COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPPINGS AND APPLICATIONS IN DYNAMIC PROGRAMMING

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Abstract. In this note, we establish a common fixed point theorem for a quadruple of self mappings on a complete metric space satisfying weak compatibility and a generalized Φ-contraction. Our main result improves and extends some known results. As an application, we use our main result to obtain common solutions of certain functional equations arising in dynamic programming. We also discuss an illustrative example to validate all the conditions of the main result in dynamic programming.

Keywords and phrases: common fixed points, weakly compatible mappings, dynamic programming.

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1. Introduction

In 1986, the notion of compatible mappings which generalizes commuting mappings, was introduced by Jungck [6]. Further, in 1998, the more general class of mappings called weakly compatible mappings was introduced by Jungck and Rhoades [7]. Recall that self mappings S and T of a metric space (X, d) are called weakly compatible if $Sx = Tx$ for some $x \in X$ implies that $STx = TSx$.

Bellman and Lee [1] initiated the basic form of the functional equation arising in dynamic programming as follows:

$$f(x) = \sup_{y \in D}\{A(x, y, f(a(x, y)))\}, \quad x \in S.$$

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In 1984, Bhakta and Mitra [2] obtained some existence theorem for the following functional equation which arises in multistage decision process related to dynamic programming

$$f(x) = \sup_{y \in D}\{r(x,y) + f(c(x,y))\}, \quad x \in S.$$

In 2003, Liu and Ume [8] provided sufficient conditions which insure the existing and uniqueness for solution for the functional equation

$$f(x) = opt_{y \in D}\{u[p(x,y) + f(a(x,y))] + v\ opt[q(x,y), f(b(x,y))]\}, \quad x \in S.$$

Several existence and uniqueness results of solution and common solution for some functional equations and systems of functional equations in dynamic programming are discussed by Liu et al. [9].

In 2010, Singh and Mishra[13] established coincidence and fixed point theorems for a new class of contractive, nonexpansive and hybrid contractions mappings. Applications regarding the existence of solutions of certain functional equations are also discussed.

Recently, Jiang et al. [5] studied the properties of solutions of the following functional equation arising in dynamic programming of multistage decision process:

$$f(x){=}opt_{y\in D}\{p(x,y), q(x,y)f(a(x,y)), r(x,y), f(b(x,y)), s(x,y)f(c(x,y))\}, \forall x{\in}S.$$

Bondar et al. [3] proved some common fixed point theorems for two pairs of mappings and some applications are given in dynamic programming.

Most recently, Pathak et al. [10] introduced the following two functional equations arising in dynamic programming of multistage decision process:

$$f(x){=}opt_{y\in D} opt\{p(x,y) + A(x,y,f(a(x,y))), q(x,y))\}, \forall x \in S,$$

and

$$f(x){=}opt_{y\in D} opt\{p_1(x,y) + q(x,y)f(a(x,y)), p_2(x,y) + r(x,y)f(b(x,y))\}, \forall x \in S.$$

In this paper, we prove some common fixed point theorem for a quadruple of self mappings of a complete metric space satisfying weak compatibility condition and a generalized Φ-contraction. Subsequently, we use our main theorem to obtain common solutions of certain functional equations arising in dynamic programming.

2. Preliminaries

In what follows, we denote by Φ the collection of all the functions $\varphi : [0,\infty) \to [0,\infty)$ which are upper semicontinuous from the right, non-decreasing and satisfy $\lim_{s\to t+}\sup \varphi(s) < t$, $\varphi(t) < t$, for all $t > 0$.

Let X denote a metric space endowed with metric d and let $\mathbb{N}$ denote the set of natural numbers.

Now, let A, B, S and T be self-mappings of X such that

$$A(X) \subset T(X) \quad and \quad B(X) \subset S(X) \tag{2.1}$$

$$\begin{aligned}
&[d^p(Ax, By) + a\ d^p(Sx, Ty)]d^p(Ax, By) \\
&\leq a\ \max\{d^p(Ax, Sx)d^p(By, Ty), d^q(Ax, Ty)d^{q'}(By, Sx)\} \\
&+ \max\Big\{\varphi_1(d^{2p}(Sx, Ty)), \varphi_2(d^r(Ax, Sx)d^{r'}(By, Ty)), \\
&\varphi_3(d^s(Ax, Ty)d^{s'}(By, Sx)), \\
&\varphi_4\Big(\frac{1}{2}[d^l(Ax, Ty)]d^{l'}(Ax, Sx) + d^l(By, Sx)\Big)d^{l'}(By, Ty)\Big\},
\end{aligned} \tag{2.2}$$

for all $x, y \in X, \varphi_i \in \Phi$ $(i = 1, 2, 3, 4)$, $a, p, q, q', r, r', s, s', l, l' \geq 0$ and $2p = q+q' = r + r' = s + s' = l + l' \leq 1$. Condition (2.2) is commonly called a generalized Φ-contraction.

Now, we pick $x_0 \in X$. Since $A(X) \subset T(X)$, we can choose a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Again, since $B(X) \subset S(X)$ for $x_1 \in X$, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$. Continuing in this way, we can construct a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \ \text{ and } \ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \quad (n \in \mathbb{N} \cup \{0\}). \tag{2.3}$$

First, we prove the following lemmas:

Lemma 2.1. *Let us suppose* $d_n = d(y_n, y_{n-1}), n \in \mathbb{N}$. *Then,* $\lim_{n\to\infty} d_n = 0$.

Proof. In (2.2), putting $x = x_{2n}$ and $y = x_{2n+1}$ and using (2.3), we get

$$\begin{aligned}
&[d^p(Ax_{2n}, Bx_{2n+1}) + a\ d^p(Sx_{2n}, Tx_{2n+1})]d^p(Ax_{2n}, Bx_{2n+1}) \\
&\leq a\ max\Big\{d^p(Ax_{2n}, Sx_{2n})d^p(Bx_{2n+1}, Tx_{2n+1}), d^q(Ax_{2n}, Tx_{2n+1}) \\
&d^{q'}(Bx_{2n+1}, Sx_{2n})\} + \max\{\varphi_1(d^{2p}(Sx_{2n}, Tx_{2n+1})), \varphi_2(d^r(Ax_{2n}, Sx_{2n}) \\
&d^{r'}(Bx_{2n+1}, Tx_{2n+1})), \varphi_3(d^s(Ax_{2n}, Tx_{2n+1})d^{s'}(Bx_{2n+1}, Sx_{2n})), \\
&\varphi_4\Big(\frac{1}{2}[d^l(Ax_{2n}, Tx_{2n+1})d^{l'}(Ax_{2n}, Sx_{2n}) + d^l(Bx_{2n+1}, Sx_{2n})d^{l'}(Bx_{2n+1}, Tx_{2n+1})]\Big)\Big\},
\end{aligned}$$

or

$$\begin{aligned}
[d^p_{2n+1} + ad^p_{2n}]d^p_{2n+1} &\leq a\ \max\{d^p_{2n+1}d^p_{2n}, 0\} + \max\Big\{\varphi_1(d^{2p}_{2n}), \varphi_2(d^r_{2n}d^{r'}_{2n+1}), \\
&\qquad \varphi_3(0), \varphi_4\big(\frac{1}{2}[d^l_{2n+1} + d^l_{2n}d^{l'}_{2n+1}]\big)\Big\} \\
&\leq ad^p_{2n+1}d^p_{2n} + \max\Big\{\varphi_1(d^{2p}_{2n}), \varphi_2(d^r_{2n}d^{r'}_{2n+1}), \\
&\qquad \varphi_3(0), \varphi_4\Big(\frac{1}{2}[d^l_{2n+1}d^{l'}_{2n+1} + d^l_{2n}d^{l'}_{2n+1}]\Big)\Big\},
\end{aligned}$$

which implies

$$d_{2n+1}^{2p} \leq \max\left\{\varphi_1(d_{2n}^{2p}), \varphi_2(d_{2n}^{r} d_{2n+1}^{r'}), \varphi_3(0), \varphi_4\Big(\frac{1}{2}[d_{2n+1}^{l+l'} + d_{2n}^{l} d_{2n+1}^{l'}]\Big)\right\}. \tag{2.4}$$

If $d_{2n+1} > d_{2n}$ then, we have

$$d_{2n+1}^{2p} \leq \max\left\{\varphi_1(d_{2n+1}^{2p}), \varphi_2(d_{2n+1}^{r+r'}), \varphi_3(0), \varphi_4\Big(\frac{1}{2}[d_{2n+1}^{l+l'} + d_{2n+1}^{l+l'}]\Big)\right\} \leq \varphi_i(d_{2n+1}^{2p}) \quad (i = 1, 2, 4).$$

This, together with a well known result of Chang [4], which states that, if $\varphi_i \in \Phi$, where $i \in I$ (some indexing set), then there exists a $\varphi \in \Phi$ such that $\max\{\varphi_i, i \in I\} \leq \varphi(t)$, for all $t > 0$, imply $d_{2n+1}^{2p} < d_{2n+1}^{2p}$, a contradiction. Consequently, we have $d_{2n+1} \leq d_{2n}$, for all $n \in \mathbb{N}$, and

$$d_{2n+1} \leq \varphi(d_{2n}) \ \text{ for all } \ n \in \mathbb{N} \ \text{ and some } \varphi \in \Phi. \tag{2.5}$$

Similarly, for $x = x_{2n+2}$ and $y = x_{2n+1}$, we have

$$d_{2n+2}^{2p} \leq \max\left\{\varphi_1(d_{2n+1}^{2p}), \varphi_2(0), \varphi_3(0), \varphi_4\left(\frac{1}{2}[d_{2n+2}^{l+l'} + d_{2n+1}^{l} d_{2n+2}^{l'}]\right)\right\}. \tag{2.6}$$

A similar argument applied to (2.6) will give

$$d_{2n+2} \leq \varphi(d_{2n+1}) \text{ for all } n \in \mathbb{N}, \tag{2.7}$$

where $\varphi \in \Phi$ is assumed to be same as in the previous case. Therefore, for all $n \in \mathbb{N}$, we have $d_{n+1} \leq \varphi(d_n)$, and by Lemma 2 of [4], we have $\lim_{n\to\infty} d_n = 0$.

Lemma 2.2. *The sequence* $\{y_n\}$ *defined in* (2.3) *is a Cauchy sequence.*

Proof. We prove that the subsequence $\{y_{2n}\}$ of the sequence $\{y_n\}$ is a Cauchy sequence. On the contrary, let us suppose that $\{y_{2n}\}$ is not Cauchy. Then, there exists an $\epsilon > 0$ such that for each even integer 2k there exist even integers $2m(k)$, $2n(k)$ ($n \in \mathbb{N}$) with $2m(k) > 2n(k) \geq 2k$, such that

$$d(y_{2n(k)}, y_{2m(k)}) > \epsilon \ \text{ and } \ d(y_{2n(k)}, y_{2m(k)-2)} \leq \epsilon, \tag{2.8}$$

that is, $2m(k)$ is the least positive even integer such that $2m(k) > 2n(k)$ and

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon.$$

Hence, for each even integer $2k$, we have

$$\begin{aligned} \epsilon &< d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}) \\ &< \epsilon + d_{2m(k)-1} + d_{2m(k)}. \end{aligned}$$

Hence, by Lemma 2.1 and (2.8) it follows that

$$\lim_{n\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{2.9}$$

By making use of the triangle inequalities, for $\rho \in [0,1]$, we have

$$\begin{aligned} d^\rho(y_{2m(k)+2}, y_{2n(k)+1}) &\leq d^\rho(y_{2m(k)+2}, y_{2m(k)+1}) + d^\rho(y_{2m(k)+1}, y_{2m(k)}) \\ &\quad + d^\rho(y_{2m(k)}, y_{2n(k)}) + d^\rho(y_{2n(k)}, y_{2n(k)+1}), \end{aligned}$$

or

$$d^\rho(y_{2m(k)+2}, y_{2n(k)+1}) - d^\rho(y_{2m(k)}, y_{2n(k)}) \leq d_{2m(k)+2} + d_{2m(k)+1} + d_{2n(k)+1}.$$

And

$$\begin{aligned} d^\rho(y_{2m(k)}, y_{2n(k)}) &\leq d^\rho(y_{2m(k)}, y_{2m(k)+1}) + d^\rho(y_{2m(k)+1}, y_{2m(k)+2}) \\ &\quad d^\rho(y_{2m(k)+2}, y_{2n(k)+1}) + d^\rho(y_{2n(k)+1}, y_{2n(k)}), \end{aligned}$$

or

$$d^\rho(y_{2m(k)}, y_{2n(k)}) - d^\rho(y_{2m(k)+2}, y_{2n(k)+1}) \leq d^\rho_{2m(k)+1} + d^\rho_{2m(k)+2} + d^\rho_{2n(k)+1}.$$

Thus, we obtain

$$|d^\rho(y_{2m(k)}, y_{2n(k)}) - d^\rho(y_{2m(k)+2}, y_{2n(k)+1})| \leq d^\rho_{2m(k)+1} + d^\rho_{2m(k)+2} + d^\rho_{2n(k)+1}. \tag{2.10}$$

Similarly we have

$$|d^\rho(y_{2m(k)+1}, y_{2n(k)}) - d^\rho(y_{2n(k)}, y_{2m(k)})| \leq d^\rho_{2m(k)+1}, \tag{2.11}$$

$$|d^\rho(y_{2m(k)+1}, y_{2n(k)+1}) - d^\rho(y_{2n(k)}, y_{2m(k)})| \leq d^\rho_{2n(k)+1} + d^\rho_{2m(k)+1}, \tag{2.12}$$

and

$$|d^\rho(y_{2m(k)+2}, y_{2n(k)}) - d^\rho(y_{2n(k)}, y_{2m(k)})| \leq d^\rho_{2m(k)+1} + d^\rho_{2m(k)+2}. \tag{2.13}$$

By Lemma 2.2 and inequalities (2.10)–(2.13), we have

$$\begin{aligned} \lim_{k\to\infty} d^\rho(y_{2m(k)+2}, y_{2n(k)+1}) &= \lim_{k\to\infty} d^\rho(y_{2m(k)+1}, y_{2n(k)}) \\ &= \lim_{k\to\infty} d^\rho(y_{2m(k)+1}, y_{2n(k)+1}) \\ &= \lim_{k\to\infty} d^\rho(y_{2m(k)+2}, y_{2n(k)}) \\ &= \epsilon. \end{aligned} \tag{2.14}$$

Now, using (2.2) with $x = x_{2m(k)+2}$ and $y = x_{2n(k)+1}$ along with (2.3) and a rearrangement, we obtain

$$
\begin{aligned}
&[d^p(Ax_{2m(k)+2}, Bx_{2n(k)+1}) + a\ d^p(Sx_{2m(k)+2}, Tx_{2n(k)+1})]d^p(Ax_{2m(k)+2}, Bx_{2n(k)+1}) \\
&\qquad \leq a \max\{d^p(Ax_{2m(k)+2}, Sx_{2m(k)+2})d^p(Bx_{2n(k)+1}, Tx_{2n(k)+1}), \\
&\qquad\quad d^q(Ax_{2m(k)+2}, Tx_{2n(k)+1})d^{q'}(Bx_{2n(k)+1}, Sx_{2m(k)+2})\} \\
&\qquad\quad + \max\Big\{\varphi_1(d^{2p}(Sx_{2m(k)+2}, Tx_{2n(k)+1})), \\
&\qquad\quad \varphi_2(d^r(Ax_{2m(k)+2}, Sx_{2m(k)+2})d^{r'}(Bx_{2n(k)+1}, Tx_{2n(k)+1})), \\
&\qquad\quad \varphi_3(d^s(Ax_{2m(k)+2}, Tx_{2n(k)+1})d^{s'}(Bx_{2n(k)+1}, Sx_{2m(k)+2})), \\
&\qquad\quad \varphi_4\Big(\frac{1}{2}[d^l(Ax_{2m(k)+2}, Tx_{2n(k)+1})d^{l'}(Ax_{2m(k)+2}, Sx_{2m(k)+2}) \\
&\qquad\quad + d^l(Bx_{2n(k)+1}, Sx_{2m(k)+2})d^{l'}(Bx_{2n(k)+1}, Tx_{2n(k)+1})]\Big)\Big\},
\end{aligned}
$$

or

$$
\begin{aligned}
&[d^p(y_{2m(k)+2}, y_{2n(k)+1}) + a\ d^p(y_{2m(k)+1}, y_{2n(k)})]d^p(y_{2m(k)+2}, y_{2n(k)+1}) \\
&\qquad \leq a \max\{d^p(y_{2m(k)+2}, y_{2m(k)+1})d^p(y_{2n(k)+1}, y_{2n(k)}), \\
&\qquad\quad d^q(y_{2m(k)+2}, y_{2n(k)+1})d^{q'}(y_{2n(k)+1}, y_{2m(k)+1})\} \\
&\qquad + \max\Big\{\varphi_1(d^{2p}(y_{2m(k)+1}, y_{2n(k)})), \\
&\qquad\quad \varphi_2(d^r(y_{2m(k)+2}, y_{2m(k)+1})d^{r'}(y_{2n(k)+1}, y_{2n(k)})), \\
&\qquad\quad \varphi_3(d^s(y_{2m(k)+2}, y_{2n(k)})d^{s'}(y_{2n(k)+1}, y_{2m(k)+1})), \\
&\qquad\quad \varphi_4\Big(\frac{1}{2}[d^l(y_{2m(k)+2}, y_{2n(k)})d^{l'}(y_{2m(k)+2}, y_{2m(k)+1}) \\
&\qquad\quad + d^l(y_{2n(k)+1}, y_{2m(k)+1})d^{l'}(y_{2n(k)+1}, y_{2n(k)})]\Big)\Big\}.
\end{aligned}
$$

Letting $k \to \infty$ and using Lemma 2.1, (2.9) and (2.14) and the fact that $\varphi_i \in \Phi$ $(i = 1, 2, 3, 4)$, we have $\epsilon^{2p} + a\epsilon^{2p} \leq a\ \epsilon^{q+q'} + \max\{\varphi_1(\epsilon^{2p}), \varphi_2(0), \varphi_3(\epsilon^{s+s'}), \varphi_4(0)\}$, or $\epsilon^{2p} \leq \max\{\varphi_1(\epsilon^{2p}), \varphi_2(0), \varphi_3(\epsilon^{s+s'}), \varphi_4(0)\}$, or $\epsilon^{2p} \leq \varphi(\epsilon^{2p}) < \epsilon^{2p}$, a contradiction. Hence, $\{y_{2n}\}$ is a Cauchy sequence in X. This proves that $\{y_n\}$ is Cauchy in X.

3. Main results

The following theorems are our main results of this section.

Theorem 3.1. *Let A, B, S and T be self mappings of a complete metric space X satisfying* (2.1) *and* (2.2). *If the pairs (A, S) and (B, T) are weakly compatible and $T(X)$ or $S(X)$ is closed, then A, B, S and T have a unique common fixed point in X.*

Proof. Since X is complete, it follows from Lemma 2.2 that the sequence $\{y_n\}$ converges to a point z in X. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the same limit z.

Now, suppose that $T(X)$ is closed. Then, since $\{Tx_{2n+1}\} \subset T(X)$, there exists a point $u \in X$ such that $z = Tu$. Then, by using (2.2) with $x = x_{2n}$ and $y = u$, we have

$$\begin{aligned}
&[d^p(Ax_{2n}, Bu) + ad^p(Sx_{2n}, Tu)]d^p(Ax_{2n}, Bu)\\
&\quad \le a \max\Big\{d^p(Ax_{2n}, Sx_{2n})d^p(Bu, Tu), d^q(Ax_{2n}, Tu)d^{q'}(Bu, Sx_{2n})\Big\}\\
&\quad + \max\Big\{\varphi_1(d^{2p}(Sx_{2n}, Tu)), \varphi_2(d^r(Ax_{2n}, Sx_{2n})d^{r'}(Bu, Tu)),\\
&\qquad \varphi_3(d^s(Ax_{2n}, Tu)d^{s'}(Bu, Sx_{2n})),\\
&\quad \varphi_4\Big(\frac{1}{2}[d^l(Ax_{2n}, Tu)d^{l'}(Ax_{2n}, Sx_{2n}) + d^l(Bu, Sx_{2n}))d^{l'}(Bu, Tu)]\Big)\Big\},
\end{aligned}$$

letting $k \to \infty$, we obtain

$$\begin{aligned}
[d^p(z, Bu) + a\ d^p(z, z)]d^p(z, Bu) \quad &\le a \max\{d^p(z, z)d^p(Bu, z), d^q(z, z)d^{q'}(Bu, z)\}\\
&+ \max\Big\{\varphi_1(d^{2p}(z, z)), \varphi_2(d^r(z, z)d^{r'}(Bu, z)),\\
&\quad \varphi_3(d^s(z, z)d^{s'}(Bu, z)), \varphi_4\big(\frac{1}{2}[d^l(z, z)d^{l'}(z, z)\\
&+ d^l(Bu, z)d^{l'}(Bu, z)]\big)\Big\},
\end{aligned}$$

or

$$d^{2p}(z, Bu) \le \max\Big\{\varphi_1(0), \varphi_2(0), \varphi_3(0), \varphi_4\big(\frac{1}{2}d^{l+l'}(Bu, z)\big)\Big\},$$

or

$$\begin{aligned}
d^{2p}(z, Bu) \quad &\le \max\Big\{\varphi_1(d^{2p}(z, Bu)), \varphi_2(d^{r+r'}(z, Bu)),\\
&\qquad \varphi_3(d^{s+s'}(z, Bu)), \varphi_4\big(\frac{1}{2}d^{l+l'}(Bu, z)\big)\Big\}\\
&\le \varphi(d^{2p}(z, Bu))\\
&< d^{2p}(z, Bu),
\end{aligned}$$

a contradiction. This implies that $z = Bu$. Therefore, $Tu = z = Bu$. Hence, it follows by the weak compatibility of the pair (B, T) that $BTu = TBu$, that is $Bz = Tz$.

Now, we shall show that z is a common fixed point of B and T. For this put $x = x_{2n}$ and $y = z$ in (2.2), we have

$$\begin{aligned}
&[d^p(Ax_{2n}, Bz) + a\ d^p(Sx_{2n}, Tz)]d^p(Ax_{2n}, Bz)\\
&\qquad \le a \max\Big\{d^p(Ax_{2n}, Sx_{2n})d^p(Bz, Tz), d^q(Ax_{2n}, Tz)d^{q'}(Bz, Sx_{2n})\}\\
&\qquad + \max\{\varphi_1(d^{2p}(Sx_{2n}, Tz)), \varphi_2(d^r(Ax_{2n}, Sx_{2n})d^{r'}(Bz, Tz)),\\
&\qquad \varphi_3(d^s(Ax_{2n}, Tz)d^{s'}(Bz, Sx_{2n})),\\
&\qquad \varphi_4\big(\frac{1}{2}[d^l(Ax_{2n}, Tz)d^{l'}(Ax_{2n}, Sx_{2n}) + d^l(Bz, Sx_{2n}))d^{l'}(Bz, Tz)\big)\Big\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$[d^p(z,Bz)+ad^p(z,Tz)]d^p(z,Bz) \le a\max\{d^p(z,z)d^p(Bz,Tz), d^q(z,Tz)d^{q'}(Bz,z)\}$$
$$+\max Big\{\varphi_1(d^{2p}(z,Tz)), \varphi_2(d^r(z,z)d^{r'}(Bz,Tz)), \varphi_3(d^s(z,Tz)d^{s'}(Bz,z)),$$
$$\varphi_4\Big(\frac{1}{2}[d^l(z,Tz)d^{l'}(z,z)+d^l(Bz,z)d^{l'}(Bz,Tz)]\Big)\Big\},$$

or

$$d^{2p}(z,Bz)+a\ d^{2p}(z,Bz)$$
$$\le a\ d^{q+q'}(Bz,z)+\max\{\varphi_1(d^{2p}(z,Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z,Bz)), \varphi_4(0)\},$$

or

$$(1+a)d^{2p}(z,Bz) \le a\ d^{q+q'}(Bz,z)\}$$
$$+\max\{\varphi_1(d^{2p}(z,Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z,Bz)), \varphi_4(0)\},$$

or

$$d^{2p}(z,Bz) \le \frac{a}{1+a}d^{q+q'}(Bz,z)$$
$$+\frac{1}{1+a}\max\{\varphi_1(d^{2p}(z,Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z,Bz)), \varphi_4(0)\}$$
$$< d^{2p}(z,Bz),$$

a contradiction. So $z = Bz = Tz$. Thus z is a common fixed point of B and T.

Similarly, we can prove that z is a common fixed point of A and S. Thus, z is the common fixed point of A, B, S and T. The uniqueness of z as a common fixed point of A, B, S and T can easily be verified.

Remark 3.2 If we assume $S(X)$ to be closed then the above theorem also remains valid. We find the same result if $A(X)$ or $B(X)$ is assumed to be closed by (2.1).

Remark 3.3 Our Theorem 3.1 extends Theorem 2.1 of Pathak et al.[7].

In Theorem 3.1, if we put $a = 0$ and $\varphi_i(t) = ht$ (i=1, 2, 3, 4), where $0 < h < 1$, we get the following corollary:

Corollary 3.4. *Let A, B, S and T be self mappings of a complete metric space X satisfying* (2.1) *and* (2.2′).

$$(2.2') \quad d^{2p}(Ax,By) \le h\ \max\Big\{(d^{2p}(Sx,Ty), d^r(Ax,Sx)d^{r'}(By,Ty), d^s(Ax,Ty)$$
$$d^{s'}(By,Sx)), \frac{1}{2}[d^l(Ax,Ty)d^{l'}(Ax,Sx)+d^l(By,Sx))d^{l'}(By,Ty)\Big\}$$

for all $x, y \in X, \varphi_i \in \Phi$ $(i = 1,2,3,4)$, $a, p, q, q', r, r', s, s', l, l' \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l' \le 1$. If the pairs (A,S) and (B,T) are weakly compatible and $T(X)$ or $S(X)$ is closed, then A, B, S and T have a unique common fixed point in X.

Especially, when $\max\{d^{2p}(Sx,Ty), d^r(Ax,Sx)d^{r'}(By,Ty), d^s(Ax,Ty)d^{s'}(By,Sx)),$ $\frac{1}{2}[d^l(Ax,Ty)d^{l'}(Ax,Sx)+d^l(By,Sx))d^{l'}(By,Ty)\} = d^{2p}(Sx,Ty)$, we get Corollary 3.9 of Pathak et al.[9].

In Theorem 3.1, if we take $S = T = I_X$ (the identity mapping on X), then we have the following corollary:

Corollary 3.5. *Let A and B be self mappings of a complete metric space X satisfying the following condition:*

$$\begin{aligned}
&[d^p(Ax,By) + a\ d^p(x,y)]d^p(Ax,By) \\
&\quad \le a\ \max\{d^p(Ax,x)d^p(By,y), d^q(Ax,y)d^{q'}(By,x)\} \\
&\quad + \max\Big\{\varphi_1(d^{2p}(x,y)), \varphi_2(d^r(Ax,x)d^{r'}(By,y)), \varphi_3(d^s(Ax,y)d^{s'}(By,x)), \\
&\quad \varphi_4\big(\frac{1}{2}[d^l(Ax,y)d^{l'}(Ax,x) + d^l(By,x))d^{l'}(By,y)\big)\Big\}
\end{aligned}$$

for all $x,y \in X, \varphi_i \in \Phi$ $(i = 1,2,3,4)$, $a,p,q,q',r,r',s,s',l,l' \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l' \le 1$, then A and B have a unique common fixed point in X.

As an immediate consequences of Theorem 3.1 with $S = T$, we have the following:

Corollary 3.6. *Let A, B, and S be self-mappings of X such that*

$$A(X) \cup B(X) \subset S(X) \tag{2.1$'$}$$

$$\begin{aligned}
&[d^p(Ax,By) + a\ d^p(Sx,Sy)]d^p(Ax,By) \\
&\quad \le a \max\{d^p(Ax,Sx)d^p(By,Sy), d^q(Ax,Sy)d^{q'}(By,Sx)\} \\
&\quad + \max\Big\{\varphi_1(d^{2p}(Sx,Sy)), \varphi_2(d^r(Ax,Sx)d^{r'}(By,Sy)), \\
&\qquad \varphi_3(d^s(Ax,Sy)d^{s'}(By,Sx)), \varphi_4\big(\frac{1}{2}[d^l(Ax,Sy)d^{l'}(Ax,Sx) \\
&\quad +d^l(By,Sx))d^{l'}(By,Sy)\big)\Big\}
\end{aligned} \tag{2.2$''$}$$

for all $x,y \in X, \varphi_i \in \Phi$ $(i = 1,2,3,4), a,p,q,q',r,r',s,s',l,l' \ge 0$ and $0 \le 2p = q+q' = r+r' = s+s' = l+l' \le 1$. If the pairs (A,S) and (B,S) are weakly compatible and $S(X)$ is closed, then A, B and S have a unique common fixed point in X.

The following theorem is an immediate consequence of Theorem 3.1.

Theorem 3.7. *Let S, T and A_n $(n \in \mathbb{N})$ be self mappings of a complete metric space X. Suppose further that the pairs (A_{2n-1}, S) and (A_{2n}, T) are weakly compatible for any $n \in \mathbb{N}$ and*

$$A_{2n-1}(X) \subset T(X) \quad \textit{and} \quad A_{2n}(X) \subset S(X).$$

If $S(X)$ or $T(X)$ is closed and that for any $i \in N$, the following condition is satisfied for all $x, y \in X$

$$[d^p(A_ix, A_{i+1}y){+}a\ d^p(Sx, Ty)]d^p(A_ix, A_{i+1}y) \leq a \max\{d^p(A_ix, Sx)d^p(A_{i+1}y, Ty), d^q(A_ix, Ty)d^{q'}(A_{i+1}y, Sx)\} + max\Big\{\varphi_1(d^{2p}(Sx, Ty)), \varphi_2(d^r(A_ix, Sx)d^{r'}(A_{i+1}y, Ty)), \varphi_3(d^s(A_ix, Ty)d^{s'}(A_{i+1}y, Sx)), \varphi_4\big(\frac{1}{2}[d^l(A_ix, Ty)d^{l'}(A_ix, Sx) + d^l(A_{i+1}y, Sx))d^{l'}(A_{i+1}y, Ty)\big)\Big\}$$

where $\varphi_i \in \Phi$ $(i = 1, 2, 3, 4)$, $a, p, q, q', r, r', s, s', l, l' \geq 0$ and $0 \leq 2p = q + q' = r + r' = s + s' = l + l' \leq 1$, then S, T and $A_n(n \in \mathbb{N})$ have a common fixed point in X.

4. Applications to existence theorems for functional equations arising in dynamic programming

Throughout this section, we assume that X and Y are Banach spaces, $S \subset X$ is the state space and $D \subset Y$ is the decision space. Let $\mathbb{R} = (-\infty, \infty)$ and $B(S)$ denote the set of all bounded real valued functions on S.

The basic form of the functional equation of dynamic programming is given by Bellman and Lee [1] as follows:

$$f(x) = opt_y H(x, y, f(T(x, y))),$$

where x and y represent the state and decision vectors respectively, T represents the transformation of the process and $f(x)$ represents the optimal return with initial state x (where opt denotes max or min).

In this section, we study the existence and uniqueness of a common solution of the following functional equations arising in dynamic programming.

$$f_i(x) = \sup_{y \in D} H_i(x, y, f_i(T(x, y))), x \in S, \tag{4.1}$$

$$g_i(x) = \sup_{y \in D} F_i(x, y, g_i(T(x, y))), x \in S, \tag{4.2}$$

where $T : S \times D \to S$ and $H_i, F_i : S \times D \times \mathbb{R} \to \mathbb{R}$, i =1, 2.

Suppose the mappings A_i and T_i (i =1, 2) are defined by

$$\begin{aligned} A_i h(x) &= \sup_{y \in D} H_i(x, y, h(T(x, y)), \\ T_i k(x) &= \sup_{y \in D} F_i(x, y, k(T(x, y))), \end{aligned} \tag{4.3}$$

for all $x \in S; h, k \in B(S), i = 1, 2$.

Now, we present our main theorems of this section.

Theorem 4.1. *Suppose that the following conditions are satisfied:*

(i) *H_i and F_i are bounded for $i = 1, 2$.*

(ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq M^{-1}(a \max\{|T_1h(t) - A_1h(t)|.|T_2k(t) - A_2k(t)|, |T_1h(t) - A_2k(t)|.|T_2k(t) - A_1h(t)|\} + \max\{\varphi_1(|T_1h(t) - T_2k(t)|), \varphi_2(|T_1h(t) - A_1h(t)|), \varphi_3(|T_2k(t) - A_2k(t)|), \varphi_4(\frac{1}{2}[|T_1h(t) - A_2k(t)|) + (|T_2k(t) - A_1h(t)|])\})$, *for all $(x, y) \in S \times D$, $k \in B(S)$, $t \in S$, $a \geq 0$, where*

$$M = [1 + a \sup_{t \in S} |T_1k(t) - T_2h(t)|], \;\; \varphi_i \in \Phi \;\; (i = 1, 2, 3, 4)$$

and the mappings A_i and $T_i (i = 1, 2)$ are defined as in (4.3).

(iii) *For sequences $\{h_n\}, \{k_n\} \subset B(S)$ and $h, k \in B(S)$ with*

$$\lim_{n\to\infty} \sup_{x\in S} |h_n(x) - h(x)| = 0, \;\; \lim_{n\to\infty} \sup_{x\in S} |k_n(x) - k(x)| = 0,$$

there exist $h_i, k_i \in B(S)$ such that $k = T_2h_i$ and $h = T_1k_i$ for $i = 1$ or 2.

(iv) *For any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$ such that $A_1h(x) = T_2k_2(x)$, $A_2h(x) = T_1k_1(x)$, $x \in S$.*

(v) *For any $h, k \in B(S)$, with $A_1h = T_1h$, we have $T_1A_1h = A_1T_1h$ and with $A_2k = T_2k$, we have $T_2A_2k = A_2T_2k$.*

Then, the system of functional equations (4.1) and (4.2) have a unique common solution in $B(S)$.

Proof. Obviously, $B(S)$ endowed with the metric

$$d(h, k) = \sup_{x\in D} |h(x) - k(x)| \;\;\; for \;\; any \;\;\; h, k \in B(S)$$

is a complete metric space. Moreover, by condition (i), A_i and T_i are self mappings of $B(S)$ and by condition (iv) it is clear that

$$A_1(B(S)) \subset T_2(B(S)) \;\; \text{and} \;\; A_2(B(S)) \subset T_1(B(S)).$$

Also, by condition (v), the pairs (A_i, T_i) are weakly compatible for i = 1, 2. Moreover, by (4.3) and (i) we have for any $\eta > 0$ there exist $y_i \in D$ $(i = 1, 2)$ such that

$$A_ih_i(x) < H_i(x_i, y_i, h_i(x)) + \eta, \tag{4.4}$$

where $x_i = T(x, y_i)$, $i = 1, 2$. Also,

$$A_1h_1(x) \geq H_1(x, y_2, h_1(x_2)), \tag{4.5}$$

$$A_2h_2(x) \geq H_2(x, y_2, h_2(x_1)), \tag{4.6}$$

Then, from (4.4), (4.5), (4.6) and (ii), we have

$$
\begin{aligned}
(4.7)\quad & A_1h_1(x) - A_2h_2(x) \le H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \eta \\
& \le |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \eta \\
& \le M^{-1}\Big(a\ \max\{|T_1h_1(x_1) - A_1h_1(x_1)|.|T_2h_2(x_1) - A_2h_2(x_1)|, \\
& |T_1h_1(x_1) - A_2h_2(x_1)|.|T_2h_2(x_1) - A_1h_1(x_1)|\} \\
& + \max\Big\{\varphi_1(|T_1h_1(x_1) - T_2h_2(x_1)|), \varphi_2(|T_1h_1(x_1) - A_1h_1(x_1)|), \\
& \varphi_3(|T_2h_2(x_1) - A_2h_2(x_1)|), \varphi_4(\frac{1}{2}[|T_1h_1(x_1) - A_2h_2(x_1)| \\
& + |T_2h_2(x_1) - A_1h_1(x_1)|]) \Big\}\Big), \\
& \le M^{-1}\Big(a\ \max\{d(T_1h_1, A_1h_1)d(T_2h_2, A_2h_2), \\
& d(T_1h_1, A_2h_2)d(T_2h_2, A_1h_1)\} \\
& + \max\Big\{\varphi_1(d(T_1h_1, T_2h_2)), \varphi_2(d(T_1h_1, A_1h_1)), \\
& \varphi_3(d(T_2h_2, A_2h_2)), \varphi_4(\frac{1}{2}[d(T_1h_1, A_2h_2) + d(T_2h_2, A_1h_1)])\Big\}\Big) + \eta.
\end{aligned}
$$

From (4.4), (4.5) and (ii), we have

$$
\begin{aligned}
(4.8)\quad & A_1h_1(x) - A_2h_2(x) \\
& \ge -M^{-1}\Big(a \max\{d(T_1h_1, A_1h_1)d(T_2h_2, A_2h_2), d(T_1h_1, A_2h_2)d(T_2h_2, A_1h_1)\} \\
& \quad + \max\Big\{\varphi_1(d(T_1h_1, T_2h_2)), \varphi_2(d(T_1h_1, A_1h_1)), \\
& \quad \varphi_3(d(T_2h_2, A_2h_2)), \varphi_4(\frac{1}{2}[d(T_1h_1, A_2h_2) + d(T_2h_2, A_1h_1)])\Big\}\Big) - \eta.
\end{aligned}
$$

Using (4.7) and (4.8), we obtain

$$
\begin{aligned}
(4.9)\quad & |A_1h_1(x) - A_2h_2(x)| \\
& \le M^{-1}\Big(a \max\{d(T_1h_1, A_1h_1)d(T_2h_2, A_2h_2), d(T_1h_1, A_2h_2)d(T_2h_2, A_1h_1)\} \\
& \quad + \max\Big\{\varphi_1(d(T_1h_1, T_2h_2)), \varphi_2(d(T_1h_1, A_1h_1)), \varphi_3(d(T_2h_2, A_2h_2)), \\
& \quad \varphi_4(\frac{1}{2}[d(T_1h_1, A_2h_2) + (d(T_2h_2, A_1h_1))])\Big\}\Big) + \eta.
\end{aligned}
$$

Since (4.9) is true for any $x \in S$ and $\eta > 0$ is arbitrary, by taking sup over all $x \in S$ we have,

$$
\begin{aligned}
& [1 + a\ d(T_1h_1, T_2h_2)]d(A_1h_1, A_2h_2) \le \Big(a \max\{d(T_1h_1, A_1h_1)d(T_2h_2, A_2h_2), \\
& \quad d(T_1h_1, A_2h_2)d(T_2h_2, A_1h_1)\} + \max\Big\{\varphi_1(d(T_1h_1, T_2h_2)), \\
& \quad \varphi_2(d(T_1h_1, A_1h_1)), \varphi_3(d(T_2h_2, A_2h_2)), \varphi_4(\frac{1}{2}[d(T_1h_1, A_2h_2) + (d(T_2h_2, A_1h_1))])\Big\}\Big).
\end{aligned}
$$

Therefore, condition (2.2) is satisfied by mappings A_1, A_2, T_1 and T_2 and hence by Theorem 3.1, they have a common fixed point $h^* \in B(S)$, i.e. $h^*(x)$ is a unique common solution of the functional equations (4.1) and (4.2).

As an immediate consequence of Theorem 4.1 and Corollary 3.5, we have the following:

Theorem 4.2. *Suppose the following conditions are satisfied:*

(i) H_i *is bounded for* $i = 1, 2$.

(ii) $|H_1(x,y,h(t)) - H_2(x,y,k(t))| \leq N^{-1}(a \max\{|h(t) - A_1h(t)|.|k(t) - A_2k(t)|, |h(t) - A_2k(t)|.|k(t) - A_1h(t)|\} + \max\{\varphi_1(|h(t) - k(t)|), \varphi_2(|h(t) - A_1h(t)|), \varphi_3(|k(t) - A_2k(t)|), \varphi_4(\frac{1}{2}[|h(t) - A_2k(t)| + |k(t) - A_1h(t)|]\}))$, *for all* $(x,y) \in S \times D$, $h,k \in B(S)$, $t \in S$, $a \geq 0$, *where* $N = [1 + a\ sup_{t\in S}|h(t) - k(t)|]$, $\varphi_i \in \Phi$ $(i = 1,2,3,4)$ *and the mappings* A_i *are defined as in* (4.3).

Then, the functional equations (4.1) *and* (4.2) *have a unique common solution in* $B(S)$.

Now, we furnish an example to validate Theorem 4.1.

Example 4.3. Let $X = Y = \mathbb{R}$ be two Banach spaces endowed with the standard norm $\|\cdot\|$ defined by $\|x\| = |x|$ for all $x \in \mathbb{R}$. Let $S = [0,1] \subset X$ be the state space, $D = [1,\infty) \subset Y$ the decision space and T represents the transformation of the process. Define $T : S \times D \to S$ by

$$T(x,y) = \frac{x}{y^2+1} \text{ for all } x \in S, y \in D.$$

For any $h,k \in B(S)$, define $f_i, g_i : S \to R$ (i =1, 2) by

$$f_i(x) = g_i(x) = \frac{1}{4}\Big(\frac{x}{x+1} + 1\Big).$$

Define $H_i, F_i : S \times D \times \mathbb{R} \to \mathbb{R}$ $(i = 1,2)$ by

$$H_i(x,y,t) = \frac{1}{4}\left[\frac{x}{x+1}\sin\left(t \cdot \frac{y}{y+1}\right) + 1\right]$$

and

$$F_i(x,y,t) = \frac{1}{4}\left[\frac{x}{x+1}\cos\left(t \cdot \frac{y}{y+1}\right) + 1\right].$$

Clearly, $\|H_i\| \leq \frac{1}{2}$ and $\|F_i\| \leq \frac{1}{2}$. By varying y over D and taking supremum, we see that H_i yield f_i and H_i yield g_i, respectively $(i = 1,2)$, as defined above. Define mappings A_i and T_i (i =1, 2) by

$$A_1h(x) = \sup_{y\in D} H_1(x,y,h(T(x,y)), \quad A_2k(x) = \sup_{y\in D} H_2(x,y,k(T(x,y))),$$
$$T_1h(x) = \sup_{y\in D} F_1(x,y,h(T(x,y))), \quad T_2k(x) = \sup_{y\in D} F_2(x,y,k(T(x,y)),$$

for all $x \in S; h,k \in B(S)$.

Then, we see that

$$\begin{aligned}
A_1h(x) &= \sup_{y\in D} H_1(x,y,h(T(x,y)) = \sup_{y\in D} H_1\left(x,y,h\left(\frac{x}{y^2+1}\right)\right)\\
&= \sup_{y\in D}\frac{1}{4}\left[\frac{x}{x+1}\sin\left(h\left(\frac{x}{y^2+1}\right)\frac{y}{y+1}\right)+1\right] = \frac{1}{4}\left(\frac{x}{x+1}+1\right) = f_1(x),\\
A_2k(x) &= \sup_{y\in D} H_2(x,y,k(T(x,y)) = \sup_{y\in D} H_2\left(x,y,k\left(\frac{x}{y^2+1}\right)\right)\\
&= \sup_{y\in D}\frac{1}{4}\left[\frac{x}{x+1}\sin\left(k(\frac{x}{y^2+1})\frac{y}{y+1}\right)+1\right] = \frac{1}{4}\left(\frac{x}{x+1}+1\right) = f_2(x),\\
T_1h(x) &= \sup_{y\in D} F_1(x,y,h(T(x,y)) = \sup_{y\in D} F_1\left(x,y,h\left(\frac{x}{y^2+1}\right)\right)\\
&= \sup_{y\in D}\frac{1}{4}\left[\frac{x}{x+1}\cos\left(h(\frac{x}{y^2+1})\frac{y}{y+1}\right)+1\right] = \frac{1}{4}\left(\frac{x}{x+1}+1\right) = g_1(x),\\
T_2k(x) &= \sup_{y\in D} F_2(x,y,k(T(x,y)) = \sup_{y\in D} F_2\left(x,y,k\left(\frac{x}{y^2+1}\right)\right)\\
&= \sup_{y\in D}\frac{1}{4}\left[\frac{x}{x+1}\cos\left(k(\frac{x}{y^2+1})\frac{y}{y+1}\right)+1\right] = \frac{1}{4}\left(\frac{x}{x+1}+1\right) = g_2(x),
\end{aligned}$$

for all $x \in S; h, k \in B(S)$. Also, we see that

$$M = \left[1 + a\ \sup_{t\in S}|T_1k(t) - T_2h(t)|\right] = \left[1 + a\ \sup_{t\in S}|g_1(x) - g_2(x)|\right] = 1.$$

Further, for any $k \in B(S)$, define

$$h_n(x) = \left(1-\frac{1}{n}\right)h(x) \quad\text{and}\quad k_n(x) = \left(1-\frac{1}{n+1}\right)k(x)$$

so that

$$\lim_{n\to\infty}\sup_{x\in S}|h_n(x)-h(x)| = 0 \quad\text{and}\quad \lim_{n\to\infty}\sup_{x\in S}|k_n(x)-k(x)| = 0.$$

Now, we observe that, for $\varphi_1(t) = \dfrac{t}{2}$,

(i) H_i and F_i are bounded for $i = 1, 2$.

(ii) $|H_1(x,y,h(t)) - H_2(x,y,k(t))|$

$$\begin{aligned}
&= \left|\frac{1}{4}\left[\frac{x}{x+1}\sin\left(h(t)\frac{y}{y+1}\right)+1\right] - \frac{1}{4}\left[\frac{x}{x+1}\sin\left(k(t)\frac{y}{y+1}\right)+1\right]\right|\\
&= \frac{1}{4}\left|\frac{x}{x+1}\right|\left|\sin\left(h(t)\frac{y}{y+1}\right) - \sin\left(k(t)\frac{y}{y+1}\right)\right|\\
&= \frac{1}{4}\left|\frac{x}{x+1}\right|\cdot 2\left|\sin\left(\frac{h(t)\frac{y}{y+1} - k(t)\frac{y}{y+1}}{2}\right)\right|\left|\cos\left(\frac{h(t)\frac{y}{y+1} + k(t)\frac{y}{y+1}}{2}\right)\right|
\end{aligned}$$

$$\leq \frac{1}{4}\left|\frac{x}{x+1}\right|\left|\frac{y}{y+1}\right||h(t)-k(t)|$$
$$\leq \frac{1}{4}\left|\frac{x}{x+1}\right||h(t)-k(t)| \leq \frac{1}{4}\varphi_1(h(t)-k(t)).$$

Finally, for any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$ such that

$$A_1h(x) = T_2k_2(x),\ A_2h(x) = T_1k_1(x), \quad x \in S.$$

Also, for any $h, k \in B(S)$, with $A_1h = T_1h$, we have $T_1A_1h = A_1T_1h$ and, with $A_2k = T_2k$, we have $T_2A_2k = A_2T_2k$. Thus, all the assumption of Theorem 4.1 are satisfied. So, the system of equations (4.1) and (4.2) have a unique common solution in $B(S)$.

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