ON BOUNDEDNESS AND CONTNUITY OF JORDAN, ORDINARY AND QUADRATIC PRODUCT IN ALTERNATIVE SEMI-PRIME ALGEBRAS

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Abstract. In this work we prove that, if A is an alternative semi-prime algebra, which is considered as a complete convex bornological vector space (respectively, completely bornological locally convex space) and its bornology has a net, then there is equivalent between separating boundedness (resp. separating continuity) of Jordan, ordinary product and quadratic product. If A is again topological, then the boundedness is global and if A is Fréchet space, there is an equivalence between the continuity of these three products.

Keywords: bornological algebras, boundedness, Jordan product, quadratic product, derivation, alternative semi-prime algebra, Mackey-convergence, separating space, net in bornological space.

1. Introduction

Let A be an alternative semi-prime $\mathbb{K}$-algebra with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

In [5], A. Rodriguez Palacios proved that if A is Banach space then the continuity of Jordan product implies those of ordinary product. After, he extended this results in case of Fréchet space by similar techniques.

The goal of this work is to extend this result to class of alternative algebras which are also convex bornological vector space (cbs), [3], [4].

The paper is organized as follows. In the next section we recall some preliminaries. In Section 3, we extend some results defined in normed spaces onto bornological vector spaces case. In particular, we extend the notion of separating space of a linear map between Banach spaces [7] to the case of linear map T

between bornological vector spaces (Definition 3.1). We give the necessary conditions for an operator that is bounded (Corollary 3.1). In Section 4, we prove some properties of derivations which are necessary in the following. Next, in Section 5, inspiring of Rodriguez Palacios's technical, we prove the main theorem (Theorem 5.1) and, consequently, we generalize Rodriguez Palacios's theorem (Corollary 5.3). Also, in Section 6, we prove the equivalence between the boundedness of Jordan product and the boundedness of quadratic product (Theorem 6.1).

Furthermore, we study the continuity problem of this products (Theorem 6.2). In the Fréchet case, we conclude the equivalence between the global continuity of ordinary product and quadratic product (Corollary 6.2).

2. Preliminaries

Recall that a bornology on a set X is a family $\mathcal{B}$ of subset X such that $\mathcal{B}$ is a covering of X, hereditary under inclusion and stable under finite union.

The pair $(X, \mathcal{B})$ is called bornological set.

A subfamily $\mathcal{B}'$ of $\mathcal{B}$ is said to be base of bornology $\mathcal{B}$, if every element of $\mathcal{B}$ is contained in an element of $\mathcal{B}'$.

Let X and Y be two bornological set, a map $X \longrightarrow Y$ is called bounded if the image of every bounded subset of X is bounded in Y.

A bornology $\mathcal{B}$ on $\mathbb{K}$-vector space E is said to be vector bornology on E if the maps $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda.x$ are bounded.
We called a bornological vector space(b.v.s) any pair $(X, \mathcal{B})$ consisting of a vector space E and a vector bornology $\mathcal{B}$ on E.

A vector bornology on a vector space is called a convex vector bornology if it is stable under the formation of convex hull.

A bornological vector space is said a convex bornological vector space (cbvs) if its bornology is convex.

A sequence $(x_n)_{n\geq 0}$ in bornological vector space (b.v.s) E is said Mackey-convergent to 0 (or converge bornologically to 0) if there exists a bounded set $B \subset E$ such that

$$(\forall \varepsilon > 0)\ (\exists n_0 \in \mathbb{N})/\ (n \geq n_0) \text{ implies } (x_n \in \varepsilon B).$$

If E is (cbvs) then $(x_n)_{n\geq 0}$ is Mackey-convergent to 0 if there exists a bounded disk $B \subset E$ such that $(x_n)_{n\geq 0} \subset E_B$ and $(x_n)_{n\geq 0}$ converges to 0 in E_B.

A set B in a (bvs) E is said M-closed (or b-closed) if every sequence $(x_n)_{n\geq 0} \subseteq B$ Mackey convergent in E its limit belongs to B.

Let E be a (cbvs) and A is a disk in E. A is called a completant disk if the space (E_A, p_A) is a Banach space.

A (cbvs) E is called a complete convex bornological vector space if its bornology has a base consisting of completant disks.

Let E be a (bvs) and $A \subset E$, the bornological closure (briefly b-closure or M-closure) of A denoted $\overline{A}$ is the intersection of all bornologically closed subsets of E containing A.

If $(E,\mathcal{B})$ is a (cbs) then, there exists on E a locally convex topology denoted t_E.

Let (E,T) be a topological vector space (tvs) we denotes by b_E its von Neumann bornology given by the topology T. Then (E,b_E) is a (cbvs).

Generally we have $T \neq t_{b_E}$, if there is equality, we say that the topology T is bornological.

Let $(E,\mathcal{B})$ be a (cbvs). The bornology $\mathcal{B}$ is said topological if $\mathcal{B} = b_{t_E}$.

Let E and F be two (lcs) and $u : E \to F$ a bounded linear map. If the topology of E is bornological then u is continuous.

Let (E,T) be a (lcs), E is called completely bornological if there exists a complete convex bornological space (E_1, B) such that $T = t_{b_{E_1}}$ algebraically and topologically. Evidently, in this case E is bornological.

Since every Fréchet space is completely bornological, then it is bornological [2].

Recall that if E is a (cbvs), a net in E is a family $\mathcal{R}$ of disks of E, $V_{n_1,\dots,n_k}$ with $k, n_1, n_2, \dots, n_k \in \mathbb{N}$, satisfying the condition

$$E = \bigcup_{n_1 \geq 1} V_{n_1} \text{ and } V_{n_1,\dots,n_{k-1}} = \bigcup_{n_k \geq 1} V_{n_1,\dots,n_k} \text{ for all } k > 1.$$

If $\mathcal{B}$ is a separated convex bornology on E, we say that $\mathcal{R}$ and $\mathcal{B}$ are compatible if the following two properties are verified:

(i) For every sequence $(n_k)_{k\geq 1}$ of integers, there exists a sequence $(\alpha_k)_{k\geq 1}$ of positive reals such that, for each $f_k \in V_{n_1,\dots,n_k}$ and for each $\mu_k \in [0,\alpha_k]$ the series $\sum_{k=1}^{\infty} \mu_k f_k$ converges bornologically in $(E,\mathcal{B})$ and its sum satisfies $\sum_{k=1}^{\infty} \mu_k f_k \in V_{n_1}, \dots, n_{k_0}$ for every $k_0 \in \mathbb{N}$.

(ii) For every pair $(n_k, \lambda_k)_{k\geq 1}$ consisting of a sequence $(n_k)_{k\geq 1}$ of positive integers and a sequence $(\lambda_k)_{k\geq 1}$ of a positive reals, the set $\bigcap_{k\geq 1} \lambda_k V_{n_1,\dots,n_k}$ is bounded in $(E,\mathcal{B})$.

We say that a convex bornological space $(E,\mathcal{B})$ has a net, or that its bornology has a net, if there exists in E a net $\mathcal{R}$ compatible with $\mathcal{B}$. In this case we also say that $\mathcal{R}$ is a net in $(E,\mathcal{B})$ and that $(E,\mathcal{B})$ is a space with a net, see [1], [2].

Recall that every bornology of a (cbvs) having a countable base has a net. Consequently the von Neumann bornology b_E of a Fréchet space has a net.

The bornologically closed graph theorem. [4] *Let* $(E,\mathcal{B})$ *be a complete* (cbvs) *and* $(E',\mathcal{B}')$ *be a* (cbvs) *such that* $\mathcal{B}'$ *has a net. Then, every linear map* $u : E \to E'$ *with a bornologically closed graph in* $E \times E'$ *is bounded.*

Let $\mathbb{K}$ be a commutative field of characteristic 0. An algebra over $\mathbb{K}$ is a $\mathbb{K}$-vector space A with a bilinear map $(x,y) \mapsto x.y$ of $A \times A$ into A. If this product is associative (resp commutative), we say that the algebra is associative (resp. commutative).

Let $x, y \in A$, define the following maps:

$$R_x(y) = yx$$
$$L_x(y) = xy$$
$$U_x(y) = 2x(xy) - x^2y$$
$$U_{x,y} = \frac{1}{2}(U_{x+y} - U_x - U_y)$$

It is well-known that $x \mapsto U_x$ is quadratic. $U_x(y)$ is called a quadratic product of x and y.

Let A be an algebra, we denote by A^+ the algebra A equipped with its vector space structure and the product $\circ$ defined by

$$x \circ y = \frac{1}{2}(xy + yx) \text{ for all } x,\, y \in A.$$

The product $\circ$ is called the Jordan product.

In A^+, we have the remarkable identities

$$x \circ y = y \circ x \quad \text{and} \quad (x^2 \circ y) \circ x = x^2 \circ (y \circ x).$$

Let A be an algebra, A is called Alternative algebra if

$$\forall x, y \in A,\ x(xy) = x^2y \text{ and } (yx)x = yx^2.$$

It follows that $U_x(y) = xyx$.

Denotes $[x, y] = xy - yx$, it is easy to verify that:

$$[x, [x, y]] \quad = \quad 4(y \circ (x \circ x) - (y \circ x) \circ x) \quad \text{and}$$
$$[x, y \circ z] \quad = \quad y \circ [x, z] + [x, y] \circ z.$$

An alternative algebra A is semi-prime if $\{0\}$ is the only two sided ideal J of A with $J^2 = 0$. This is equivalent to $aAa = 0$ implies that $a = 0$.

On the other hand, an alternative algebra A is semi-prime if, and only if, A^+ is semi-prime.

A linear mapping D on an algebra A is called a derivation if:

$$D(xy) = D(x)y + xD(y) \text{ for all } x,\, y \in A$$

Let D be a derivation on an algebra A, we have the Leibnitz rule:

$$\forall x, y \in A,\ D^n(xy) = \sum_{i=0}^{n} C_n^i D^i(x) D^{(n-i)}(y).$$

For details, we can see [3].

3. Separating space and properties

In [7], Sinclair studied the necessary conditions for continuity of homomorphisms, derivations and pair of operators acting on a Banach space.

The aim of the present paper is to extend some of this results in case of bornological vector space (bvs) and consequently obtains some techniques to answer the boundedness problem for linear operators.

We extend naturally the notion of separating space of some linear operator S between (bvs) X and (bvs) Y. The notion of separating space characterizes the continuity of linear operators (Corollary 3.1).

Definition 3.1. Let X and Y be two bornological vector spaces, let T be a linear map of X into Y. We called the separating space of T, the subset $\sigma(T)$ of Y defined by:

$$\sigma(T) = \{y \in Y/\exists(x_n)_n \subset X : x_n \stackrel{M}{\to} 0 \text{ and } Tx_n \stackrel{M}{\to} y\}$$

Proposition 3.1. *Let X and Y be two bornological vector spaces and $T : X \to Y$ a linear map. Then $\sigma(T) = \{0\}$ if, and only if, the graph of T is b-closed.*

Proof. Let $G(T)$ be the graph of T and suppose that $\sigma(T) = \{0\}$.

Let $(x_n, T(x_n))_{n\geq 0} \subset G(T)$ such that $(x_n, T(x_n)) \stackrel{M}{\to} (x, y) \in X \times Y$.

Then, $x_n \stackrel{M}{\to} x$ and $T(x_n) \stackrel{M}{\to} y$, but $T(x_n - x) = T(x_n) - T(x)$.

Hence $T(x_n - x) \stackrel{M}{\to} y - T(x)$.

Consequently, $(y - T(x)) \in \sigma(T)$ and $y = T(x)$.

Conversely, assume that $G(T)$ is b-closed.

Let $y \in \sigma(T)$. There exists a sequence $(x_n)_{n\geq 0} \subset X$ such that $x_n \stackrel{M}{\to} 0$ and $T(x_n) \stackrel{M}{\to} y$. Since $(x_n, T(x_n))_n \subset G(T)$ and $(x_n, T(x_n))_{n\geq 0} \stackrel{M}{\to} (0, y)$, then $(0, y) \in G(T)$. Therefore $y = 0$. ∎

Corollary 3.1. *Let X be a complete convex bornological vector space* (ccbvs) *and Y be a bornological vector space* (bvs) *such that its bornology has a net, let $T : X \to Y$ a linear map. Then, T is bounded if, and only if, $\sigma(T) = \{0\}$.*

Proof. Using the bornologically closed graph theorem and Proposition 3.1. ∎

4. Properties of derivations

Proposition 4.2. *Let A be an alternative algebra, assume that A is considered as a bornological vector space* (bvs)*. Let D be a derivation on A. If the ordinary product of A is separating bounded, then the separating space $\sigma(D)$ of D is two-sided ideal in A.*

Proof. Let $a \in A$ and $b \in \sigma(D)$. Then there exists a sequence $(b_n)_{n\geq 0} \subset A$ such that $b_n \stackrel{M}{\to} 0$ and $D(b_n) \stackrel{M}{\to} b$. By definition of derivation we have

$$\begin{aligned} D(ab_n) &= aD(b_n) + D(a)b_n \\ D(b_na) &= b_nD(a) + D(b_n)a \end{aligned}$$

Since the ordinary product of A is separating bounded, by taking the limit we have: $D(ab_n) \stackrel{M}{\to} ab$, $D(b_na) \stackrel{M}{\to} ba$, $ab_n \stackrel{M}{\to} 0$ and $b_na \stackrel{M}{\to} 0$.

Therefore, $ab \in \sigma(T)$ and $ba \in \sigma(D)$. ∎

Proposition 4.3. *Let A be an alternative algebra, assume that A is considered as a bornological vector space* (bvs). *Let D be a derivation on A such that its square is bounded. If the ordinary product of A is separating bounded, then $[\sigma(D)]^2 = \{0\}$.*

Proof. Let $a \in A$ and $b \in \sigma(D)$. Then there exists $(b_n)_{n\geq 0} \subset A$ such that

$$b_n \overset{M}{\to} 0 \text{ and } D(b_n) \overset{M}{\to} b.$$

By Leibnitz rule, it follows that

$$D^2(ab_n) = aD^2(b_n) + 2D(a).D(b_n) + D^2(a)b_n.$$

Since D^2 is bounded and the ordinary product of A is separating bounded, taking the limit in sense of Mackey we obtain

$$(\forall a \in A),\ (\forall b \in \sigma(D)),\ D(a)b = 0.$$

This shows that $[\sigma(D)]^2 = \{0\}$. ∎

Corollary 4.2. *Let A be an alternative semi-prime algebra, assume that A is considered as a complete convex bornological vector space* (cbvs) *and its bornology has a net. If the ordinary product of A is separating bounded, then every derivation having a bounded square is bounded.*

Proof. By Corollary 3.1, it suffices to prove that $\sigma(D) = \{0\}$.

By Proposition 4.1, $\sigma(D)$ is two sided ideal, and from Proposition 4.2 we have

$$[\sigma(D)]^2 = \{0\}.$$

Since A is semi-prime, then $\sigma(D) = \{0\}$. ∎

5. Boundedness and continuity of Jordan and ordinary products

Theorem 5.1. *Let A be an alternative semi-prime algebra, assume that A is considered as a complete convex bornological vector space and its bornology has a net. Then, the Jordan product is separating bounded if, and only if, the ordinary product is separating bounded.*

Proof. Clearly, if the ordinary product is separating bounded then Jordan product is separating bounded.

Conversely, assume that the Jordan product is separating bounded.

Since A is semi-prime, A^+ is again semi-prime [5].

Consider an arbitrary element $a \in A$, and define the map D_a on A:

$$D_a(x) = [a, x] = ax - xa \text{ for all } x \in A$$

D_a is derivation in A^+, indeed:

$$D_a(xoy) = [a, x \circ y] = x \circ [a, y] + [a, x] \circ y = x \circ D_a(y) + D_a(x) \circ y.$$

D_a^2 is bounded, indeed:

$$D_a^2(x) = [a, [a, x]] = 4(x \circ (a \circ a) - (x \circ a) \circ a).$$

Since A^+ satisfies the conditions of corollary 4.1, it follows that D_a^+ is bounded.

On the other hand, consider the bilinear mapping φ defined on $A \times A$ by

$$\varphi(x, y) = [x, y].$$

Therefore, φ is separating bounded.

Since $ab = a \circ b + \frac{1}{2}\varphi(a, b)$, then ordinary product is separating bounded. ∎

Remark 5.1. For example of complete convex bornological vector space (cbvs) such its bornology has a net, we take Fréchet space and complete convex bornological vector space with a countable base.

Lemma 5.1. *Let E be a Mackey-complete convex bornological vector space, F be a topological convex bornological space and G be a bornological vector space. Let $\psi : E \times F \to G$ be a bilinear map separating bounded. Then, ψ is globally bounded.*

Proof. It is immediate by Theorems 5 and 8 of [1]. ∎

Corollary 5.3. *Let A be an alternative semi-prime algebra, assume that A is considered as a complete topological convex bornological vector space and its bornology has a net. Then, the Jordan product is bounded if, and only if, the ordinary product is bounded.*

Proof. It is immediate application of Lemma 5.1 with $E = F = G = A$. ∎

Corollary 5.4. *Let A be an alternative semi-prime algebra, assume that A is considered as a completely bornological locally convex space and its bornology has a net. Then, the Jordan product is separating continuous if, and only if, the ordinary product is separating continuous.*

Proof. Assume that Jordan product is separating continuous. Then it is is separating bounded. Since A is completely bornological, there is on A a complete convex bornology $\mathcal{B}$. By Theorem 5.1, it follows that ordinary product is separating bounded. Consequently, it is separating continuous.

A similar argument shows that if the ordinary product is separating continuous then also the Jordan product is separating continuous. ∎

Corollary 5.5. ([Rodriguez-Palacios's theorem]) *Let A be an alternative semi-prime algebra, assume that A is considered as a Fréchet space. Then, the Jordan product is separating continuous if, and only if, the ordinary product is separating continuous.*

Proof. Since A is a Fréchet space, then it is completely bornological and its bornology has a net [2].

We conclude the result by Corollary 5.2 and Theorem 2.17 of [6]. ∎

6. Boundedness and continuity of quadratic product

Theorem 6.2. *Let A be unital alternative algebra, assume that A is considered as a bornological vector space. Then, the Jordan product is bounded if, and only if, the quadratic product is bounded.*

Proof. Since A is unital alternative algebra, then

$$(x, y \in A),\, U_x(y) = U_x^+(y) = 2x \circ (x \circ y) - (x \circ x) \circ y$$

Therefore, Jordan product is bounded implies that the quadratic product

$$(x, y) \mapsto U_x(y)$$

is bounded.

Conversely, assume that the quadratic product is bounded. Consider the bilinear map defined by:

$$U_{a,b} = \frac{1}{2}(U_{a+b} - U_a - U_b).$$

Denotes by e the unit of A and let $x \in A$. We have:

$$R_x^+ = L_x^+ = U_{x,e} = \frac{1}{2}(U_{x+e} - U_x - U_e).$$

Hence

$$(\forall x, y \in A)\ x \circ y = \frac{1}{2}(U_{x+e}(y) - U_x(y) - U_e(y)).$$

Since A is bornological vector space the map $x \mapsto x + e$ is bounded, consequently the Jordan-product is bounded. ∎

Corollary 6.6. *Let A be an unital alternative algebra, assume that A is considered as a complete bornological vector space (respectively, topological) and its bornology has a net. Then, the ordinary product is separating bounded (respectively, bounded) if, and only if, the quadratic product is separating bounded (respectively, bounded).*

Theorem 6.3. *Let A be an unital alternative algebra, assume that A is considered as a bornological locally convex space. Then, the Jordan product is separating continuous if, and only if, the quadratic product is separating continuous.*

Proof. Denotes by b_A the von Neumann bornology of a locally convex topology of A. It follows that (A, b_A) is a convex bornological space. We have

$$(\forall x,\, y \in A),\ U_x(y) = U_x^+(y) = 2x \circ (x \circ y) - (x \circ x) \circ y.$$

From this, if the Jordan product is separating continuous then the quadratic product is separating continuous.

Conversely, assume that quadratic product is separating continuous. For a fix $y \in A$ consider the map $\varphi_y : x \mapsto U_x(y)$. Thus, φ_y is separating continuous.

We claim that φ_y is bounded.

Let B be a bounded subset of A and W a neighbourhood of 0 in A. Then there is a neighbourood V of 0 and a positive real number α such that

$$\varphi_y(V) \subset W \text{ and } B \subset \alpha V.$$

But φ_y is quadratic, hence $\varphi_y(\alpha x) = \alpha^2 \varphi_y(x)$.

Therefore, $\varphi_y(B) \subset \alpha^2 W$, and consequently φ_y is bounded.

On the other hand, the map $y \mapsto U_x(y)$ is linear and continuous, then it is bounded. We conclude that, $(x, y) \mapsto U_x(y)$ is separating bounded.

By Theorem 5.1, we conclude that the Jordan product is separating bounded.

By hypothesis A is bornological, then the Jordan product is separating continuous. So the proof is complete. ∎

An immediate consequence, we have the following theorem.

Corollary 6.7. *Let A be unital alternative semi-prime algebra, assume that A is considered as a Fréchet space. Then, the ordinary product is continuous if, and only if, the quadratic product is continuous.*

Proof. It is clear that if the ordinary product is continuous then the quadratic product $(x, y) \mapsto U_x(y) = xyx$ is continuous.

Conversely, suppose that the quadratic product is continuous. Since A is a Fréchet space then it is bornological. Thus, by Theorem 6.2, the Jordan product is separating continuous, and, consequently, it is continuous [6, Theorem 2.17]. By Corollary 6.1, we conclude that the ordinary product is continuous. ∎

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