\gamma\right)v}{\left(p-\gamma\right)u+\gamma-\left(\nu+p\right)}.$$

Then, clearly, $\Phi\left(u,v\right)$ satisfies all the conditions of Lemma 1. Hence $h_i\left(z\right)\in P$ $\left(i=1,2\right)$ and consequently $h\left(z\right)\in P_k$ for $z\in U$, which implies that $F_{\nu,p}(f)(z)\in\sum S_{k,p}\left(\alpha,\beta,\mu;\gamma\right)$. This completes the proof of Theorem 5.

Next, we derive an inclusion property for the subclass $\sum C_{k,p}\left(\alpha,\beta,\mu;\gamma\right)$ involving $F_{\nu,p}(f)(z)$, which is given by the following theorem.

Theorem 6. *If* $0\leq\gamma<p$, $k\geq 2$, $\nu>0$ *and* $f\in\sum C_{k,p}\left(\alpha,\beta,\mu;\gamma\right)$, *then*

$$F_{\nu,p}(f)(z)\in\sum C_{k,p}\left(\alpha,\beta,\mu;\gamma\right).$$

Proof. By applying Theorem 5, it follows that

$$\begin{aligned}&f\in\sum C_{k,p}\left(\alpha,\beta,\mu;\gamma\right)\\ &\quad\Longleftrightarrow-\frac{zf^{'}}{p}\in\sum S_{k,p}\left(\alpha,\beta,\mu;\gamma\right)\\ &\quad\Longrightarrow F_{\nu,p}(f)(z)\left(-\frac{zf^{'}}{p}\right)\in\sum S_{k,p}\left(\alpha,\beta,\mu;\gamma\right)\\ &\quad\Longleftrightarrow-\frac{z\left(F_{\nu,p}(f)(z)\right)^{'}}{p}\in\sum S_{k,p}\left(\alpha,\beta,\mu;\gamma\right)\\ &\quad\Longleftrightarrow F_{\nu,p}(f)(z)\in\sum C_{k,p}\left(\alpha,\beta,\mu;\gamma\right).\end{aligned}$$

This completes the proof of Theorem 6.

Using (2.16) instead of (1.11) and the techniques of the proof of Theorems 3 and 4, respectively, we can prove the following theorems, respectively.

Theorem 7. *If* $0 \leq \gamma, \zeta < p$, $k \geq 2$, $\nu \geq 0$ *and* $f \in \sum V_{k,p}(\alpha, \beta, \mu; \gamma, \zeta)$, *then*

$$F_{\nu,p}(f)(z) \in \sum V_{k,p}(\alpha, \beta, \mu; \gamma, \zeta).$$

Theorem 8. *If* $0 \leq \gamma, \zeta < p$, $k \geq 2$, $\nu \geq 0$ *and* $f \in \sum V^*_{k,p}(\alpha, \beta, \mu; \gamma, \zeta)$, *then*

$$F_{\nu,p}(f)(z) \in \sum V^*_{k,p}(\alpha, \beta, \mu; \gamma, \zeta).$$

Remark 1.

(i) Using (1.12) instead of (1.11), in our results, we can obtain new results corresponding to the operator $H^{\alpha}_{p,\beta,\mu}$.

(ii) Putting $\mu = 1$, in the above results, we obtain the corresponding results for different classes associated with the operator $H^{\alpha}_{p,\beta}$ defined in (1.13).

Acknowledgements. The authors would like to thank the referees of the paper for their helpful suggestions.

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Accepted: 20.01.2013