CENTRALIZERS ON SEMIPRIME GAMMA RINGS

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Abstract. Let M be a 2-torsion free semiprime Γ-ring satisfying a certain assumption and let $T : M \to M$ be an additive mapping such that

$$T(x\alpha y\beta x) \quad = x\alpha T(y)\beta x$$

holds for all $x, y \in M$, and $\alpha, \beta \in \Gamma$. Then we prove that T is a centralizer. We also show that T is a centralizer if M contains a multiplicative identity 1.

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1. Introduction

Let M and Γ be additive abelian groups. If there exists a mapping $(x, \alpha, y) \to x\alpha y$ of $M \times \Gamma \times M \to M$, which satisfies the conditions

(i) $x\alpha y \in M$

(ii) $(x + y)\alpha z{=}x\alpha z{+}y\alpha z$, $x(\alpha + \beta)z{=}x\alpha z{+}x\beta z$, $x\alpha(y + z){=}x\alpha y{+}x\alpha z$

(iii) $(x\alpha y)\beta z{=}x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

then M is called a Γ-ring.

Every ring M is a Γ-ring with $M{=}\Gamma$. However a Γ-ring need not be a ring. Gamma rings, more general than rings, were introduced by Nobusawa [11]. Bernes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa.

Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U(U\Gamma M \subset U)$. If U is both a left and a right ideal , then we say U is an ideal of M. Suppose again that M is a Γ-ring. Then M is said to be a 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. An ideal P_1 of a Γ-ring M is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime if $a\Gamma M\Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, M is said to be commutative Γ-ring if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) = \{x \in M : x\alpha y = y\alpha x$ for all $\alpha \in \Gamma, y \in M\}$ is called the centre of the Γ-ring M.

If M is a Γ-ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x, y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta \text{ and}$$
$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta \ ,$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

We consider the following assumption:

(A) $x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M$, and $\alpha, \beta \in \Gamma$.

According to the assumption (A), the above two identities reduce to

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta \text{ and}$$
$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta,$$

which we extensively used.

An additive mapping $T : M \to M$ is a left (right) centralizer if

$$T(x\alpha y) = T(x)\alpha y \quad (T(x\alpha y) = x\alpha T(y))$$

holds for all $x, y \in M$ and $\alpha \in \Gamma$. A centralizer is an additive mapping which is both a left and a right centralizer. For any fixed $a \in M$ and $\alpha \in \Gamma$, the mapping $T(x) = a\alpha x$ is a left centralizer and $T(x) = x\alpha a$ is a right centralizer. We shall restrict our attention on left centralizer, since all results of right centralizers are the same as left centralizers.

An additive mapping $D : M \to M$ is called a derivation if

$$D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$$

holds for all $x, y \in M$, and $\alpha \in \Gamma$ and is called a Jordan derivation if

$$D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$$

for all $x \in M$ and $\alpha \in \Gamma$.

An additive mapping $T : M \to M$ is Jordan left(right) centralizer if

$$T(x\alpha x) = T(x)\alpha x(T(x\alpha x) = x\alpha T(x))$$

for all $x \in M$, and $\alpha \in \Gamma$.

Every left centralizer is a Jordan left centralizer but the converse is not in general true.

An additive mappings $T : M \to M$ is called a Jordan centralizer if

$$T(x\alpha y + y\alpha x) = T(x)\alpha y + y\alpha T(x),$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

Bernes [1], Luh [9] and Kyuno [8] studied the structure of Γ-rings and obtained various generalizations of corresponding parts in ring theory.

Borut Zalar [15] worked on centralizers of semiprime rings and prove that Jordan centralizers and centralizers of this rings coincide. Joso Vukman [12], [13], 14] developed some remarkable results using centralizers on prime and semiprime rings.

Y. Ceven [5] worked on Jordan left derivations on completely prime Γ-rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime Γ-ring that makes the Γ-ring commutative with an assumption. With the same assumption, he showed that every Jordan left derivation on a completely prime Γ-ring is a left derivation on it.

In [6], Fazlul Hoque and A.C. Paul proved that every Jordan centralizer of a 2-torsion free semiprime Γ-ring is a centralizer. Here they also gave an example of a Jordan centralizer which is not a centralizer.

In this paper, we develop some results of J. Vukman [14] in Γ-rings. If M is a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and if $T : M \to M$ is an additive mapping such that

$$T(x\alpha y\beta x) = x\alpha T(y)\beta x \tag{1}$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then T is a centralizer. Also, we prove that T is a centralizer if M contains a multiplicative identity 1.

2. Centralizers of Semiprime Gamma Rings

For proving our main results, we need the following Lemmas:

Lemma 2.1 *Suppose M is a semiprime Γ-ring satisfying the assumption* (A). *Suppose that the relation $a\alpha x\beta b + b\alpha x\beta c = 0$ holds for all $x \in M$, some $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Then $(a + c)\alpha x\beta b = 0$ is satisfied for all $x \in M$ and $\alpha, \beta \in \Gamma$.*

Proof. Putting $x = x\beta b\alpha y$ in the relation

$$a\alpha x\beta b + b\alpha x\beta c = 0 \tag{2}$$

We have

$$a\alpha x\beta b\alpha y\beta b + b\alpha x\beta b\alpha y\beta c = 0 \tag{3}$$

On the other hand, a right multiplication by $\alpha y\beta b$ of (2) gives

$$a\alpha x\beta b\alpha y\beta b + b\alpha x\beta c\alpha y\beta b = 0. \tag{4}$$

Subtracting (4) from (3), we have

$$b\alpha x\beta(b\alpha y\beta c - c\alpha y\beta b) = 0. \tag{5}$$

Putting $x = y\beta c\alpha x$ in (5) gives

$$b\alpha y\beta c\alpha x\beta(b\alpha y\beta c - c\alpha y\beta b) = 0 \tag{6}$$

Left multiplication by $c\alpha y\beta$ of (5) gives

$$c\alpha y\beta b\alpha x\beta(b\alpha y\beta c - c\alpha y\beta b) = 0 \tag{7}$$

Subtracting (7) from (6), we obtain

$$(b\alpha y\beta c - c\alpha y\beta b)\alpha x\beta(b\alpha y\beta c - c\alpha y\beta b) = 0$$

which gives

$$b\alpha y\beta c = c\alpha y\beta b, \tag{8}$$

$y \in M$ and $\alpha, \beta \in \Gamma$. Therefore, $b\alpha x\beta c$ can be replaced by $c\alpha x\beta b$ in (2), which gives

$$a\alpha x\beta b + c\alpha x\beta b = 0$$

i.e.

$$(a + c)\alpha x\beta b = 0$$

Hence, the proof is complete. ∎

Lemma 2.2 *Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption* (A) *and let $T : M \to M$ be an additive mapping. Suppose that*

$$T(x\alpha y\beta x) = x\alpha T(y)\beta x$$

holds for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Then

(i) $[[T(x), x]_\alpha, x]_\beta = 0$

(ii) $x\beta[T(x), x]_\alpha\gamma x = 0$

(iii) $x\beta[T(x), x]_\alpha = 0$

(iv) $[T(x), x]_\alpha\beta x = 0$

(v) $[T(x), x]_\alpha = 0.$

Proof. We prove that (i)

$$[[T(x), x]_\alpha, x]_\beta = 0. \tag{9}$$

For linearization, we put $x + z$ for x in relation (1), we obtain

$$T(x\alpha y\beta z + z\alpha y\beta x) = x\alpha T(y)\beta z + z\alpha T(y)\beta x. \tag{10}$$

Replacing y for x and z for y in (10), we have

$$T(x\alpha x\beta y + y\alpha x\beta x) = x\alpha T(x)\beta y + y\alpha T(x)\beta x. \tag{11}$$

For $z = (x\alpha)^2 x$ relation (10) reduces to

$$\begin{aligned} &T(x\alpha y\beta(x\alpha)^2 x + (x\alpha)^2 x\alpha y\beta x) \\ &\quad = x\alpha T(y)\beta(x\alpha)^2 x + (x\alpha)^2 x\alpha T(y)\beta x. \end{aligned} \tag{12}$$

Putting $y = x\alpha y\beta x$ in (11), we obtain

$$\begin{aligned} &T((x\alpha)^2 x\beta y\beta x + x\beta y\beta(x\alpha)^2 x) \\ &\quad = x\alpha T(x)\beta x\alpha y\beta x + x\alpha y\beta x\alpha T(x)\beta x. \end{aligned} \tag{13}$$

The substitution $x\alpha x\beta y + y\alpha x\beta x$ for y in the relation (1) gives

$$T((x\alpha)^2 x\beta y\beta x + (x\alpha)^2 x\beta y\beta x) = x\alpha T(x\alpha x\beta y + y\beta x\alpha x)\beta x$$

which gives because of (11),

$$\begin{aligned} &T((x\alpha)^2 x\beta y\beta x + x\beta y\beta(x\alpha)^2 x) \\ &\quad = (x\alpha)^2 T(x)\beta y\beta x + x\alpha y\beta T(x)\alpha x\beta x. \end{aligned} \tag{14}$$

Combining (13) with (14), we arrive at

$$x\alpha[T(x), x]_\alpha\beta y\beta x - x\alpha y\beta[T(x), x]_\alpha\beta x = 0. \tag{15}$$

Using (8) in the above relation, we have

$$\begin{aligned} x\alpha y\beta T(x)\alpha x\beta x - x\alpha y\beta x\alpha T(x)\beta x - x\alpha[T(x), x]_\alpha\beta y\beta x &= 0 \\ T(x)\alpha x\alpha y\beta x\beta x - x\alpha T(x)\beta x\alpha y\beta x - x\alpha[T(x), x]_\alpha\beta y\beta x &= 0 \\ T(x)\alpha x\beta x\alpha y\beta x - x\alpha T(x)\beta x\alpha y\beta x - x\alpha[T(x), x]_\alpha\beta y\beta x &= 0 \\ (T(x)\alpha x - x\alpha T(x))\beta x\alpha y\beta x - x\alpha[T(x), x]_\alpha\beta y\beta x &= 0 \\ [T(x), x]_\alpha\beta x\alpha y\beta x - x\beta[T(x), x]_\alpha\alpha y\beta x &= 0 \\ ([T(x), x]_\alpha\beta x - x\beta[T(x), x]_\alpha)\alpha y\beta x &= 0 \\ [[T(x), x]_\alpha, x]_\beta\alpha y\beta x &= 0. \end{aligned} \tag{16}$$

Let $y = y\alpha[T(x), x]_\alpha$ in (16), we have

$$[[T(x), x]_\alpha, x]_\beta \alpha y \alpha [T(x), x]_\alpha \beta x = 0. \tag{17}$$

Right multiplication of (16) by $\alpha[T(x), x]_\alpha$ gives

$$[[T(x), x]_\alpha, x]_\beta \alpha y \beta x \alpha [T(x), x]_\alpha = 0. \tag{18}$$

Subtracting (18) from (17) one obtains

$$[[T(x), x]_\alpha, x]_\beta \alpha y \alpha [[T(x), x]_\alpha, x]_\beta = 0.$$

Since M is semiprime, so (9) follows i.e.

$$[[T(x), x]_\alpha, x]_\beta = 0.$$

Now, we prove the relation (ii):

$$x\beta[T(x), x]_\alpha \gamma x = 0. \tag{19}$$

The linearization of (9) gives

$$\begin{aligned}[][[T(x), x]_\alpha, y]_\beta &+ [[T(y), x]_\alpha, x]_\beta + [[T(y), x]_\alpha, y]_\beta + [[T(x), y]_\alpha, x]_\beta \\ &+ [[T(x), y]_\alpha, y]_\beta + [[T(y), y]_\alpha, x]_\beta = 0.\end{aligned}$$

Putting $x = -x$ in the above relation, we have

$$\begin{aligned}[][[T(x), x]_\alpha, y]_\beta + [[T(y), x]_\alpha, x]_\beta &- [[T(y), x]_\alpha, y]_\beta + [[T(x), y]_\alpha, x]_\beta \\ &- [[T(x), y]_\alpha, y]_\beta - [[T(y), y]_\alpha, x]_\beta = 0.\end{aligned}$$

Adding the above two relations, we have

$$2[[T(x), x]_\alpha, y]_\beta + 2[[T(x), y]_\alpha, x]_\beta + 2[[T(y), x]_\alpha, x]_\beta = 0.$$

Since M is 2-torsion free semiprime Γ-ring, so, we have

$$[[T(x), x]_\alpha, y]_\beta + [[T(x), y]_\alpha, x]_\beta + [[T(y), x]_\alpha, x]_\beta = 0. \tag{20}$$

Putting $x\beta y\gamma x$ for y in (20) and using (1), (9),(20) and assumption (A), we have

$$\begin{aligned}
0 &= [[T(x), x]_\alpha, x\beta y\gamma x]_\beta + [[T(x), x\beta y\gamma x]_\alpha, x]_\beta + [[x\beta T(y)\gamma x, x]_\alpha, x]_\beta \\
&= x\beta[[T(x), x]_\alpha, y]_\beta \gamma x \\
&\quad + [[T(x), x]_\alpha \beta y\gamma x + x\beta[T(x), y]_\alpha \gamma x + x\beta y\gamma[T(x), x]_\alpha, x]_\beta \\
&\quad + [x\beta[T(y), x]_\alpha \gamma x, x]_\beta \\
&= x\beta[[T(x), x]_\alpha, y]_\beta \gamma x + [T(x), x]_\alpha \beta[y, x]_\beta \gamma x + x\beta[[T(x), y]_\alpha, x]_\beta \gamma x \\
&\quad + x\gamma[y, x]_\beta \beta[T(x), x]_\alpha + x\beta[[T(y), x]_\alpha, x]_\beta \gamma x \\
&= [T(x), x]_\alpha \beta[y, x]_\beta \gamma x + x\gamma[y, x]_\beta \beta[T(x), x]_\alpha \\
&= [T(x), x]_\alpha \beta y\beta x\gamma x - x\gamma x\beta y\beta[T(x), x]_\alpha \\
&\quad + x\gamma y\beta x\beta[T(x), x]_\alpha - [T(x), x]_\alpha \beta x\beta y\gamma x.
\end{aligned}$$

Therefore, we have

$$[T(x),x]_\alpha\beta y\beta x\gamma x - x\gamma x\beta y\beta[T(x),x]_\alpha + x\gamma y\beta x\beta[T(x),x]_\alpha$$
$$-[T(x),x]_\alpha\beta x\beta y\gamma x = 0,$$

for all $x, y \in M$, $\alpha, \beta \in \Gamma$, which reduces because of (10) and (15) to

$$[T(x),x]_\alpha\beta y\beta x\gamma x - x\gamma x\beta y\beta[T(x),x]_\alpha = 0.$$

Left multiplication of the above relation by $x\beta$ gives

$$x\beta[T(x),x]_\alpha\beta y\beta x\gamma x - x\beta x\gamma x\beta y\beta[T(x),x]_\alpha = 0.$$

One can replace in the above relation according to (15), $x\beta[T(x),x]_\alpha\beta y\beta x$ by $x\beta y\beta[T(x),x]_\alpha\beta x$ which gives

$$x\beta y\beta[T(x),x]_\alpha\beta x\gamma x - x\beta x\beta x\gamma y\beta[T(x),x]_\alpha = 0. \tag{21}$$

Left multiplication of the above relation by $T(x)\alpha$ gives

$$T(x)\alpha x\beta y\beta[T(x),x]_\alpha\beta x\gamma x - T(x)\alpha x\beta x\beta x\gamma y\beta[T(x),x]_\alpha = 0. \tag{22}$$

The substitution $T(x)\alpha y$ for y in (21), we have

$$x\beta T(x)\alpha y\beta[T(x),x]_\alpha\beta x\gamma x - x\beta x\beta x\gamma T(x)\alpha y\beta[T(x),x]_\alpha = 0. \tag{23}$$

Subtracting (23) from (22), we obtain

$$[T(x),x]_\alpha\beta y\beta[T(x),x]_\alpha\beta x\gamma x - [T(x),x\beta x\gamma x]_\alpha\beta y\beta[T(x),x]_\alpha = 0.$$

From the above relation and Lemma 2.1, it follows that

$$([T(x),x\beta x\gamma x]_\alpha - [T(x),x]_\alpha\beta x\gamma x)\beta y\beta[T(x),x]_\alpha = 0,$$

which reduces to

$$(x\beta[T(x),x]_\alpha\gamma x + x\beta x\gamma[T(x),x]_\alpha)\beta y\beta[T(x),x]_\alpha = 0.$$

Relation (9) makes it possible to write $[T(x),x]_\alpha\gamma x$ instead of $x\gamma[T(x),x]_\alpha$, which means that $x\beta x\gamma[T(x),x]_\alpha$ can be replaced by $x\beta[T(x),x]_\alpha\gamma x$ in the above relation. Thus we have

$$x\beta[T(x),x]_\alpha\gamma x\beta y\beta[T(x),x]_\alpha = 0.$$

Right multiplication of the above relation by γx and substitution $y\beta x$ for y gives finally,

$$x\beta[T(x),x]_\alpha\gamma x\beta y\beta x\beta[T(x),x]_\alpha\gamma x = 0.$$

Hence, by semiprimeness of M, we have

$$x\beta[T(x),x]_\alpha\gamma x \quad = 0.$$

Next, we prove the relation (iii):

$$x\beta[T(x),x]_\alpha = 0,\ x \in M,\ \alpha \in \Gamma. \tag{24}$$

First, putting $y\alpha x$ for y in (15), gives because of (19)

$$x\alpha[T(x),x]_\alpha\beta y\alpha x\beta x = 0. \tag{25}$$

The substitution $y\alpha T(x)$ for y in (25), we have

$$x\alpha[T(x),x]_\alpha\beta y\alpha T(x)\alpha x\beta x = 0. \tag{26}$$

Right multiplication of (25) by $\alpha T(x)$,

$$x\alpha[T(x),x]_\alpha\beta y\alpha x\beta x\alpha T(x) = 0. \tag{27}$$

Subtracting (27) from (26) we have

$$\begin{aligned} x\alpha[T(x),x]_\alpha\beta y\alpha(T(x)\alpha x\beta x - x\beta x\alpha T(x)) = 0 \\ \Rightarrow x\alpha[T(x),x]_\alpha\beta y\alpha[T(x),x\beta x]_\alpha = 0 \\ \Rightarrow x\alpha[T(x),x]_\alpha\beta y\alpha([T(x),x]_\alpha\beta x + x\beta[T(x),x]_\alpha) = 0 \\ \Rightarrow x\beta[T(x),x]_\alpha\alpha y\alpha([T(x),x]_\alpha\beta x + x\beta[T(x),x]_\alpha) = 0. \end{aligned}$$

According to (9), one can replace $[T(x),x]_\alpha\beta x$ by $x\beta[T(x),x]_\alpha$, which gives

$$x\beta[T(x),x]_\alpha\alpha y\alpha x\beta[T(x),x]_\alpha) = 0,\ x,y \in M,\alpha,\ \beta \in \Gamma.$$

Hence, by semiprimeness of M,

$$x\beta[T(x),x]_\alpha = 0,\ x,y \in M,\alpha,\ \beta \in \Gamma.$$

Finally, we prove the relation (v):

$$[T(x),x]_\alpha = 0. \tag{28}$$

From (9) and (24), it follows that

$$[T(x),x]_\alpha\beta x = 0,\ x \in M,\ \alpha,\beta \in \Gamma.$$

The linearization of the above relation gives(see how relation (20) was obtained from (9)),

$$[T(x),x]_\alpha\beta y + [T(x),y]_\alpha\beta x + [T(y),x]_\alpha\beta x = 0.$$

Right multiplication of the above relation by $\beta[T(x),x]_\alpha$ gives because of (24),

$$[T(x),x]_\alpha\beta y\beta[T(x),x]_\alpha = 0,$$

which implies

$$[T(x),x]_\alpha = 0. \qquad \blacksquare$$

Lemma 2.3 *Let M be Γ-ring satisfying the assumption* (A) *and let $T : M \to M$ be an additive mapping such that $T(x\alpha y\beta x) = x\alpha T(y)\beta x$ holds for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Then*

$$x\alpha(T(x\alpha y + y\alpha x) - T(y)\alpha x - x\alpha T(y))\beta x \quad = 0. \tag{29}$$

Proof. The substitution $x\alpha y + y\alpha x$ for y in (1) gives

$$T(x\alpha x\alpha y\beta x + x\alpha y\alpha x\beta x) = x\alpha T(x\alpha y + y\alpha x)\beta x. \tag{30}$$

On the other hand, we obtain by putting $z = x\alpha x$ in (10), we have

$$T(x\alpha x\alpha y\beta x + x\alpha y\beta x\alpha x) = x\alpha T(y)\alpha x\beta x + x\alpha x\alpha T(y)\beta x,$$

i.e.,

$$T(x\alpha x\alpha y\beta x + x\alpha y\alpha x\beta x) = x\alpha T(y)\alpha x\beta x + x\alpha x\alpha T(y)\beta x. \tag{31}$$

By comparing (30) and (31), we have

$$x\alpha(T(x\alpha y + y\alpha x) - T(y)\alpha x - x\alpha T(y))\beta x = 0.$$

Let $G_\alpha(x, y) = T(x\alpha y + y\alpha x) - T(y)\alpha x - x\alpha T(y)$. Then, it is clear that

$$x\alpha G_\alpha(x, y)\beta x = 0 \text{ and } G_\alpha(x, y) = G_\alpha(y, x).$$

Replacing x for y and using (29), we have

$$y\alpha G_\alpha(x, y)\beta y = 0.$$

We can also prove easily the following results:

(i) $G_\alpha(x + z, y) = G_\alpha(x, y) + G_\alpha(z, y)$

(ii) $G_\alpha(x, y + z) = G_\alpha(x, y) + G_\alpha(x, z)$

(iii) $G_{\alpha+\beta}(x, y) = G_\alpha(x, y) + G_\beta(x, y)$

(iv) $G_\alpha(-x, y) = -G_\alpha(x, y)$

(v) $G_\alpha(x, -y) = -G_\alpha(x, y)$. ∎

Lemma 2.4 *Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption* (A) *and let $T : M \to M$ be an additive mapping. Suppose that*

$$T(x\alpha y\beta x) = x\alpha T(y)\beta x$$

holds for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Then

(a) $[G_\alpha(x,y),x]_\alpha = 0$

(b) $G_\alpha(x,y) = 0$.

Proof. First we prove the relation (a):

$$[G_\alpha(x,y),x]_\alpha = 0. \tag{32}$$

The linearization of (28) gives

$$[T(x),y]_\alpha + [T(y),x]_\alpha = 0,\ x,y \in M,\ \alpha \in \Gamma. \tag{33}$$

Putting $x\alpha y + y\alpha x$ for y in the above relation and using (28), we obtain

$$\begin{aligned}
&[T(x), x\alpha y + y\alpha x]_\alpha + [T(x\alpha y + y\alpha x), x]_\alpha = 0\\
&\quad \Rightarrow x\alpha[T(x),y]_\alpha + [T(x),y]_\alpha \alpha x + [T(x\alpha y + y\alpha x),x]_\alpha = 0\\
&\quad \Rightarrow [T(x\alpha y + y\alpha x),x]_\alpha + x\alpha[T(x),y]_\alpha + [T(x,y]_\alpha \alpha x = 0.
\end{aligned}$$

According to (33), one can replace $[T(x),y]_\alpha$ by $-[T(y),x]_\alpha$ in the above relation. We have, therefore,

$$[T(x\alpha y + y\alpha x),x]_\alpha - x\alpha[T(y),x]_\alpha - [T(y),x]_\alpha \alpha x = 0,$$

which can be written in the form

$$[T(x\alpha y + y\alpha x) - T(y)\alpha x - x\alpha T(y), x]_\alpha = 0,$$

i.e.,

$$[G_\alpha(x,y),x]_\alpha = 0.$$

The proof is, therefore, complete. ■

Finally, we prove the relation (b):

$$G_\alpha(x,y) \quad = 0. \tag{34}$$

From (29) one obtains (see how (20) was obtained from (9))

$$x\alpha G_\alpha(x,y)\beta z + x\alpha G_\alpha(z,y)\beta x + z\alpha G_\alpha(x,y) \quad = 0.$$

Right multiplication of the above relation by $G_\alpha(x,y)\alpha x$ gives because of (29),

$$x\alpha G_\alpha(x,y)\beta z\beta G_\alpha(x,y)\alpha x \quad = 0. \tag{35}$$

Relation (32) makes it possible to replace in (35), $x\alpha G_\alpha(x,y)$ by $G_\alpha(x,y)\alpha x$. Thus, we have

$$G_\alpha(x,y)\alpha x\beta z\beta G_\alpha(x,y)\alpha x \quad = 0. \tag{36}$$

Therefore, by semiprimeness of M,

$$G_\alpha(x,y)\alpha x = 0. \tag{37}$$

Of course, we also have

$$x\alpha G_\alpha(x,y) = 0. \tag{38}$$

The linearization of (37) with respect to x gives

$$G_\alpha(x,y)\alpha z + G_\alpha(z,y)\alpha x = 0.$$

Right multiplication of the above relation by $\alpha G_\alpha(x,y)$ gives because of (38),

$$G_\alpha(x,y)\alpha z\alpha G_\alpha(x,y) = 0,$$

which gives

$$G_\alpha(x,y) = 0,$$

i.e.,

$$T(x\alpha y + y\alpha x) = T(y)\alpha x + x\alpha T(y). \tag{39}$$

Hence, the proof is complete. ■

Theorem 2.1 *Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption* (A) *and let $T : M \to M$ be an additive mapping. Suppose that*

$$T(x\alpha x\beta x) = x\alpha T(x)\beta x$$

holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then T is a centralizer.

Proof. In particular, for $y = x$, the relation (39) reduces to

$$2T(x\alpha x) = T(x)\alpha x + x\alpha T(x).$$

Combining the above relation with (28), we arrive at

$$2T(x\alpha x) = 2T(x)\alpha x, x \in M, \alpha \in \Gamma$$

and

$$2T(x\alpha x) = 2x\alpha T(x),\ x \in M,\ \alpha \in \Gamma.$$

Since M is 2-torsion free, so we have

$$T(x\alpha x) = T(x)\alpha x,\ x \in M,\ \alpha \in \Gamma$$

and

$$T(x\alpha x) = x\alpha T(x),\ x \in M,\ \alpha \in \Gamma.$$

By Theorem 2.1 in [6], it follows that T is a left and also right centralizer which completes the proof of the theorem. ■

Putting $y = x$ in relation (1), we obtain

$$T(x\alpha x\beta x) = x\alpha T(x)\beta x,\ x \in M,\ \alpha, \beta \in \Gamma. \tag{40}$$

The question arises whether in a 2-torsion free semiprime Γ-ring the above relation implies that T is a centralizer. Unfortunately, we were unable to answer it affirmative if M has an identity element.

Theorem 2.2 *Let M be a 2-torsion free semiprime Γ-ring with identity element 1 satisfying the assumption* (A) *and let $T : M \to M$ be an additive mapping. Suppose that*

$$T(x\alpha x\beta x) = x\alpha T(x)\beta x$$

holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then T is a centralizer.

Proof. Putting $x + 1$ for x in relation (40), one obtains after some calculations

$$3T(x\alpha x) + 2T(x) = T(x)\beta x + x\alpha T(x) + x\alpha a\beta x + a\alpha x + x\beta a,$$

where a stands for $T(1)$.

Putting $-x$ for x in the relation above and comparing the relation so obtained with the above relation we have

$$6T(x\alpha x) = 2T(x)\beta x + 2x\alpha T(x) + 2x\alpha a\beta x \tag{41}$$

and

$$2T(x) = a\alpha x + x\beta a. \tag{42}$$

We shall prove that $a \in Z(M)$. According to (42) one can replace $2T(x)$ on the right side of (41) by $a\alpha x + x\beta a$ and $6T(x\alpha x)$ on the left side by $3a\alpha x\beta x + 3x\beta x\alpha a$, which gives, after some calculation,

$$a\alpha x\beta x + x\beta x\alpha a - 2x\alpha a\beta x = 0.$$

The above relation can be written in the form

$$[[a, x]_\alpha, x]_\beta = 0;\ x \in M,\ \alpha, \beta \in \Gamma. \tag{43}$$

The linearization of the above relation gives

$$[[a, x]_\alpha, y]_\beta + [[a, y]_\alpha, x]_\beta = 0. \tag{44}$$

Putting $y = x\alpha y$ in (44), we obtain because of (43) and (44),

$$\begin{aligned}
0 &= [[a, x]_\alpha, x\alpha y]_\beta + [[a, x\alpha y]_\alpha, x]_\beta \\
&= [[a, x]_\alpha, x]_\beta \beta y + x\alpha[[a, x]_\alpha, y]_\beta + [[a, x]_\alpha \alpha y + x\alpha[a, y]_\alpha, x]_\beta \\
&= x\alpha[[a, x]_\alpha, y]_\beta + [[a, x]_\alpha \alpha y, x]_\beta + [x\alpha[a, y]_\alpha, x]_\beta \\
&= x\alpha[[a, x]_\alpha, y]_\beta + [[a, x]_\alpha, x]_\beta \beta y + [a, x]_\alpha \beta[y, x]_\alpha + x\alpha[[a, y]_\alpha, x]_\beta \\
&= [a, x]_\alpha \beta[y, x]_\alpha.
\end{aligned}$$

The substitution $y\beta a$ for y in the above relation gives

$$[a, x]_\alpha \beta y \beta [a, x]_\alpha \quad = 0,$$

whence it follows $a \in Z(M)$, which reduces (42) to the form $T(x) = a\alpha x$, $x \in M$, $\alpha \in \Gamma$. The proof of the theorem is complete. ∎

We conclude with the following conjecture:

Let M be a semiprime Γ-ring with suitable torsion restrictions. Suppose there exists an additive mapping $T : M \to M$ such that

$$T((x\alpha)^m (x\beta)^n x) = (x\alpha)^m T(x)(\beta x)^n$$

holds for all $x \in M$, $\alpha, \beta \in \Gamma$, where $m \geq 1$, $n \geq 1$ are some integers. Then T is a centralizer.

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