ON HYPERRINGS ASSOCIATED WITH BINARY RELATIONS ON SEMIHYPERGROUP

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Abstract. In this paper we construct a class of hyperrings associated with binary relations on a semihypergroup. We establish a connection between the constructed hyperring $(H, +_{\rho_1}, \circ_{\rho_2})$ and the hyperring of multiendomorphisms of hypergroup $(H, +_{\rho_1})$. Also, we analyze subclasses of the constructed class, which are associated with partial orderings on a set of multimappings.

1. Introduction

The hyperstructure theory was introduced by F. Marty at the 8^{th} Congress of Scandinavian Mathematicians held in 1934. A semihypergroup $(H, \circ)$ is a nonempty set H equipped with a hyperoperation $\circ$, that is a map $\circ : H \times H \to P^*(H)$, where $P^*(H)$ denotes the family of all nonempty subsets of H, and for all $(x, y, z) \in H^3$: $x \circ (y \circ z) = (x \circ y) \circ z$. A semihypergroup is called a hypergroup in the same sense of Marty [1] if for every $a \in H : a \circ H = H \circ a = H$. In the above definitions, if $A, B \in P^*(H)$, then $A \circ B$ is given by:

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b$$

$x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$.

A comprehensive review of the theory of hyperstructures appears in Corsini [2], Corsini and Leoreanu [3] and Vougiouklis [4].

Similar to hypergroups, that are algebraic structures more general than groups, the hyperrings extend the classical notion of rings, substituting both or only one of the binary operations of addition and multiplication by hyperoperations. The principal notion of hyperring theory can be found in Krasner [5], Davvaz [6], [7], [8], Dasic [9], Rota [10], Spartalis [11] and Vougiouklis [12].

The association between hyperstructures and binary relations had been studied mainly in [13], [14], [15], [16], [17], [18]. Chvalina [13], [14] and Hort [19] use ordered structures for the construction of semigroups and hypergroups. Rosenberg [17] associated with any binary relation ρ of the full domain, a hypergrupoid H_ρ and found conditions on ρ, such that H_ρ is a hypergroup. Corsini and Leoreanu [20] study hypergroups and binary relations.

In Section 3 of this paper we obtain a class of strongly distributive hyperrings associated with binary relations on semihypergroup. We investigate their morphisms and we also establish connection between the constructed hyperring $(H, +_{\rho_1}, \circ_{\rho_2})$ and the hyperring of multiendomorphisms of a hypergroup $(H, +_{\rho_1})$. Schweizer and Sklar [21], has given a set of postulates designed to describe the algebraic behaviour of ordinary functions under any one of the three operations: addition, multiplication or composition. These postulates define a system which is a partially ordered semigroup with identity, where semigroup operation and the partial order relation are related with certain conditions. We show that these systems generate a subclass of a class that we constructed in Section 3.

2. Preliminaries

In this Section, first we recall some notions, notations and results.

Definition 2.1. A multivalued system $(H, +, \cdot)$ is a hyperring if:

1) $(H, +)$ is a hypergroup
2) $(H, \cdot)$ is a semihypergroup
3) $(\cdot)$ is a distributive with respect to $(+)$, i.e. for all $(x, y, z) \in H^3$ we have:
    a) $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$

    and

    b) $(y + z) \cdot x \subseteq y \cdot x + z \cdot x$.

If in conditions 3a) and 3b) the equality is valid, then the hyperring is called strongly distributive.

Definition 2.2. Let $(A, +, \cdot)$ and $(B, +', \cdot')$ be two hyperrings. A map $f : A \to B$ is called an inclusion homomorphism if the following conditions are satisfied:

$$f(x + y) \subseteq f(x) +' f(y)$$

and

$$f(x \cdot y) \subseteq f(x) \cdot' f(y)$$

for all $x, y \in A$. A map f is called a strong homomorphism if in the previous conditions the equality is valid.

Definition 2.3 (and Theorem). *Let $(H,+)$ be a commutative hypergroup, and $F(H)$ the set of multiendomorphisms of H, i.e.*

$$F(H)=\Big\{h:H\to P^*(H)\Big|(\forall x,y\in H)\bigcup_{u\in x+y}h(u)\subseteq h(x)+h(y)\Big\}.$$

Let's define for all $(f,g)\in F(H)\times F(H)$:

$$f\oplus g=\big\{h\in F(H)|(\forall x\in H)h(x)\subseteq f(x)\oplus g(x)\big\}$$

$$f\odot g=\Big\{h\in F(H)\Big|(\forall x\in H)h(x)\subseteq f(g(x))=\bigcup_{u\in g(x)}f(u)\Big\}.$$

Then the structure $(F(H),\oplus,\odot)$ is a hyperring. This hyperring is called a hyperring of multiendomorphisms of hypergroup $(H,+)$. The each binary relation ρ on a set H, a partial hypergroupoid $H_\rho=(H;\circ)$ is associated [17], as follows:

$$\forall(x,z)\in H^2, x\circ x=\{y\in H|(x,y)\in\rho\}, x\circ z=x\circ x\cup z\circ z.$$

By a partial hypergrupoid we mean a non-empty set H endowed with a function from $H\times H$ to the set of subsets of H.

Let

$$D(\rho)=\{x\in H|\exists y\in H:(x,y)\in\rho\}$$

$$R(\rho)=\{x\in H|\exists z\in H:(z,x)\in\rho\}.$$

An element $z\in H$ is called an outer element of ρ if there exists $y\in H$ such that $(y,z)\notin\rho^2$.

Theorem 2.1. [[17]] *H_ρ is a hypergroup if and only if:*

1) $H=D(\rho)$;

2) $H=R(\rho)$;

3) $\rho\subseteq\rho^2$;

4) *if z is an outer element of ρ, then* $\forall x\in H,(x,z)\in\rho^2\Longrightarrow(x,z)\in\rho$.

3. Hyperrings associated with binary relations on semihypergroup

In this Section we construct a class of hyperrings associated with binary relations on semihypergroup. We establish connection between the constructed hyperring $(H,+_{\rho_1},\circ_{\rho_2})$ and the hyperring of multiendomorphisms of hypergroup $(H,+_{\rho_1})$. Also, we analyze morphisms of obtained class and subclasses which are associated with partial orderings on a set of multimappings.

Theorem 3.1. *Let $(H, \circ)$ be a semihypergroup equipped with binary relations ρ_1 and ρ_2 such that $\rho_1 \subseteq \rho_2$. Let $\rho_i (i = 1, 2)$ be a reflexive and transitive relation such that, for all $x, y, z \in H$,*

$$(1) \qquad \begin{aligned} (x, y) \in \rho_i \Longrightarrow \ & (\forall b \in y \circ z)\, (\exists a \in x \circ z)\, (a, b) \in \rho_i \quad \text{and} \\ & (\forall b \in z \circ y)\, (\exists a \in z \circ x)\, (a, b) \in \rho_i . \end{aligned}$$

We define hyperoperations $+_{\rho_1}$ and $\circ_{\rho_2}$ on H, as follows:

$$x +_{\rho_1} y = \{z \mid (x, z) \in \rho_1 \quad or \quad (y, z) \in \rho_1\}$$

and

$$x +_{\rho_2} y = \{z \mid \exists a \in x \circ y, (a, z) \in \rho_2\}$$

for all $(x, y) \in H \times H$.

The structure $(H, +_{\rho_1}, \circ_{\rho_2})$ is a strongly distributive hperring.

Proof. By Theorem 2.1, $(H, +_{\rho_1})$ is a hypergroup. Let us prove that $(H, \circ_{\rho_2})$ is a semihypergroup. Since ρ_2 is reflexive, then for any $a \in x \circ y$ it holds $a \in x \circ_{\rho_2} y$ i.e. $x \circ_{\rho_2} y \neq \emptyset$.

Let $x, y, z \in H$. Set:

$$L = (x \circ_{\rho_2} y) \circ_{\rho_2} z = \bigcup_{u \in x \circ_{\rho_2} y} u \circ_{\rho_2} z$$

and

$$D = x \circ_{\rho_2} (y \circ_{\rho_2} z) = \bigcup_{v \in y \circ_{\rho_2} z} x \circ_{\rho_2} v.$$

Suppose $w \in L$. Then there exists $u \in x \circ_{\rho_2} y$ such that $w \in u \circ_{\rho_2} z$. Thus, there exists $a \in x \circ y$ such that $(a, u) \in \rho_2$ and there exists $c \in u \circ z$ such that $(c, w) \in \rho_2$. By condition (1) from $(a, u) \in \rho_2$ and $c \in u \circ z$ it follows that there exists $c' \in a \circ z$ such that $(c', c) \in \rho_2$. Since ρ_2 is transitive, then $(c', w) \in \rho_2$. On the other hand, $c' \in a \circ z \subseteq (x \circ y) \circ z = x \circ (y \circ z)$ and so $c' \in x \circ v$ for some $v \in y \circ z \subseteq y \circ_{\rho_2} z$. So, $w \in x \circ_{\rho_2} v$ while $v \in y \circ_{\rho_2} z$ i.e. $w \in D$.

Thus, $L \subseteq D$. Similarly, we obtain $D \subseteq L$.

Now, we prove the right distributivity of $\circ_{\rho_2}$ with respect to $+_{\rho_1}$.

Let $x, y, z \in H$. Set:

$$L = (x +_{\rho_1} y) \circ_{\rho_2} z = \cup u \circ_{\rho_2} z, \quad while \quad u \in x +_{\rho_1} y$$

and

$$D = (x \circ_{\rho_2} z) +_{\rho_1} (y \circ_{\rho_2} z) = \cup a +_{\rho_1} b, \quad while \quad a \in x \circ_{\rho_2} z \quad and \quad b \in y \circ_{\rho_2} z.$$

If $w \in L$ then there exists $v \in u \circ z$ such that $(v, w) \in \rho_2$ for some $u \in x +_{\rho_1} y$. We have two possibilities:

(1) If $(x, u) \in \rho_1$ then by condition (1), there exists $a \in x \circ z$ such that $(a, v) \in \rho_1 \subseteq \rho_2$. As ρ_2 is transitive we obtain $(a, w) \in \rho_2$. Since $a \in x \circ z$ then $w \in x \circ_{\rho_2} z$. For any $b \in y \circ_{\rho_2} z$ is holds $w \in w +_{\rho_1} b$, and so $w \in D$.

(2) If $(y, u) \in \rho_1$ we can similarly prove that $w \in D$. Thus, $L \subseteq D$.

Suppose now $w \in D$. Then, there exist $a, b \in H$ such that $w \in a +_{\rho_1} b$ while $a \in x \circ_{\rho_2} z$ and $b \in y \circ_{\rho_2} z$. That means, $(a, w) \in \rho_1$ or $(b, w) \in \rho_1$ while $(a', a) \in \rho_2$ and $(b', b) \in \rho_2$ for some $a' \in x \circ z$ and $b' \in y \circ z$.

If $(a, w) \in \rho_1 \subseteq \rho_2$, since $(a', a) \in \rho_2$ we obtain $(a', w) \in \rho_2$ while $a' \in x \circ z$ i.e. $w \in x \circ_{\rho_2} z$. As $x \in x +_{\rho_1} y$ it follows $w \in L$.

If $(b, w) \in \rho_1$ similarly we obtain $w \in L$. Thus, $D \subseteq L$.

The left distributivity of $\circ_{\rho_2}$ with respect to $+_{\rho_1}$ can be proved in a similar way. Thus, $(H, +_{\rho_1}, \circ_{\rho_2})$ ia a strongly distributive hyperring. ∎

Corollary 3.1. *Let $(H, \cdot)$ be a semigroup equipped with binary relations ρ_1 and ρ_2 such that $\rho_1 \subseteq \rho_2$. Let ρ_i $(i = 1, 2)$ be a reflexive and transitive relation such that for all $x, y, z \in H$, $(x, y) \in \rho_i$ implies $(x \cdot z, y \cdot z) \in \rho_i$ and $(z \cdot x, z \cdot y) \in \rho_i$.*

If we define hyperoperations $+_{\rho_1}$ and $\circ_{\rho_2}$ on H as follows:

$$x +_{\rho_1} y = \{z | (x, z) \in \rho_1 \quad \text{or} \quad (y, z) \in \rho_1\}$$

and

$$x \circ_{\rho_2} y = \{z | (x \cdot y, z) \in \rho_2\}$$

for all $(x, y) \in H \times H$, then the structure $(H, +_{\rho_1}, \circ_{\rho_2})$ is a strongly distributive hyperring.

Throughout the following text the quadruple $(H, \circ, \rho_1, \rho_2)$ will denote a semihypergroup $(H, \circ)$ equipped with binary relations ρ_1 and ρ_2 such that ρ_1 and ρ_2 satisfy the conditions of Theorem 3.1.

Definition 3.1. Let the triples (H_1, ρ_1, ρ_2) and $(H_2, \sigma_1, \sigma_2)$ denote the nonempty set H_1 equipped with binary relations ρ_1, ρ_2 and nonempty set H_2 with binary relations σ_1, σ_2.

(a) The map $\alpha : H_1 \to H_2$ is said to be isotone if

$$x \rho_i y \Longrightarrow \alpha(x) \delta_i \alpha(y)$$

for all $x, y \in H_1$ and $i \in \{1, 2\}$.

(b) The map $\alpha : H_1 \to H_2$ is said to be strongly isotone if

$$\alpha(x) \sigma_i y \Longleftrightarrow (\exists x' \in H_1) x \rho_i x' \wedge \alpha(x') = y$$

for all $(x, y) \in H_1 \times H_2$ and $i \in \{1, 2\}$.

The next theorem is generalization of Theorem 4.5 [22].

Theorem 3.2. *Let $(H_1, +_{\rho_1}, \circ_{\rho_2})$ be a hyperring associated with $(H_1, \circ, \rho_1, \rho_2)$ and $(H_2, +_{\sigma_1}, \circ_{\sigma_2})$ be a hyperring associated with $(H_2, \circ, \sigma_1, \sigma_2)$. If $f : (H_1, \circ) \to (H_2, \circ)$ is an isotone (strongly isotone) homomorphism of semihypergroups $(H_1, \circ)$ and $(H_2, \circ)$, then f is an inclusion (strong) homomorphism of a hyperring $(H_1, +_{\rho_1}, \circ_{\rho_2})$ into hyperring $(H_2, +_{\sigma_1}, \circ_{\sigma_2})$.*

Proof. Let $f : (H_1, \circ) \to (H_2, \circ)$ be an isotone homomorphism and $x, y \in H_1$. If $w \in f(x +_{\rho_1} y)$ then there exists $z \in H_1$ such that $w = f(z)$ while $(x, z) \in \rho_1$ or $(y, z) \in \rho_1$. Since f is isotone, then $(f(x), f(z) = w) \in \sigma_1$ or $(f(y), f(z) = w) \in \sigma_1$ and so $w \in f(x) +_{\sigma_1} f(y)$.

Now, let $w \in f(x \circ_{\rho_2} y)$. Then there exist $a, z \in H_1$ such that $a \in x \circ y, (a, z) \in \rho_2$ and $w = f(z)$. As f is an isotone homomorphism then $f(a) \in f(x) \circ f(y)$ while $(f(a), f(z) = w) \in \sigma_2$ and so $w \in f(x) \circ_{\sigma_2} f(y)$. Thus $f : (H_1, +_{\rho_1}, \circ_{\rho_2}) \to (H_2, +_{\sigma_1}, \circ_{\sigma_2})$ is an inclusion homomorphism.

If $f : (H_1, \circ) \to (H_2, \circ)$ is a strongly isotone homomorphism, then obviously f is isotone and as we proved for all $x, y \in H_1$ it holds: $f(x +_{\rho_1} y) \subseteq f(x) +_{\sigma_1} f(y)$ and $f(x \circ_{\rho_2} y) \subseteq f(x) \circ_{\sigma_2} f(y)$. We will prove the converse inclusion.

Suppose $w \in f(x) +_{\sigma_1} f(y)$. Then $(f(x), w) \in \sigma_1$ or $(f(y), w) \in \sigma_1$. It implies that there exists $z \in H_1$ such that $(x, z) \in \rho_1$ or $(y, z) \in \rho_1$ while $f(z) = w$, as f is strongly isotone. Thus, $w = f(z) \in f(x +_{\rho_1} y)$. So, $f(x) +_{\sigma_1} f(y) \subseteq f(x +_{\rho_1} y)$.

Now, let $w \in f(x) \circ_{\sigma_2} f(y)$. Then there exists $u \in f(x) \circ f(y) = f(x \circ y)$ such that $(u, w) \in \sigma_2$. So, there exists $z \in x \circ y$ such that $u = f(z)$ while $(u, w) \in \sigma_2$. As f is strongly isotone, there exists $a \in H_1$ such that $(z, a) \in \rho_2$ and $w = f(a)$. Thus, $a \in x \circ_{\rho_2} y$ and $w = f(a) \in f(x \circ_{\rho_2} y)$. Therefore $f(x) \circ_{\sigma_2} f(y) \subseteq f(x \circ_{\rho_2} y)$.

This completes the proof. ∎

Let $(H, \circ)$ be a semihypergroup and ρ be a binary relation on H. If $\{A, B\} \subset P^*(H)$ we write $A \tilde{\rho} B$ to denote that: $(\forall a \in A)(\exists b \in B)$ such that $(b, a) \in \rho$ and $(\forall b \in B)(\exists a \in A)$ such that $(a, b) \in \rho$.

Theorem 3.3. *Let $(H, +_{\rho_1}, \circ_{\rho_2})$ be a hyperring associated with $(H, \circ, \rho_1, \rho_2)$ and $(F(H), \oplus, \odot)$ be a hyperring of multiendomorphisms of hypergroup $(H, +_{\rho_1})$. If we define a mapping $\Psi : (H, +_{\rho_1}, \circ_{\rho_2}) \to (F(H), \oplus, \odot)$ by*

$$\Psi(a) = f_a, \forall a \in H,$$

where $f_a : H \to P^*(H)$ is defined by:

$$f_a(x) = a \circ_{\rho_2} x, \forall x \in H,$$

then:

(1) *Ψ is an inclusion homomorphism.*

(2) *If there exists at least one element $x \in H$ such that*

$$a \circ x \tilde{\rho}_2 b \circ x \Longrightarrow a = b$$

for all $a, b \in H$, then Ψ is injective.

Proof. As $(H, +_{\rho_1})$ is a commutative hypergroup there exists the hyperring of its multiendomorphisms $(F(H), \oplus, \odot)$.

Let $a \in H$. We verify that $f_a \in F(H)$. Let $x, y \in H$.

Then:

$$\bigcup_{u\in x+_{\rho_1}y} f_a(u) = \bigcup_{u\in x+_{\rho_1}y} a\circ_{\rho_2}u = a\circ_{\rho_2}(x+_{\rho_1}y) = (a\circ_{\rho_2}x)+_{\rho_1}(a\circ_{\rho_2}y) = f_a(x)+_{\rho_1}f_a(y).$$

(1) Let $a, b \in H$. Set:

$$L = \Psi(a +_{\rho_1} b) = \{f_c|(a,c) \in \rho_1 \quad or \quad (b,c) \in \rho_1\}$$

and

$$D = \Psi(a) \oplus \Psi(b) = f_a \oplus f_b = \{h|(\forall x)h(x) \subseteq f_a(x) +_{\rho_1} f_b(x)\}.$$

Let $c \in H$ such that $(a,c) \in \rho_1$ and $x \in H$. Then:

$$f_c(x) = c \circ_{\rho_2} x = \{y|\exists z \in c \circ x, (z,y) \in \rho_2\}.$$

If $y \in f_c(x)$, then $(z,y) \in \rho_2$ for some $z \in c \circ x$ and as $(a,c) \in \rho_1$ there exists $w \in a \circ x$ such that $(w,z) \in \rho_1 \subseteq \rho_2$. By transitivity of ρ_2 we obtain $(w,y) \in \rho_2$ i.e. $y \in a \circ_{\rho_2} x \subseteq f_a(x) +_{\rho_1} f_b(x)$.

If $c \in H$ such that $(b,c) \in \rho_1$, similarly we obtain $f_c \in D$. Thus, $L \subseteq D$.

Now, assume:

$$L = \Psi(a \circ_{\rho_2} b) = \{f_c|c \in a \circ_{\rho_2} b\} = \{f_c|\exists d \in a \circ b, (d,c) \in \rho_2\}$$

and

$$D = \{h|(\forall x)h(x) \subseteq f_a(f_b(x))\}.$$

Let $c \in H$ such that $(d,c) \in \rho_2$ for some $d \in a \circ b$ and $x \in H$. If $y \in f_c(x)$ then $(z,y) \in \rho_2$ for some $z \in c \circ x$, as $(d,c) \in \rho_2$ and $z \in c \circ x$, there exists $w \in d \circ x$ such that $(w,z) \in \rho_2$ and by transitivity of ρ_2 we obtain $(w,y) \in \rho_2$. Thus, $y \in d \circ_{\rho_2} x \subseteq (a \circ b) \circ_{\rho_2} x \subseteq (a \circ_{\rho_2} b) \circ_{\rho_2} x = a \circ_{\rho_2} (b \circ_{\rho_2} x) = f_a(f_b(x))$. Therefore, $f_c(x) \subseteq f_a(f_b(x))$ i.e. $f_c \in D$.

So, $L \subseteq D$.

(2) Let $a \neq b$. Then there exists $x \in H$ such that one of the following is valid:

(i) $(\exists c \in a \circ x)(\forall d \in b \circ x)(d,c) \notin \rho_2$

or

(ii) $(\exists d \in b \circ x)(\forall c \in a \circ x)(c,d) \notin \rho_2$.

If (i) is valid, then there exists $c \in a \circ x$ such that $c \notin b \circ_{\rho_2} x = f_b(x)$. Since $c \in a \circ x \subseteq a \circ_{\rho_2} x = f_a(x)$, it follows $f_a(x) \nsubseteq f_b(x)$.

If (ii) is valid, then similarly $f_b(x) \nsubseteq f_a(x)$. Hence, $f_a \neq f_b$ i.e. $\Psi(a) \neq \Psi(b)$.

Remark 3.1. If $(H, +_{\rho_1}, \circ_{\rho_2})$ is a hyperring associated with $(H, \cdot, \rho_1, \rho_2)$ where $(H, \cdot)$ is a semigroup with at least one reductive element and ρ_2 is the partial order then Ψ is an inclusion monomorphism of a hyperring $(H, +_{\rho_1}, \circ_{\rho_2})$ into $(F(H), \oplus, \odot)$.

Now, we deal with some classes of hyperrings associated with partial orderings on a semihypergroup (semigroup). Schweizer and Sklar [21] postulated a system $(S, \cdot, \leqslant)$ where $(S, \cdot)$ is semigroup with identity where partial order $\leqslant$ satisfies certain conditions, such that these systems are natural generalization of certain algebras of functions. In that case, the set of mappings from the reals to the reals is partially ordered by restriction, i.e. ordinary set inclusion, where the function being regarded as sets of ordered pairs of numbers. We show that such systems generate a suitable subclasses of a class constructed in Theorem 3.1.

Theorem 3.4. *Let $(H, \circ)$ be a semigroup with identity e and let $\leqslant$ be the partial order on H, connected with semigroup operation with next conditions:*

(1) *If $a \leqslant b$ then there exists an element $e_1 \leqslant e$ such that $a = b \circ e_1$*

(2) *If $e_2 \leqslant e$ then $a \circ e_2 \leqslant a$ and $e_2 \circ a \leqslant a$.*

Then, there exists a hyperring $(H, +_{\leqslant}, \circ_{\leqslant})$ associated with $(H, \cdot, \leqslant, \leqslant)$.

Proof. From Theorem 1 [21] it follows that the quadruple $(H, \circ, \leqslant, \leqslant)$ satisfies the conditions of Corollary 3.1. ∎

Example 1.

(a) Denote by $P^*(\mathbb{R})$ the set of all nonempty subsets of reals $\mathbb{R}$. Let $M(\mathbb{R})$ be the set of all multimappings from relas to reals, i.e.

$$M(\mathbb{R}) = \{f : A \to P^*(\mathbb{R}) \mid A \subseteq \mathbb{R}, A \neq \emptyset\}.$$

We can consider function f as the set $\big\{(x, f(x)) \mid x \in Dom f\big\}$. Denote by φ the empty multimapping, i.e. the multimapping that contains no ordered pairs. If $f, g \in M(\mathbb{R})$ and $Dom f \cap Dom g \neq \emptyset$, define:

$$f + g = \big\{(x, f(x) + g(x)) | x \in Dom f \cap Dom g\big\}.$$

If $f, g \in M(\mathbb{R}) \cup \varphi$ and $Dom f \cap Dom g = \emptyset$, we define:

$$f + g = \varphi.$$

Obviously, $(M(\mathbb{R}) \cup \varphi, +)$ is a semigroup with identity element $o : \mathbb{R} \to \mathbb{R}$ defined by $o(x) = 0$ for all $x \in \mathbb{R}$.

If we define:

$$f \leqslant g \Longleftrightarrow Dom f \subseteq Dom g \quad \text{and} \quad (\forall x \in Dom f) f(x) = g(x),$$

then $M(\mathbb{R} \cup \varphi, +, \leqslant)$ satisfies conditions of Theorem 3.4. The restriction of o which has the same domain as f serves as element $e_1 \leqslant o$ such that $f \leqslant g$ implies $f = g + e_1$. Condition (2) of Theorem 3.4 is, obviously, satisfied. So, there exists a hyperring associated with $(M(\mathbb{R}) \cup \varphi, +, \leqslant, \leqslant)$.

It can be noted that condition (1) in Theorem 3.4 is symmetric in this case. Generally, it is not valid (see [21]).

(b) Now, we will define a hyperoperation $\oplus$ on a set $M(\mathbb{R}) \cup \varphi$, as follows:

$$f \oplus g = \big\{u \big| Domu = Domf \cap Domg, \quad and \quad \forall x \in Domu, u(x) \subseteq f(x)+g(x)\big\}$$

for all $f, g \in M(\mathbb{R})$ such that $Domf \cap Domg \neq \emptyset$, and $f \oplus g = \varphi$ for all $f, g \in M(\mathbb{R}) \cup \varphi$ such that $Domf \cap Domg = \emptyset$.

Then, $(M(\mathbb{R}) \cup \varphi, \oplus)$ is a semihypergroup. If we define a binary relation $\leqslant$ as in a Example 1(a), then, obviously, $\leqslant$ is a partial order. Let us verify that $(M(\mathbb{R}) \cup \varphi, \oplus, \leqslant, \leqslant)$ satisfies condition (1) of Theorem 3.1. Let $f \leqslant g$ and $b \in g \oplus h$.

If $Domg \cap Domh = \emptyset$ then $Domf \cap Domh = \emptyset$ and $b = \varphi$. If we put $a = \varphi$ then $a \in f \oplus h$.

If $Domg \cap Domh \neq \emptyset$ and $b \in g \oplus h$ then $Domb = Domg \cap Domh$ and for all $x \in Domb$ it holds $b(x) \subseteq g(x) + h(x)$. Now, we have two possibilities:

(1) If $Domf \cap Domh = \emptyset$ then we put $a = \varphi$ and obviously $a \in f \oplus h$ and $a \leqslant b$.

(2) If $Domf \cap Domh = A \neq \emptyset$, then we put $A = Doma$ and $a(x) = b(x)$ for all $x \in A$.

As $Domf \subseteq Domg$, then $A \subseteq Domb$ and we obtain $a \leqslant b$.

As $a(x) = b(x) \subseteq g(x) + h(x) = f(x) + h(x)$ for all $x \in A$, it follows $a \in f \oplus h$. As $\oplus$ is commutative hyperoperation it follows that condition (1) of Theorem 3.1 is satisfied. Thus, there exists a hyperring associated with $(M(\mathbb{R}) \cup \varphi, \oplus, \leqslant, \leqslant)$.

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