INTEGRAL FILTERS AND INTEGRAL BL-ALGEBRAS

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Abstract. In this paper, we introduce the concepts of integral filters and integral BL-algebras. With respect to concepts, we give some related results. In particular, we prove that an integral BL-algebra is a perfect, local, directly indecomposable BL-algebra and SBL-algebra. Also, we give some relations among integral filters and some types of filters in BL-algebras, such as prime, primary, perfect, fantastic, positive implicative and obstinate filters.

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1. Introduction

BL-algebras are the algebraic structure for Hájek basic logic [8] in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Godel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general"many-valued logic with truth values in $[0, 1]$ and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on $[0; 1]$. Most familiar example of a BL-algebra is the unit interval $[0, 1]$ endowed with the structure induced by a continuous t-norm. In 1958, Chang [3] introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. Turunen [14] introduced the notion of an implicative filter and a *Boolean* filter and proved that these notions are equivalent in BL-algebras. *Boolean* filters are an important class of filters, because the quotient BL-algebra induced by these filters are *Boolean* algebras. In this paper we study integral BL-algebras and BL-algebras with Godel negation such that they are local BL-algebras, but the converse is not correct in general. So it is important for us studying this class of BL-algebras. At first, we prove some theorems about local and integral BL-algebras. Also, we show integral MV-algebras are trivial. Moreover, we define integral filters and we prove that a BL-algebra is integral BL-algebra if and only if $\{1\}$ is integral filter. Also, we show in finite BL-algebras, integral filters and perfect filters are equal. In the follow, we define BL-algebras with Godel negation and we prove that these class of BL-algebras are equal to the class integral BL-algebras and in a linearly ordered BL-algebra, a BL-algebra is a SBL-algebra if and only if is a integral BL-algebra. Finally, we study the relationship between integral filters and obstinate filters and we prove that an integral and fantastic filter is a obstinate filter.

2. Preliminaries

In this section, we recollect some definitions and theorems which will be used in the following, and we shall not cite them every time they are used.

Definition 2.1. ([8]) A *BL-algebra* is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) such that

$(BL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
$(BL2)$ $(L, \odot, 1)$ is a commutative monoid,
$(BL3)$ $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,
$(BL4)$ $x \wedge y = x \odot (x \rightarrow y)$,
$(BL5)$ $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

We denote $x^n = \overbrace{x \odot ... \odot x}^{n-times}$, if $n > 0$ and $x^0 = 1$.

A BL-algebra L is called a*Godel algebra* if $x^2 = x$, for all $x \in L$ and L is called an*MV-algebra* if, $(x^-)^- = x$, for all $x \in L$, where $x^- = x \rightarrow 0$.

Lemma 2.2. ([4],[5],[8]) *In any BL-algebra the following hold:*

$(BL6)$ $x \leq y$ *if and only if* $x \rightarrow y = 1$.
$(BL7)$ $0 \rightarrow x = 1$ *and* $x \rightarrow x = 1$.
$(BL8)$ $x \odot y \leq x \wedge y$.
$(BL9)$ $x^- = 1$ *if and only if* $x = 0$.
$(BL10)$ $x^- = x^{---}$.
$(BL11)$ $x \odot x^- = 0$, $1 \odot x = x$ *and* $0 \odot x = 0$.
$(BL12)$ $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$.
$(BL13)$ $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
$(BL14)$ $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$.
$(BL15)$ $(x \wedge y)^- = x^- \vee y^-$.
$(BL16)$ $x \odot y = 0$ *if and only if* $x \leq y^-$.

for all $x, y, z \in L$.

We briefly review some types of filters and related theorems that, we refer the reader to [4],[5],[8],[9], [13],[14],[15],[16], for more details.

Definition 2.3. Let L be a BL-algebra and F be a nonempty subset of L. Then

(i) F is called a *filter* of L, if $x \odot y \in F$, for all $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$.

(ii) proper filter F is called a *prime filter* of L, if $x \vee y \in F$ implies $x \in F$ or $y \in F$, for all $x, y \in L$.

(iii) proper filter F is called a *maximal filter* of L, if it is not properly contained in any other proper filter of L.

(iv) proper filter F is called a *primary filter*, if for all $x, y \in L$, $(x \odot y)^{-} \in F$ implies $(x^n)^{-} \in F$ or $(y^n)^{-} \in P$, for some $n \in \mathbb{N} \cup \{0\}$.

(v) filter F is called a *Boolean filter*, if $x^{-} \vee x \in F$, for all $x \in L$.

(vi) F is called an *implicative filter*, if $1 \in F$ and $x \to (y \to z) \in F$ and $x \to y \in F$, then $x \to z \in F$, for all $x, y, z \in L$.

(vii) F is called a *positive implicative filter*, if $1 \in F$ and $x \to ((y \to z) \to y) \in F$ and $x \in F$, then $y \in F$, for all $x, y, z \in L$.

(viii) F is called a *fantastic filter*, if $1 \in F$ and $z \to (y \to x) \in F$ and $z \in F$, then $((x \to y) \to y) \to x \in F$, for all $x, y, z \in L$.

Definition 2.4. Let L be a BL-algebra. Then L is called a *local* BL-algebra, if it has a unique maximal filter.

Note. A filter F of BL-algebra L is maximal and implicative if and only if $x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$, for all $x, y \in L$. This filter is called an *obstinate* filter.

Theorem 2.5. ([11],[12]) *Let F of BL-algebra L. Then*

(i) *F is a fantastic filter of L if and only if $((x \to 0) \to 0) \to x \in F$, for all $x \in L$.*

(ii) *if F is an obstinate filter of L, then F is a fantastic filter of L.*

Theorem 2.6. *[8] Let F be a filter of BL-algebra L. Then the binary relation $\equiv_F$ which is defined by*

$$x \equiv_F y \ \text{ if and only if } \ x \to y \in F \ \text{ and } y \to x \in F$$

is a congruence relation on L. Define $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ on L/F, the set of all congruence classes of L, as follows:

$$[x] \cdot [y] = [x \odot y], \ [x] \rightharpoonup [y] = [x \to y], \ [x] \sqcup [y] = [x \vee y], \ [x] \sqcap [y] = [x \wedge y].$$

Then $(L/F, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a BL-algebra which is called quotient BL-algebra with respect to F.

Theorem 2.7. ([4],[13]) *Let P be a proper filter of BL-algebra L. Then P is a prime filter if and only if L/P is a BL-chain.*

Definition 2.8. ([5]) Let L be a BL-algebra and $x \in L$. If there exists a smallest positive integer number n such that $x^n = 0$, then we say the order of x is n and we denote by $ord(x) = n$ and we say is $ord(x) = \infty$, if no such n exists.

Definition 2.9. ([13]) Let L_1 and L_2 be two BL-algebras. Then the map $f : L_1 \to L_2$ is called a BL-algebra *homomorphism* if and only if it satisfies the following conditions, for every $x, y \in L_1$:

(i) $f(0) = 0$,

(ii) $f(x \odot y) = f(x) \odot f(y)$,

(iii) $f(x \to y) = f(x) \to f(y)$.

If f is a bijective, then the homomorphism f is called BL-algebra *isomorphism.* In this case we write $L_1 \cong L_2$.

The following theorems and definitions are from [6],[16] and the reader can refer to it, for more details.

Theorem 2.10. *Let L be a BL-algebra. The following are equivalent:*

(i) *L is a local BL-algebra,*

(ii) *$M(L) = \{x \in L \mid x^n \neq 0, \forall n \in \mathbb{N} \cup \{0\}\} = \{x \in L \mid ord(x) = \infty\}$ is the unique maximal filter of L,*

(iii) *for all $x \in L$, $ord(x) < \infty$ or $ord(x^-) < \infty$.*

Theorem 2.11. *Let P be a filter of BL-algebra L. The following are equivalent:*

(i) *L/P is a local BL-algebra,*

(ii) *P is a primary filter.*

Theorem 2.12. *Let L be a BL-algebra. The following are equivalent:*

(i) *L is a Godel algebra,*

(ii) *Any filter of L is an implicative filter of L.*

Definition 2.13. A BL-algebra L is called *directly indecomposable* if and only if L is nontrivial and whenever $L \cong L_1 \times L_2$ then either L_1 or L_2 is trivial.

Note. Let $B(L)$ be the *Boolean* algebra of all complemented elements in the distributive lattice $L(L)$.

Theorem 2.14. *Let L be a BL-algebra. Then*

(i) *L is directly indecomposable if and only if $B(L) = \{0, 1\}$.*

(ii) *L is directly indecomposable if L is local.*

Definition 2.15. Let L be a BL-algebra. Then

(i) L is a *perfect* BL-algebra, if it is local and for any $x \in L$, $ord(x) < \infty$ implies $ord(x^-) = \infty$.

(ii) proper filter P of L is called *perfect filter*, if for all $x \in L$, $(x^n)^- \in P$, for some $n \in \mathbb{N} \cup \{0\}$ if and only if $((x^-)^m)^- \notin P$, for all $m \in \mathbb{N} \cup \{0\}$.

Theorem 2.16. *Let P be a filter of BL-algebra L. The following are equivalent:*

(i) *L/P is a perfect BL-algebra,*

(ii) *P is a perfect filter.*

Theorem 2.17. *Any perfect filter of BL-algebra L is a primary filter.*

Theorem 2.18. *[1, 6] Let L be a BL-algebra. Then*

(i) *a local BL-algebra L is a* perfect *BL-algebra if and only if $MV(L)$ is a perfect BL-algebra, where $MV(L) = \{x \in L \mid x^{--} = x\}$.*

(ii) *the only finite perfect MV-algebra is $\{0, 1\}$.*

Definition 2.19. [7] An SBL-algebra is a BL-algebra L verifying this further condition (strict axiom): $(x \odot y)^- = (x^-) \vee (y^-)$

From now on, in this paper $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ (or simply) L is a BL-algebra, unless otherwise state.

3. Integral BL-algebras

In this section we study a class of BL-algebra that called integral BL-algebra and we give some related results.

Definition 3.1. L is called an *integral BL-algebra*, if $x \odot y = 0$, then $x = 0$ or $y = 0$, for all $x, y \in L$.

Example 3.2. [10] Let $L = \{0, a, b, c, 1\}$. Define $\wedge$,$\vee$,$\odot$ and $\rightarrow$ on L as follows:

$\vee$	0	c	a	b	1
0	0	c	a	b	1
c	c	c	a	b	1
a	a	a	a	1	1
b	b	b	1	b	1
1	1	1	1	1	1

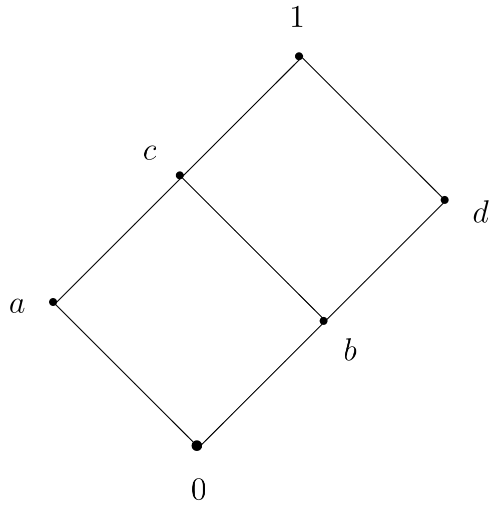
$\wedge$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is an integral BL-algebra.

Example 3.3. [10] Let $L = \{0, a, b, c, d, 1\}$. Then L by the following diagram is a bounded lattice.



Now, let $\rightarrow$ and $\odot$ are defined on L as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL-algebra, which is not an integral BL-algebra. Since $a \odot b = 0$ for $a, b \neq 0$.

Theorem 3.4. *Let P be a filter of L and L/P be an integral BL-algebra. Then P is a primary filter.*

Proof. Let $(x \odot y)^- \in P$, for $x, y \in L$. Then $(x \odot y) \rightarrow 0 \in P$. Since by $(BL7)$, $0 \rightarrow (x \odot y) = 1 \in P$, then $[x \odot y] = 0$ and so $[x] \cdot [y] = [x \odot y] = [0]$. Now, since L/P is an integral BL-algebra then $[x] = [0]$ or $[y] = [0]$. Hence, $x^- = x \rightarrow 0 \in P$ or $y^- = y \rightarrow 0 \in P$ and so P is a primary filter. ∎

The following example shows that, the converse of Theorem 3.4, is not correct, in general.

Example 3.5. Let $F = \{1, d\}$, in Example 3.3. It is easy to check that F is a primary filter, but L/F is not an integral BL-algebra. Since $a \rightarrow 0 = d \in F$, then $[a] = [0]$ and so $[c] \cdot [c] = [c \odot c] = [a] = [0]$. But $[c] \neq [0]$. Therefore, L/F is not an integral BL-algebra.

Theorem 3.6. *Let L be an integral BL-algebra. Then*

(i) *L is a local and directly indecomposable BL-algebra and $B(L) = \{0, 1\}$.*

(ii) *$M(L) = L\backslash\{0\}$.*

(iii) *L is a perfect BL-algebra and $ord(x) = \infty$ when $0 \neq x \in L$.*

(iv) *If L is an Boolean algebra, then $L = \{0, 1\}$.*

Proof. (i) Since $\{1\}$ is a filter of L, then by Theorem 2.6, $L/\{1\}$ is well-defined. Now, let $f : L \to L/\{1\}$ be canonical epimorphism with $[x] = [y]$, for $x, y \in L$. Then $x \to y \in \{1\}$, $y \to x \in \{1\}$ and so by $(BL6)$, $x \leq y$ and $y \leq x$. Hence, $x = y$ and so f is isomorphism. Now, since $L \cong L/\{1\}$ and L is an integral BL-algebra, then $L/\{1\}$ is an integral BL-algebra and by Theorem 3.4, $\{1\}$ is a primary filter. Hence, by Theorem 2.11, $L/\{1\}$ is a local BL-algebra and so L is a local BL-algebra. Thus, by Theorem 2.14, L is directly indecomposable BL-algebra and $B(L) = \{0, 1\}$.

(ii) Let $0 \neq x \in L\backslash M(L)$. Then there exists minimal element $n \in \mathbb{N} \cup \{0\}$, such that $x^n = 0$ and $x^m \neq 0$, for any $m < n$. Now, since $x^{n-1} \odot x = 0$ and L is an integral BL-algebra, then $x^{n-1} = 0$ or $x = 0$, which is impossible. Hence, $M(L) \subseteq L\backslash\{0\} \subseteq M(L)$ and so $M(L) = L\backslash\{0\}$.

(iii) Since L is an integral BL-algebra, then by (i), L is a local BL-algebra. Hence, by (ii) and Theorem 2.10, for all $0 \neq x \in L$, $ord(x) = \infty$. Moreover, if $x = 0$, then $ord(x^-) = \infty$. Hence, L is a perfect BL-algebra.

(iv) Since L is an integral BL-algebra, then by (i), L is a local BL-algebra and $B(L) = \{0, 1\}$. Since L is a Boolean algebra, then $B(L) = L$. Therefore, $L = \{0, 1\}$. ■

Note. In Example 3.5, since $F = \{1, d\}$ is a primary filter, then by Theorem 2.11, L/F is a local BL-algebra, but it is not an integral BL-algebra. Therefore, the converse of Theorem 3.6(i), is not true in general.

Theorem 3.7. *If P is a maximal and implicative filter of L, then L/P is an integral BL-algebra and so P is a primary filter.*

Proof. Let P be a maximal and implicative filter of L and $[x] \cdot [y] = [0]$, for $[x], [y] \in L/P$. Then by $(BL12)$ and $(BL13)$, $x \to (y \to 0) = (x \odot y) \to 0 \in P$ and $y \to (x \to 0) = (x \odot y) \to 0 \in P$. If $x \in P$, then $y \to 0 \in P$ and so $[y] = 0$. Similarly, if $y \in P$, then $x \to 0 \in P$ and so $[x] = 0$. Now, let $x, y \in P$. Then $0 \in P$, that is impossible. If $x, y \notin P$, since P is a maximal and implicative filter, then $x \to y \in P$ and $y \to x \in P$. Since P is an implicative filter, then $x \to 0 \in P$, $y \to 0 \in P$ and so $[x] = 0$ and $[y] = 0$. Hence, L/P is an integral BL-algebra and so by Theorem 3.4, P is a primary filter. ■

Lemma 3.8. *Let P be a primary filter of L and $[x] \cdot [y] = [0]$, for $[x], [y] \in L/P$. Then $[x]$ or $[y]$ is nilpotent.*

Proof. Let $[x] \cdot [y] = [0]$, for $[x], [y] \in L/P$. Then $(x \odot y)^- \in P$. Since P is a primary filter, then $(x^n)^- \in P$ or $(y^n)^- \in P$, for some $n \in \mathbb{N} \cup \{0\}$. Therefore, $[x]^n = [0]$ or $[y]^n = [0]$ and so $[x]$ or $[y]$ is nilpotent. ■

Theorem 3.9. *Let L be a Godel algebra L and P be a filter of L. Then*

(i) *P is a primary filter of L if and only if L/P is an integral BL-algebra.*

(ii) *L is an integral BL-algebra if and only if L is a Local BL-algebra.*

Proof. (i) Let P be a primary filter of L and $[x]\cdot[y]=[0]$, for $[x],[y]\in L/P$. Then by Lemma 3.8, there exist $m,n\in\mathbb{N}$, such that $[x]^n=[0]$ or $[y]^m=[0]$ and so $[x^n]=[0]$ or $[y^m]=[0]$. Now, since L is a Godel algebra, then $x^n=x$ and $y^n=y$ and so $[x]=[0]$ or $[y]=[0]$. Conversely, if L/P is an integral BL-algebra, then by Theorem 3.4, P is a primary filter of L.

(ii) Let L be an integral BL-algebra. Then by Theorem 3.6(i), L is a local BL-algebra. Conversely, let L be a local BL-algebra. Then by $L\cong L/\{1\}$, $L/\{1\}$ is a local BL-algebra and by Theorem 2.11, $\{1\}$ is a primary filter. Therefore, by (i), $L/\{1\}$ is an integral BL-algebra and so L is an integral BL-algebra. ∎

Note. In [7], it is proved that, a linearly ordered BL-algebra L is a SBL-algebra if and only if the negation $\neg x=(x\to 0)$ is a Godel negation.

Definition 3.10. L is called a BL-algebra with Godel negation, if $\underset{St_r}{\to}(\{0\})=L\backslash\{0\}$, where $\underset{St_r}{\to}(\{0\})=\{a\in L\mid a\to 0=0\}$.

In [2], it is proved that $\underset{St_r}{\to}(\{0\})$ is a filter of L.

Lemma 3.11. *Let L be a BL-algebra with Godel negation. Then $x\to 0=0$ or $x\to 0=1$, for all $x\in L$.*

Proof. If $x=0$, then $x\to 0=1$ and if $0\neq x\in L$, then $x\in\underset{St_r}{\to}(\{0\})$ and so $x\to 0=0$, for all $x\in L$. ∎

Example 3.12. BL-algebra in the Example 3.2, is a BL-algebra with Godel negation.

Example 3.13. Let $L=\{0,a,b,1\}$ be a chain such that $0<a<b<1$ and operations $x\wedge y=\min\{x,y\}$, $x\vee y=\max\{x,y\}$, $\odot$ and $\to$ are defined on L as the following tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L,\vee,\wedge,\odot,\to,0,1)$ is a BL-algebra but it is not a BL-algebra with Godel negation.

Theorem 3.14. *Let L be a BL-algebra with Godel negation and F be a proper filter of L. Then*

(i) *if F is a fantastic filter of L, then $F = L\backslash\{0\}$,*

(ii) *if F is a positive implicative(Boolean) filter of L, then $F = L\backslash\{0\}$.*

(ii) *if F is a obstinate filter of L, then $F = L\backslash\{0\}$.*

Proof. (i) Let $0 \neq x \in L$ and F be a proper fantastic filter of L. Then, by Theorem 2.5, $((x \to 0) \to 0) \to x \in F$. Since L is a BL-algebra with Godel negation, then $x \to 0 = 0$, and so by $(BL7)$, we have

$$((x \to 0) \to 0) \to x = (0 \to 0) \to x = 1 \to x = x \in F$$

. Hence, $L\backslash\{0\} \subseteq F$ and so $F = L\backslash\{0\}$.

(ii) Since every positive implicative filter is a fantastic filter, then $F = L\backslash\{0\}$.

(iii) Since every obstinate filter is a fantastic filter, then $F = L\backslash\{0\}$. ∎

Corollary 3.15. *In any BL-algebra L with Godel negation, the following hold:*

(i) *every proper fantastic filter is a maximal filter,*

(ii) *every proper positive implicative filter is a maximal filter,*

(iii) *every proper obstinate filter is a maximal filter.*

Theorem 3.16. *Let L be a BL-algebra L with Godel negation. Then the proper positive implicative filters, the proper obstinate filters and the proper fantastic filters are equal and they are exactly $L\backslash\{0\}$.*

Proof. By Theorem 3.14 and Corollary 3.15, the proof is clear. ∎

4. Integral filters in BL-algebras

Definition 4.1. A proper filter P of L is called an *integral filter*, if for all $x, y \in L$,

$$(x \odot y)^- \in P \text{ implies } x^- \in P \text{ or } y^- \in P$$

Example 4.2. Let $F = \{1, a, c\}$, in the Example 3.3. Then F is an integral filter. Since, if $(x \odot y)^- \in F$, then $(x \odot y)^- = a$ or c or 1, and so $(x \odot y) = b$ or d or 0. If $(x \odot y) = b$, then $x = c$ or b and $y = d$. If $x = c$, then $c^- = b \in F$ and if $x = b$, $b^- \notin F$, then $d^- = a \in F$. By the similar way for $(x \odot y) = d$ or 0, we conclude that F is an integral filter.

Theorem 4.3. *Every integral filter is a primary filter.*

Proof. Let $n = 1$ in Definition 2.3(iv). Then the proof is clear. ∎

The following example shows that the converse of Theorem 4.3 is not true in general.

Example 4.4. Let $F = \{1, d\}$, in Example 3.3. Then it is easy to check that F is a primary filter. Now, since $(c \odot c)^- = a^- = d \in F$ and $c^- = b \notin F$. Therefore, F is not an integral filter.

Theorem 4.5. *Let $F \subseteq G$, where F and G be filters of L and F be an integral filter of L. Then G is an integral filter, too.*

Proof. Let $(x \odot y)^- \in G$, for $x, y \in L$. Then, by $(BL9)$ and $(BL11)$, $((x \odot y) \odot (x \odot y)^-)^- = 0^- = 1 \in F$. Since F is an integral filter, then we get $(x \odot y)^- \in F$ or $(x \odot y)^{--} \in F$. If $(x \odot y)^- \in F$, then $x^- \in F \subseteq G$ or $y^- \in F \subseteq G$ and so G is an integral filter. If $(x \odot y)^{--} \in F \subseteq G$, then by $(x \odot y)^- \in G$ and by $(BL11)$, we have $(x \odot y)^{--} \odot (x \odot y)^- = 0 \in G$. Therefore, $G = L$ and so G is an integral filter. ∎

Theorem 4.6. *Let P be a proper filter of L. Then P is an integral filter if and only if L/P is an integral BL-algebra.*

Proof. Let P be an integral filter and $[x] \cdot [y] = [0]$, for $[x], [y] \in L/P$. Then $(x \odot y)^- \in P$, and so $x^- \in P$ or $y^- \in P$. Hence, $[x] = 0$ or $[y] = 0$. Conversely, let $(x \odot y)^- \in P$, for $x, y \in L$. Then $[x \odot y] = [x] \cdot [y] = [0]$. Since L/P is an integral BL-algebra, then $[x] = 0$ or $[y] = 0$. Therefore, $x^- \in P$ or $y^- \in P$. ∎

The following theorem describes the relationship between integral filters and integral BL-algebras.

Theorem 4.7. *The following conditions are equivalent on L:*

(i) $\{1\}$ *is an integral filter of L,*

(ii) *any filter of L is an integral filter,*

(iii) *L is an integral BL-algebra.*

Proof. (i)⇔(ii): By Theorem 4.5, the proof is clear.

(i)⇒(iii): Since $L \cong L/\{1\}$ and $\{1\}$ is an integral filter, then by Theorem 4.6, L is an integral BL-algebra.

(iii)⇒(i): By Theorem 4.6, the proof is clear. ∎

Theorem 4.8. *L is an integral BL-algebra if and only if L has the Godel negation.*

Proof. Let L be an integral BL-algebra. Then by $(BL7)$, $0 \to 0 = 1$, and so $0 \notin \underset{St_r}{\to}(\{0\})$. Hence, $\underset{St_r}{\to}(\{0\}) \subseteq L \backslash \{0\}$. Now, let $x \in L \backslash \{0\}$. Then by $(BL11)$, $x \odot x^- = 0$ and so $(x \odot x^-)^- = 1$. Since by Theorem 4.7, $\{1\}$ is an integral filter, then $x^- = 1$ or $(x^-)^- = 1$. If $x^- = 1$, then $x \odot x^- = x \odot 1 = x$ and so $x = 0$, that it is a contradiction. Hence, $(x^-)^- = 1$ and so by $(BL10)$, $x^- = ((x^-)^-)^- = 0$. Hence, $x \in \underset{St_r}{\to}(\{0\})$ and so L is a BL-algebra with Godel negation. Conversely, let $(x \odot y) = 0$, for $x, y \in L$. Then $(x \odot y) \notin \underset{St_r}{\to}(\{0\}) = L \backslash \{0\}$. Since $\underset{St_r}{\to}(\{0\})$ is a proper filter, then $x \notin \underset{St_r}{\to}(\{0\})$ or $y \notin \underset{St_r}{\to}(\{0\})$. Therefore, $x = 0$ or $y = 0$ and so L is an integral BL-algebra. ∎

Theorem 4.9. *Every integral BL-algebra is an SBL-algebra.*

Proof. Let L be an integral BL-algebra. We will prove that $(x\odot y)^- = (x^-)\vee(y^-)$, for all $x, y \in L$. If $x\odot y = 0$, then $x = 0$ or $y = 0$ and by Lemma 3.11 and Theorem 4.8, $(x\odot y)^- = 1$ and $(x^-)\vee(y^-) = 1$. Let $x\odot y \neq 0$. Then $x \neq 0$ and $y \neq 0$, hence by Lemma 3.11 and Theorem 4.8, $(x \odot y)^- = 0$ and $(x^-) \vee (y^-) = 0$. Therefore, $(x \odot y)^- = (x^-) \vee (y^-) = 0$. Thus, L is a SBL-algebra. ∎

Corollary 4.10. *Any linearly ordered BL-algebra L is a SBL-algebra if and only if L is an integral BL-algebra.*

Proof. By Theorem 4.9, every integral BL-algebra is an SBL-algebra. Conversely, a linearly ordered BL-algebra L is a SBL-algebra if L is with Godel negation and so by Theorem 4.8, L is an integral BL-algebra. ∎

Theorem 4.11. *Let L be a BL-algebra with Godel negation. Then*

(i) $MV(L) = B(L) = \{0, 1\}$.

(ii) *If L be an MV-algebra, then $L = \{0, 1\}$.*

Proof. (i) It is clear that $0 \in MV(L)$. Now, let $0 \neq x \in MV(L)$. Since $x^- = 0$, then $x = x^{--} = 1$ and so $x = 1$. Hence, $MV(L) = \{0, 1\}$. Now, by Theorems 4.8 and 3.6(ii), $B(L) = \{0, 1\}$ and so $MV(L) = B(L) = \{0, 1\}$.

(ii) Since L is a MV-algebra, then $MV(L) = L$ and so by (i), $L = \{0, 1\}$. ∎

In the following theorem we describe the relationship between integral filters and perfect filters.

Theorem 4.12. *Let P be an integral filter of L. Then P is a perfect filter.*

Proof. Let P be an integral filter. Then, by Theorem 4.6, L/P is an integral BL-algebra and so by Theorem 3.6(iii), L/P is perfect BL-algebra. Hence, by Theorem 2.16, P is a perfect filter. ∎

Open problem. Under what suitable conditions the converse of Theorem 4.12, is correct in general? We prove the converse of Theorem 4.12, in finite BL-algebra and Godel algebra.

Lemma 4.13. *Let L be finite. Then L is an integral BL-algebra if and only if $MV(L)$ is an integral BL-algebra, where $MV(L) = \{x \in L \mid x^{--} = x\}$.*

Proof. Let L be an integral BL-algebra and $x \odot y = 0$, for $x, y \in MV(L)$. Since $MV(L) \subseteq L$, then $x = 0$ or $y = 0$ and so $MV(L)$ is an integral BL-algebra. Conversely, let $MV(L)$ is an integral BL-algebra. Since L is a finite BL-algebra and $MV(L) \subseteq L$, then $MV(L)$ is finite integral BL-algebra and so by Theorem 3.6(iii), $MV(L)$ is a finite perfect MV-algebra. Therefore, by Theorem 2.18(ii), $MV(L) = \{0, 1\}$. Now, let $x \odot y = 0$, for all $x, y \in L$. Then by $(BL16)$, $x \leq y^-$ and by $(BL10)$, $y^- \in MV(L)$. Thus, $y^- = 0$ or $y^- = 1$ and so $y = 1$ or $y = 0$. If $y = 1$, since $x \odot y = 0$, then $x = 0$ and otherwise $y = 0$. Therefore, L is an integral BL-algebra. ∎

Theorem 4.14. *Let L be finite. Then every perfect filter of L is an integral filter.*

Proof. Let P be a perfect filter of finite BL-algebra L. Then, by Theorem 2.16, L/P is a finite perfect BL-algebra. Also, by Theorem 2.18(i), $MV(L/P)$ is a finite perfect MV-algebra. Then, by Theorem 2.18(ii), $MV(L/P) = \{0, 1\}$. Hence, $MV(L/P)$ is an integral BL-algebra and by Lemma 4.13, L/P is an integral BL-algebra. Therefore, by Theorem 4.6, P is an integral filter. ∎

Theorem 4.15. *Let L be a Godel algebra. Then the concept of integral filter, perfect filter and primary filter are coincide.*

Proof. Let P be an integral filter. Then, by Theorem 4.12, P is a perfect filter. Conversely, let P is a perfect filter. Then, by Theorems 2.17 and 3.9(i), L/P is an integral BL-algebra and so by Theorem 4.6, P is an integral filter. Now, let P be a perfect filter. Then, by Theorem 2.17, P is primary filter. Conversely, let P be a primary filter. Then, by Theorem 3.9(i), L/P is an integral BL-algebra and so by Theorem 4.6, P is an integral filter. Hence, by Theorem 4.12, P is a perfect filter. ∎

Theorem 4.16. *Let L be an MV-algebra and F be an integral filter of L. Then F is a prime filter.*

Proof. Let $x \vee y \in F$, for $x, y \in L$. Since by $(BL8)$, $x^- \odot y^- \leq x^- \wedge y^-$, then by $(BL14)$, $(x^- \wedge y^-)^- \leq (x^- \odot y^-)^-$ and by $(BL15)$, $(x^- \wedge y^-)^- = x^{--} \vee y^{--} = x \vee y \in F$. Now, we have $(x^- \odot y^-)^- \in F$, $[(x^- \odot y^-)^-] = [1]$ and $[(x^- \odot y^-)]^{--} = [(x^- \odot y^-)^{--}] = [0]$. Since L is an MV-algebra, then $[x^-] \cdot [y^-] = [x^- \odot y^-] = [0]$. Since by Theorem 4.6, L/F is an integral BL-algebra, then $[x^-] = [0]$ or $[y^-] = [0]$ and so $[x] = [1]$ or $[y] = [1]$. Hence, $x \in F$ or $y \in F$, and so F is a prime filter. ∎

Note. In Theorem 4.16, L must be an MV-algebra and it is a necessary condition. Since in the Example 3.2, L is a BL-algebra, which is not a MV-algebra and $F = \{1\}$ is an integral filter of L. But it is not a prime filter, since $a \vee b = 1 \in F$ and $a, b \notin F$.

Corollary 4.17. *Every integral MV-algebra is a BL-chain.*

Proof. Let L be an integral MV-algebra. Then $\{1\}$ is an integral filter and so by Theorem 4.16, it is a prime filter. Therefore, by Theorem 2.7, $L/\{1\}$ is a BL-chain and since $L \cong L/\{1\}$, then L is a BL-chain. ∎

In what follows, we study the relations among integral filters, obstinate filters and fantastic filters. The following theorem shows that every obstinate filter is an integral filter but the converse is not true.

Theorem 4.18. *Let F be a proper obstinate filter of L. Then F is an integral filter of L.*

Proof. Let F be a proper obstinate filter, $(x \odot y)^- \in F$ but $x^- \notin F$ and $y^- \notin F$, for $x, y \in L$, by a contrary. Since F is an obstinate filter, then $(x^-)^- \in F$ and by Theorem 2.5(ii), F is a fantastic filter. Now, by Theorem 2.5(i), $((x^-)^- \to x) \in F$, and so $x \in F$. By the similar way, $y \in F$. Hence, $(x \odot y) \in F$ and so $0 \in F$, which is a contradiction. Therefore, $x^- \in F$ or $y^- \in F$ and so F is an integral filter. ∎

The following example shows that the converse of Theorem 4.18, is not correct in general.

Example 4.19. Let $F = \{1, a\}$ in Example 3.2. Since L is an integral BL-algebra, then F ia an integral filter, but it is not an obstinate filter. Because $b \notin F$ and $0 = b^- \notin F$. Also, since $((b \to 0) \to 0) \to b = b \notin F$, then by Theorem 2.5, F is not a fantastic filter.

Theorem 4.20. *Let F be an integral and fantastic filter of L. Then F is an obstinate filter of L.*

Proof. Let $x, y \notin F$, for $x, y \in L$. We will show that $x \to y \in F$ and $y \to x \in F$. Since $x \notin F$, then $x^- \in F$. Since, if $x^- \notin F$, by $(BL11)$, $(x \odot x^-)^- = 1 \in F$ and since F is an integral filter, then $(x^-)^- \in F$. Now, since F is a fantastic filter, then by Theorem 2.5(i), $((x^-)^- \to x) \in F$ and so $x \in F$ which is a contradiction. Now, by $(BL14)$, $x^- \leq x \to y$ and since F is a filter and $x^- \in F$, we get that $x \to y \in F$. By the similar way we can prove that, $y \to x \in F$. Therefore, F is an obstinate filter. ∎

The following example show that there is a fantastic filter that, is not an obstinate filter and integral filter.

Example 4.21. Let $F = \{1, a\}$ in Example 3.3. Since L is an MV-algebra, then F is an fantastic filter, which is not an obstinate filter. Because $c \notin F$ and $b = c^- \notin F$. Also, since $(b \odot b)^- = 1 \in F$ and $b^- = c \notin F$, then F is not an integral filter.

Now, by the above theorems and Lemma 3.14 [12], we conclude the following theorem:

Theorem 4.22. *Let F be a filter of L. Then the following conditions are equivalent:*

(i) *F is maximal and positive implicative filter,*

(ii) *F is maximal and implicative filter,*

(iii) *F is an obstinate filter,*

(iv) *F is an integral and fantastic filter.*

5. Conclusion

The results of this paper will be devoted to study the local BL-algebras, perfect BL-algebras and SBL-algebras which are different extension of Basic Logic. In this paper we introduced the notions of integral BL-algebras and integral filters and we prove these filters are perfect and primary but there is still an open problem: under what suitable conditions a perfect filter is an integral filter? This could be a topic of further research.

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