SOME MODULAR EQUATIONS IN THE FORM OF SCHLÄFLI[1]

M.S. Mahadeva Naika

Department of Mathematics
Bangalore University
Central College Campus
Bengaluru - 560 001
India
e-mail: msmnaika@rediffmail.com

K. Sushan Bairy

PG Department of Mathematics
Vijaya College
R.V. Road, Basavanagudi
Bengaluru - 560 004
India
e-mail: ksbairy@rediffmail.com

Abstract. On page 90 of his first notebook, S. Ramanujan records Schläfli-type modular equations for degrees 3, 5, 7, 11, 13, 17 and 19. In this paper, we establish Schläfli-type modular equations for degrees 11, 13, 17 and 19 which are recorded by Ramanujan in his first notebook. We also establish several new Schläfli-type modular equations of degrees 2, 4, 9, 15, 23, 25, 29, 31, 47 and 71. As an application, we deduce some explicit evaluations of Ramanujan-Weber class invariants.

Keywords: Schläfli-type modular equation, Class invariant.

2010 Mathematics Subject Classification: 11B65, 33D15, 11F27.

1. Introduction

In [13], Schläfli established modular equations for degrees 3, 5, 7, 11, 13, 17 and 19 which were also recorded by S. Ramanujan on page 90 of his first notebook [12]. In [11], Ramanathan gave a proof of the equation of degree 11. In [14], G.N. Watson gave a proof of Schläfli-type modular equation of degree thirteen and also examined Schläfli-type modular equations in [15]. Using the theory of modular forms, Berndt [2, pp. 231, 282, 315], [3, pp. 378–379] has verified these modular equations. Recently, William Hart [6] has proved several Schläfli-type modular equations of degrees 2, 3, 5, 7, 11, 13 and 17 by using modular forms in different levels.

[1]Research supported by DST grant SR/S4/MS:509/07, Govt. of India.

We define

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}k^{2n} = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{1.1}$$

where $0 < k < 1$ and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}z^n, \quad 0 \le |z| < 1,$$

where

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \text{ for } n \text{ a positive integer}$$

and a, b, c are complex numbers such that $c \neq 0, -1, -2, \ldots$. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.2}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (1.2). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}. \tag{1.3}$$

Ramanujan's class invariant G_n is defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) = \{4\alpha(1-\alpha)\}^{-1/24}, \tag{1.4}$$

where

$$\chi(q) = (-q;q^2)_\infty, \quad q = \exp(-\pi\sqrt{n})$$

and

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \quad |q| < 1.$$

M.S. Mahadeva Naika [8] and Mahadeva Naika and K. Sushan Bairy [9] have obtained several new explicit evaluations of the Ramanujan-Weber class invariants using modular equations.

In this paper, we establish several modular equations in the form of Schläfli for degrees 2, 4, 9, 11, 13, 15, 17, 19, 23, 25, 29, 31, 47 and 71. As an application, we obtain several explicit evaluations of Ramanujan-Weber class invariants.

2. Preliminary results

In this section, we collect several results which are useful in proving our main Schläfli-type modular equations.

Lemma 2.1. [2, Eq. (24.21), p. 215] *If β is of degree 2 over α, then*

$$\beta=\left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2. \tag{2.1}$$

Lemma 2.2. [2, Eq. (24.22), p. 215] *If β is of degree 4 over α, then*

$$\beta=\left(\frac{1-\sqrt[4]{1-\alpha}}{1+\sqrt[4]{1-\alpha}}\right)^4. \tag{2.2}$$

Lemma 2.3. [3, Entry 62, 63, 64, 65, Ch.36, pp. 387–388] *Let*

$$u = 1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}, \tag{2.3}$$

$$v = 64\left[\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}-\sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right], \tag{2.4}$$

$$w = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)}. \tag{2.5}$$

1. *If β is of degree 9 over α, then*

$$u^6-w\left(14u^3+uv\right)-3w^2=0. \tag{2.6}$$

2. *If β is of degree 13 over α, then*

$$\sqrt{u}\left(u^3+8w\right)-\sqrt{w}\left(11u^2+v\right)=0. \tag{2.7}$$

3. *If β is of degree 17 over α, then*

$$u^3-w^{1/3}\left(10u^2+v\right)+13w^{2/3}u+12w=0. \tag{2.8}$$

4. *If β is of degree 29 over α, then*

$$\sqrt{u}\left(u^2+17uw^{1/3}-9w^{2/3}\right)-w^{1/6}\left(9u^2+v-13uw^{1/3}+15w^{2/3}\right)=0. \tag{2.9}$$

Lemma 2.4. [3, Entry 53, 54, 55, 56, Ch.36, p. 385] *Let*

$$U=1\pm(\alpha\beta)^{1/8}\pm[(1-\alpha)(1-\beta)]^{1/8}, \tag{2.10}$$

$$V=4\left[(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}\pm\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\right], \tag{2.11}$$

$$W=4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \tag{2.12}$$

1. *Let U, V and W be given by* (2.10)–(2.12), *with the plus signs taken. If β is of degree* 15 *over α, then*

$$U\left(U^2-V\right)+W=0. \tag{2.13}$$

2. *Let U, V and W be given by* (2.10)–(2.12), *with the plus signs taken. If β is of degree* 31 *over α, then*

$$U^2-V=\sqrt{UW}. \tag{2.14}$$

3. *Let U, V and W be given by* (2.10)–(2.12), *with the plus signs taken. If β is of degree* 47 *over α, then*

$$U^2-V-UW^{1/3}-2W^{2/3}=0. \tag{2.15}$$

4. *Let U, V and W be given by* (2.10)–(2.12), *with the minus signs taken. If β is of degree* 71 *over α, then*

$$U^3-W^{1/3}\left(4U^2+V\right)+2UW^{2/3}-W=0. \tag{2.16}$$

Lemma 2.5. [2, Entry 7, Ch. 20, p. 363] *If β is of degree* 11 *over α, then*

$$(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}+2\left[16\alpha\beta(1-\alpha)(1-\beta)\right]^{1/12}=1. \tag{2.17}$$

Lemma 2.6. [3, Entry 58, Ch. 36, p. 386] *Let*

$$A=1-(\alpha\beta)^{1/4}-\left[(1-\alpha)(1-\beta)\right]^{1/4}, \tag{2.18}$$

$$B=16\left[(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}\right], \tag{2.19}$$

$$C=16\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}. \tag{2.20}$$

If β is of degree 19 *over α, then*

$$A^5-7A^2C-BC=0. \tag{2.21}$$

Lemma 2.7. [2, Entry 15, Ch. 20, p. 411] *If β is of degree* 23 *over α, then*

$$(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}+2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}=1. \tag{2.22}$$

Lemma 2.8. [2, Entry 15 (i), (ii), Ch. 19, p. 291] *If β is of degree* 25 *over α, then*

$$\left(\frac{\beta}{\alpha}\right)^{1/8}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/8}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}-2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}=(mm')^{1/2}, \tag{2.23}$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/8}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}-2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}=\frac{5}{(mm')^{1/2}}. \tag{2.24}$$

Lemma 2.9 (Identity Theorem)**.** *Suppose $f(z)$ is analytic in a domain D, and that $\{z_n\}$ is a sequence of distinct points converging to a point z_0 in D. If $f(z_n) = 0$ for each n, then $f(z) \equiv 0$ throughout D.*

3. Main results

In this section, we prove Schläfli-type modular equations of degrees 11, 13, 17 and 19 recorded by Ramanujan in his notebooks. We also establish several new Schläfli-type modular equations of degrees 2, 4, 9, 15, 23, 25, 29, 31, 47 and 71. First, we set

$$P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} \tag{3.1}$$

and

$$Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/24}, \tag{3.2}$$

where β is of degree n over α.

Using (3.1) and (3.2), we obtain the following lemma.

Lemma 3.1. *We have*

$$\alpha = \frac{1+r}{2} \tag{3.3}$$

and

$$\beta = \frac{1+s}{2}, \tag{3.4}$$

where, $r = \pm\sqrt{1 - \frac{P^{12}}{Q^{12}}}$ *and* $s = \pm\sqrt{1 - P^{12}Q^{12}}$.

Theorem 3.1. *If β is of degree 2 over α, then*

$$\frac{2^6}{P^8}\left[Q^{12} + \frac{1}{Q^{12}}\right] + \left[P^{12} - \frac{2^8}{P^{12}}\right] - 16\left[2P^4 - \frac{15}{P^4}\right] = 0. \tag{3.5}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

Proof. Equation (2.1) can be written as

$$1 - a - b = ab, \tag{3.6}$$

where $a = \sqrt{1-\alpha}$ and $b = \sqrt{\beta}$. Squaring both sides of equation (3.6), we find that

$$1 - 2a - 2b + a^2 + 2ab + b^2 - a^2b^2 = 0. \tag{3.7}$$

Isolating the terms containing a on one side of the equation (3.7), squaring both sides and then using (3.3) and (3.4), we deduce that

$$r - 2rs + rs^2 + 33 + s^2 - 48b + 30s - 16sb = 0. \tag{3.8}$$

Again isolating the terms containing b on one side of the equation (3.8), squaring both sides and then using (3.3) and (3.4), we find that

$$\begin{aligned} &56rsP^{12}Q^{24} + 8P^{12} - 16Q^{12} + 16Q^{12}s - 16rQ^{12} - 8P^{12}s + 16rsQ^{12} \\ &+ 80P^{12}Q^{24} - 2P^{24}Q^{36} + 4sP^{24}Q^{12} - 48rP^{12}Q^{24} - 72sP^{12}Q^{24} \\ &- 2rP^{24}Q^{36} - 8P^{24}Q^{12} + P^{36}Q^{24} = 0. \end{aligned} \tag{3.9}$$

Eliminating r and s in the same manner, we deduce that

$$\begin{aligned} &(-256Q^{12} + P^{24}Q^{12} + 64P^4 + 240Q^{12}P^8 - 32Q^{12}P^{16} \\ &+ 64Q^{24}P^4)(-15360P^{12}Q^{12} + P^{48}Q^{24} + 32Q^{24}P^{40} + 784Q^{24}P^{32} \\ &- 64Q^{36}P^{28} - 64P^{28}Q^{12} + 7168P^{24}Q^{24} - 4096Q^{36}P^{20} - 4096Q^{12}P^{20} \\ &+ 49408Q^{24}P^{16} - 15360Q^{36}P^{12} + 4096Q^{48}P^8 + 69632Q^{24}P^8 \\ &+ 4096P^8 + 16384Q^{36}P^4 + 16384Q^{12}P^4 + 65536Q^{24}) = 0. \end{aligned} \tag{3.10}$$

By examining the factors near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the first factor vanish but the second factor does not. By the Identity Theorem 2.9, the first factor vanishes identically. Hence, we obtain (3.5). ∎

Theorem 3.2. *If β is of degree 4 over α, then*

$$\begin{aligned} &P^{24} + 256\left(Q^8 + \frac{1}{Q^8}\right)\left(2575 - \frac{2^{12}}{P^{24}}\right) \\ &+ 128\left(Q^4 + \frac{1}{Q^4}\right)\left(17P^{12} + \frac{36864}{P^{12}}\right) \\ &+ \frac{16384}{P^{12}}\left[Q^{28} + \frac{1}{Q^{28}} + 32\left(Q^{20} + \frac{1}{Q^{20}}\right) + 241\left(Q^{12} + \frac{1}{Q^{12}}\right)\right] \\ &- 16384\left(Q^{16} + \frac{1}{Q^{16}}\right) - 1855488 = 0. \end{aligned} \tag{3.11}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

Proof of the identity (3.11) is similar to the proof of the identity (3.5) given above except that in place of result (2.1), result (2.2) is used.

Theorem 3.3. *If β is of degree 9 over α, then*

$$\begin{aligned} &64\left(P^{12} + \frac{1}{P^{12}}\right)\left[1 + Q^3 + \frac{1}{Q^3}\right] \\ &= 9324 + \left(Q^{18} + \frac{1}{Q^{18}}\right) + 28\left(Q^{15} + \frac{1}{Q^{15}}\right) \\ &+ 298\left(Q^{12} + \frac{1}{Q^{12}}\right) + 1548\left(Q^9 + \frac{1}{Q^9}\right) \\ &+ 4383\left(Q^6 + \frac{1}{Q^6}\right) + 7704\left(Q^3 + \frac{1}{Q^3}\right), \end{aligned} \tag{3.12}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

Proof. Using (3.1) in (2.3) – (2.5), we find that

$$u = 1 - a_1 - \frac{P^{12}}{4a_1}, \tag{3.13}$$

$$v = 64a_1 + \frac{16P^{12}}{a_1} - 16P^{12}, \tag{3.14}$$

$$w = 8P^{12}, \tag{3.15}$$

where $a_1 = \{\alpha\beta\}^{1/2}$. Using (3.13)–(3.15) in (2.6), we deduce that

$$\begin{aligned}
&428032a_2^4a_1P^{12} - 782336a_2^3a_1P^{12} - 195584a_2^2a_1P^{24}\\
&+ 6688a_2a_1P^{48} - 428032a_2^3P^{12} - 195584a_2^3a_1P^{24} - 6144a_2^2a_1P^{12}\\
&- 48896a_2^2a_1P^{36} - 81920a_2^4a_1 - 1280a_2a_1P^{36} + 48896a_2^2P^{36}\\
&+ 3840a_2^2P^{24} + 240a_2P^{48} + 121344a_2^3P^{24} + P^{72} - 24a_1P^{60}\\
&+ 782336a_2^4P^{12} - 24576a_2^5a_1 + 3840a_2^4P^{24}\\
&+ 6144a_2^5P^{12} + 1280a_2^3P^{36} + 240a_2^2P^{48}\\
&+ 24a_2P^{60} - 24576a_2^3a_1 + 4096a_2^3 + 61440a_2^4\\
&+ 61440a_2^5 + 4096a_2^6 = 0,
\end{aligned} \tag{3.16}$$

where $a_2 = \alpha\beta$. Isolating the terms involving a_1 on one side of the equation, squaring both sides and eliminating r and s, we find that

$$\begin{aligned}
&(-64P^{24}Q^{18} + 64P^{24}Q^{21} - 64Q^{18} + 64Q^{15} - 28P^{12}Q^{33} + P^{12}Q^{36}\\
&+298P^{12}Q^{6} - 28P^{12}Q^{3} + 64P^{24}Q^{15} + 64Q^{21} - 7704P^{12}Q^{21} + P^{12}\\
&+4383P^{12}Q^{12} - 1548P^{12}Q^{9} + 9324P^{12}Q^{18} - 7704P^{12}Q^{15}\\
&-1548P^{12}Q^{27} + 298P^{12}Q^{30} + 4383P^{12}Q^{24})(-3456P^{36}Q^{18} - 4096Q^{36}\\
&+196812P^{24}Q^{18} + 10882P^{24}Q^{12} + 4096P^{48}Q^{30} + 188P^{24}Q^{66}\\
&+493551P^{24}Q^{24} - 1048968P^{24}Q^{30} - 623232P^{36}Q^{30} + 778752P^{12}Q^{36}\\
&+188P^{24}Q^{6} + P^{24} + 4096P^{48}Q^{42} + P^{24}Q^{72} + 778752P^{36}Q^{36}\\
&+493551P^{24}Q^{48} + 10882P^{24}Q^{60} - 4096P^{48}Q^{36} + 163584P^{36}Q^{24}\\
&+163584P^{36}Q^{48} + 2030364P^{24}Q^{36} + 163584P^{12}Q^{48} - 3456P^{12}Q^{18}\\
&+4096Q^{30} + 4096Q^{42} - 623232P^{36}Q^{42} - 1048968P^{24}Q^{42}\\
&-623232P^{12}Q^{30} + 163584P^{12}Q^{24} + 196812P^{24}Q^{54} - 623232P^{12}Q^{42}\\
&-3456P^{36}Q^{54} - 3456P^{12}Q^{54})(-64P^{24}Q^{18} - 64P^{24}Q^{21} - 64Q^{18}\\
&-64Q^{15} + 28P^{12}Q^{33} + P^{12}Q^{36} + 298P^{12}Q^{6} + 28P^{12}Q^{3}\\
&-64P^{24}Q^{15} - 64Q^{21} + 7704P^{12}Q^{21} + P^{12} + 4383P^{12}Q^{12}\\
&+1548P^{12}Q^{9} + 9324P^{12}Q^{18} + 7704P^{12}Q^{15} + 1548P^{12}Q^{27}\\
&+298P^{12}Q^{30} + 4383P^{12}Q^{24}) = 0.
\end{aligned} \tag{3.17}$$

Putting $n = 1/9$ in (3.1) and (3.2), we find that

$$P = \frac{1}{G_9^2} \quad \text{and} \quad Q = 1. \tag{3.18}$$

Using (3.18) in (3.17), we find that the last factor vanishes for the specific value of $q = e^{-\pi\sqrt{1/9}}$, then the last factor vanishes in a neighbourhood of $q = e^{-\pi\sqrt{1/9}}$. This proves the theorem. ∎

Theorem 3.4. [12, p. 90], [3, Entry 41, p. 378] *If β is of degree* 11 *over α, then*

$$Q^6 + \frac{1}{Q^6} = 2\sqrt{2}\left(\frac{2}{P^5} - \frac{11}{P^3} + \frac{22}{P} - 22P + 11P^3 - 2P^5\right), \tag{3.19}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

Proof. Using (3.1) in (2.17), we find that

$$\{\alpha\beta\}^{1/4} + \frac{P^6}{\{16\alpha\beta\}^{1/4}} = 1 - 2P^2. \tag{3.20}$$

Squaring both sides of (3.20), we find that

$$\{\alpha\beta\}^{1/2} + \frac{P^{12}}{\{16\alpha\beta\}^{1/2}} = \left(1 - 2P^2\right)^2 - P^6. \tag{3.21}$$

Again squaring both sides of (3.21), we find that

$$\{\alpha\beta\} + \frac{P^{24}}{\{16\alpha\beta\}} = \left(\left(1 - 2P^2\right)^2 - P^6\right)^2 - \frac{P^{12}}{2}. \tag{3.22}$$

Using (3.3) and (3.4) in (3.22), we find that

$$\begin{aligned} &32P^{20}Q^{12} - 4576P^{14}Q^{12} - P^{10}Q^{24} - 352P^2Q^{12} - 352P^{18}Q^{12} \\ &+8096P^{12}Q^{12} - P^{10} + 1672P^{16}Q^{12} + 1672P^4Q^{12} - 9746P^{10}Q^{12} \\ &+8096P^8Q^{12} + 32Q^{12} - 4576P^6Q^{12} = 0. \end{aligned} \tag{3.23}$$

Simplifying the above equation, we obtain the required result (3.19).

Theorem 3.5. [12, p. 90], [3, Entry 41, p. 378] *If β is of degree* 13 *over α, then*

$$\begin{aligned} &\left(Q^7 + \frac{1}{Q^7}\right) + 13\left(Q^5 + \frac{1}{Q^5}\right) + 52\left(Q^3 + \frac{1}{Q^3}\right) \\ &+78\left(Q + \frac{1}{Q}\right) = 8\left(P^6 - \frac{1}{P^6}\right), \end{aligned} \tag{3.24}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

Proof. Using (3.13)–(3.15) in (2.7), we deduce that

$$\begin{aligned}
&11712512a_2^3a_1P^{24}+15610880a_2^2a_1P^{36}-2030144a_2^2a_1P^{48}\\
&+15610880a_2^3a_1P^{36}+P^{84}+249774080a_2^4a_1P^{12}\\
&+13590528a_2^5a_1P^{12}+16384a_2^7+53088a_2a_1P^{60}-21504a_2^2a_1P^{24}\\
&+13590528a_2^3a_1P^{12}-32482304a_2^4a_1P^{24}-28a_1P^{72}-16384a_2^3a_1\\
&+28672a_2^3P^{12}+129929216a_2^4P^{12}+32482304a_2^3P^{24}+8960a_2^2P^{36}\\
&+129929216a_2^5P^{12}-11712512a_2^4P^{24}-2928128a_2^3P^{36}-344064a_2^4a_1\\
&-573440a_2^5a_1-114688a_2^6a_1+2030144a_2^2P^{48}+28672a_2^6P^{12}\\
&+21504a_2^5P^{24}+8960a_2^4P^{36}+2240a_2^3P^{48}+336a_2^2P^{60}+114688a_2^4\\
&+573440a_2^5+344064a_2^6-2240P^{48}a_2a_1+28a_2P^{72}+336a_2P^{60}=0.
\end{aligned}\tag{3.25}$$

Isolating the terms involving a_1 on one side of the above equation, squaring on both sides and then eliminating r and s, we find that

$$\begin{aligned}
&(8Q^7P^{12}+P^6-8Q^7+52Q^4P^6+P^6Q^{14}+52P^6Q^{10}+13Q^2P^6\\
&+78Q^6P^6+13P^6Q^{12}+78P^6Q^8)(-8Q^7P^{12}+P^6+8Q^7+52Q^4P^6\\
&+78P^6Q^8+P^6Q^{14}+52P^6Q^{10}+13Q^2P^6+78Q^6P^6\\
&+13P^6Q^{12})(15574P^{12}Q^{12}+4888Q^8P^{12}+64Q^{14}P^{24}-26Q^2P^{12}\\
&+64Q^{14}-26Q^{26}P^{12}+P^{12}+273Q^{24}P^{12}+Q^{28}P^{12}+15574Q^{16}P^{12}\\
&-1508Q^{22}P^{12}-10244Q^{18}P^{12}-1508Q^6P^{12}-18044P^{12}Q^{14}\\
&+4888Q^{20}P^{12}-10244Q^{10}P^{12}+273Q^4P^{12})(1378P^{12}Q^{12}-624P^6Q^{13}\\
&+1612Q^8P^{12}+64Q^{14}P^{24}+104Q^9P^{18}-13Q^2P^{12}+64Q^{14}\\
&+624Q^{13}P^{18}+8Q^7P^{18}-2080Q^{10}P^{12}+117Q^4P^{12}-13Q^{26}P^{12}\\
&+8Q^{21}P^{18}-8Q^{21}P^6-624Q^{15}P^6+624Q^{15}P^{18}+117Q^{24}P^{12}\\
&+Q^{28}P^{12}-520Q^{22}P^{12}-2080Q^{18}P^{12}-104P^6Q^9+P^{12}-520Q^6P^{12}\\
&-974P^{12}Q^{14}+832P^6Q^{11}+104Q^{19}P^{18}+1612Q^{20}P^{12}-104Q^{19}P^6\\
&+1378Q^{16}P^{12}-832Q^{11}P^{18}-832Q^{17}P^{18}-8Q^7P^6\\
&+832Q^{17}P^6)(1378P^{12}Q^{12}+624P^6Q^{13}+1612Q^8P^{12}+64Q^{14}P^{24}\\
&-104Q^9P^{18}-13Q^2P^{12}+64Q^{14}-624Q^{13}P^{18}-8Q^7P^{18}\\
&-2080Q^{10}P^{12}+117Q^4P^{12}-13Q^{26}P^{12}-8Q^{21}P^{18}+8Q^{21}P^6\\
&+624Q^{15}P^6-624Q^{15}P^{18}+117Q^{24}P^{12}+Q^{28}P^{12}-520Q^{22}P^{12}\\
&-2080Q^{18}P^{12}+104P^6Q^9+P^{12}-520Q^6P^{12}-974P^{12}Q^{14}\\
&-832P^6Q^{11}-104Q^{19}P^{18}+1612Q^{20}P^{12}+104Q^{19}P^6+1378Q^{16}P^{12}\\
&+832Q^{11}P^{18}+832Q^{17}P^{18}+8Q^7P^6-832Q^{17}P^6)(4096Q^{28}P^{48}+P^{24}\\
&+2200848Q^{14}P^{24}-17472P^{36}Q^{38}+26Q^2P^{24}+96512Q^{20}P^{36}\\
&+655616Q^{32}P^{12}+5115604Q^{16}P^{24}+625664Q^{22}P^{36}-3328Q^{16}P^{36}\\
&+96512P^{36}Q^{36}+5115604Q^{40}P^{24}+4096Q^{28}-2309632P^{36}Q^{28}\\
&+8411936Q^{18}P^{24}-17472Q^{18}P^{36}+11103846P^{24}Q^{24}+655616Q^{24}P^{36}\\
&+167752Q^{10}P^{24}+403Q^4P^{24}+30433Q^{48}P^{24}+23542962Q^{28}P^{24}
\end{aligned}\tag{3.26}$$

$$
\begin{aligned}
&- 64Q^{42}P^{12} - 996736Q^{26}P^{12} - 64Q^{14}P^{36} + 18564676Q^{30}P^{24}\\
&+ 167752Q^{46}P^{24} - 3328P^{36}Q^{40} + 655616Q^{24}P^{12} + 2200848Q^{42}P^{24}\\
&+ 11103846Q^{32}P^{24} - 2309632Q^{28}P^{12} + 18564676Q^{26}P^{24}\\
&+ 704444Q^{12}P^{24} + 625664Q^{22}P^{12} - 64P^{36}Q^{42} - 17472Q^{18}P^{12}\\
&+ 625664Q^{34}P^{36} + 26Q^{54}P^{24} + 655616Q^{32}P^{36} + 9171448Q^{34}P^{24}\\
&+ 9835644Q^{36}P^{24} - 64P^{12}Q^{14} - 996736Q^{30}P^{12} + 4082Q^{50}P^{24}\\
&- 996736Q^{30}P^{36} - 996736P^{36}Q^{26} + 9171448Q^{22}P^{24} + 4082Q^{6}P^{24}\\
&- 17472Q^{38}P^{12} + 96512Q^{20}P^{12} + 8411936Q^{38}P^{24} + 625664Q^{34}P^{12}\\
&+ 403Q^{52}P^{24} + 96512Q^{36}P^{12} - 3328Q^{16}P^{12} - 3328Q^{40}P^{12}\\
&+ 704444Q^{44}P^{24} + 9835644Q^{20}P^{24} + 30433Q^{8}P^{24} + Q^{56}P^{24}) = 0.
\end{aligned}
$$

By examining the factors near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the first factor vanish but the other factors does not. By the Identity Theorem, the first factor vanishes identically. This proves the theorem. ∎

Theorem 3.6. *If β is of degree 15 over α, then*

$$
\begin{aligned}
&Q^{36} + \frac{1}{Q^{36}} + 2\left(Q^{24} + \frac{1}{Q^{24}}\right)\Bigg[360\sqrt{2}\left(P^{9} + \frac{1}{P^{9}}\right)\\
&+ 11580\left(P^{6} + \frac{1}{P^{6}}\right) + 40328\sqrt{2}\left(P^{3} + \frac{1}{P^{3}}\right) + 93915\Bigg]\\
&- \left(Q^{12} + \frac{1}{Q^{12}}\right)\Bigg[32\Bigg\{32\sqrt{2}\left(P^{21} + \frac{1}{P^{21}}\right) - 720\left(P^{18} + \frac{1}{P^{18}}\right)\\
&+ 2520\sqrt{2}\left(P^{15} + \frac{1}{P^{15}}\right) - 5430\left(P^{12} + \frac{1}{P^{12}}\right)\\
&+ 5195\sqrt{2}\left(P^{9} + \frac{1}{P^{9}}\right) - 13815\left(P^{6} + \frac{1}{P^{6}}\right)\\
&+ 22590\sqrt{2}\left(P^{3} + \frac{1}{P^{3}}\right)\Bigg\} + 34993\Bigg]\\
&= 16\Bigg\{49050\sqrt{2}\left(P^{3} + \frac{1}{P^{3}}\right) - 112365\left(P^{6} + \frac{1}{P^{6}}\right)\\
&+ 107730\sqrt{2}\left(P^{9} + \frac{1}{P^{9}}\right) - 34200\left(P^{12} + \frac{1}{P^{12}}\right)\\
&+ 22560\sqrt{2}\left(P^{15} + \frac{1}{P^{15}}\right) - 20192\left(P^{18} + \frac{1}{P^{18}}\right)\\
&+ 2880\sqrt{2}\left(P^{21} + \frac{1}{P^{21}}\right) - 256\left(P^{24} + \frac{1}{P^{24}}\right)\Bigg\} - 5028660,
\end{aligned}
\tag{3.27}
$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Proof of the identity (3.27) is similar to the proof of the identity (3.24) given above except that in place of result (2.7), result (2.13) is used.

Theorem 3.7. [12, p. 90], [3, Entry 41, p. 378] *If β is of degree* 17 *over α, then*

$$\begin{aligned}(3.28)\qquad &\left(Q^9+\frac{1}{Q^9}\right)+119\left(Q^3+\frac{1}{Q^3}\right)-34\left(Q^6+\frac{1}{Q^6}\right)-16\left(P^8+\frac{1}{P^8}\right)\\ &+68\left(Q^3+\frac{1}{Q^3}\right)\left(P^4+\frac{1}{P^4}\right)+136\left(P^4+\frac{1}{P^4}\right)+340=0,\end{aligned}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

The proof of the identity (3.28) is similar to the proof of the identity (3.24), except that in place of result (2.7), result (2.8) is used.

Theorem 3.8. [12, p. 90], [3, Entry 41, pp. 378, 379] *If β is of degree* 19 *over α, then*

$$\begin{aligned}(3.29)\qquad &Q^{10}+\frac{1}{Q^{10}}+114\left(Q^6+\frac{1}{Q^6}\right)+190\sqrt{2}\left(Q^4+\frac{1}{Q^4}\right)\left(P^3-\frac{1}{P^3}\right)\\ &+19\left(Q^2+\frac{1}{Q^2}\right)\left(\frac{8}{P^6}-5+8P^6\right)-4\sqrt{2}\left(\frac{4}{P^9}+\frac{19}{P^3}-19P^3-4P^9\right)=0,\end{aligned}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

Proof. Using (3.1) in (2.18)–(2.20), we find that

$$(3.30)\qquad A = 1-c_1-\frac{1}{2}\frac{P^6}{c_1},$$

$$(3.31)\qquad B = 16c_1+8\frac{P^6}{c_1}-8P^6,$$

$$(3.32)\qquad C = 8P^6.$$

where $c_1=\{\alpha\beta\}^{1/4}$. Using (3.30)–(3.32) in (2.21), we find that

$$\begin{aligned}(3.33)\qquad &-130240a_1P^{36}a_2-640a_2P^{24}a_1+5120a_1a_2^4P^6+8064a_1P^{30}a_2^2\\ &+15360a_1a_2^3P^{18}-1161088a_1P^{24}a_2^2+1024a_2^5+2078720a_2^3P^6a_1\\ &+129920a_2P^{30}a_1+119808a_2^2P^6a_1-2083840a_1a_2^3P^{12}\\ &+2966528a_2^2P^{18}a_1-1932288a_2^2P^{12}a_1+960a_1P^{42}a_2-1024a_1a_2^2\\ &+10240a_2^4+5120a_2^3+1280a_2^2P^{12}+351232a_2^2P^{18}+160a_2P^{36}\\ &-834176a_2^2P^{24}+1404928a_2^3P^6-3336704a_2^3P^{12}-10240a_1a_2^3\\ &+9152a_2P^{42}+461056a_2^2P^{30}+1844224a_2^3P^{18}+20a_1P^{54}\\ &-5120a_1a_2^4+180P^{48}a_2+3360P^{36}a_2^2+13440a_2^3P^{24}\\ &+585728a_2^4P^6+11520a_2^4P^{12}-20a_1P^{48}+P^{60}=0.\end{aligned}$$

Isolating the terms involving a_1 to one side of the above equation and squaring both sides, substituting $a_2=(1+r)(1+s)/4$ and eliminating r and s, we find that

$$
\begin{aligned}
&(-37240P^{24}Q^{28}+P^{18}+41344Q^{20}P^{30}+41344Q^{20}P^{6}-208848Q^{20}P^{24}\\
&+449388Q^{20}P^{18}-208848Q^{20}P^{12}+304Q^{8}P^{24}+12806Q^{8}P^{18}+304Q^{8}P^{12}\\
&+10944Q^{24}P^{6}+149321Q^{24}P^{18}-39824Q^{24}P^{12}+10944P^{30}Q^{24}\\
&-37240P^{12}Q^{12}-37240Q^{12}P^{24}+228P^{18}Q^{36}-512Q^{20}+P^{18}Q^{40}\\
&-39824P^{24}Q^{24}+122550Q^{12}P^{18}+304P^{24}Q^{32}-39824Q^{16}P^{12}+10944Q^{16}P^{6}\\
&-39824Q^{16}P^{24}+10944Q^{16}P^{30}+149321Q^{16}P^{18}+228Q^{4}P^{18}+12806P^{18}Q^{32}\\
&-512Q^{20}P^{36}+304P^{12}Q^{32}-37240Q^{28}P^{12}+122550Q^{28}P^{18})(-8137026944P^{24}Q^{48}\\
&+P^{36}+5807166992P^{30}Q^{28}-13769395456P^{48}Q^{36}+39178P^{36}Q^{72}\\
&-13769395456P^{24}Q^{36}+16270080P^{48}Q^{60}+100704256P^{60}Q^{48}\\
&-46751414915P^{36}Q^{48}+100704256P^{12}Q^{48}+16270080P^{24}Q^{60}\\
&-1895116544P^{24}Q^{28}-472858624P^{60}Q^{36}-8137026944P^{48}Q^{48}\\
&+15801984144P^{36}Q^{24}-1479092392P^{36}Q^{60}+16316592P^{42}Q^{64}\\
&+136087747P^{36}Q^{64}+81472P^{48}Q^{64}+74749420608P^{48}Q^{40}-472858624P^{60}Q^{44}\\
&-16857729024P^{54}Q^{40}+727401472P^{54}Q^{48}+5603328P^{66}Q^{36}-3210240P^{54}Q^{24}\\
&+421689344Q^{20}P^{30}+16270080Q^{20}P^{24}+512Q^{20}P^{18}-13769395456P^{48}Q^{44}\\
&-8618720720P^{42}Q^{24}+31548679200P^{42}Q^{48}+5603328P^{66}Q^{44}-3210240Q^{56}P^{18}\\
&+3193459200P^{18}Q^{44}+512P^{54}Q^{60}+31761710544P^{30}Q^{36}+727401472P^{18}Q^{48}\\
&+421689344P^{30}Q^{60}+5603328P^{6}Q^{44}+31761710544P^{30}Q^{44}\\
&+421689344P^{42}Q^{60}-304(Q^{8}P^{30}+Q^{8}P^{42}+P^{30}Q^{72}+P^{42}Q^{72})\\
&+39178Q^{8}P^{36}+31761710544P^{42}Q^{44}+31548679200P^{30}Q^{48}\\
&-13769395456P^{24}Q^{44}-43661221076P^{36}Q^{44}+81472Q^{16}P^{24}\\
&+136087747Q^{16}P^{36}+81472Q^{16}P^{48}+16316592Q^{16}P^{42}\\
&+31761710544P^{42}Q^{36}+419578368Q^{52}P^{18}-311296(Q^{52}P^{60}\\
&+Q^{52}P^{12})-3210240Q^{24}P^{18}+512P^{18}Q^{60}-10043781520Q^{52}P^{36}\\
&+5807166992Q^{52}P^{30}-8618720720P^{30}Q^{24}-228Q^{76}P^{36}\\
&+3193459200(P^{54}Q^{44}+P^{54}Q^{36})-42336256P^{66}Q^{40}\\
&+3193459200P^{18}Q^{36}+1803415552P^{60}Q^{40}+249431687698P^{36}Q^{40}\\
&-183111246912P^{30}Q^{40}+512Q^{20}P^{54}+74749420608P^{24}Q^{40}\\
&-16857729024P^{18}Q^{40}+1249370688P^{24}Q^{24}+262144Q^{40}\\
&-8618720720Q^{56}P^{30}+15801984144Q^{56}P^{36}+1803415552P^{12}Q^{40}\\
&+81472Q^{64}P^{24}-1895116544Q^{52}P^{48}+5807166992Q^{52}P^{42}\\
&+419578368Q^{52}P^{54}-42336256P^{6}Q^{40}-143792Q^{12}P^{30}\\
&-8137026944P^{24}Q^{32}-8137026944Q^{32}P^{48}-46751414915Q^{32}P^{36}\\
&-143792Q^{12}P^{42}+419578368P^{54}Q^{28}-10043781520P^{36}Q^{28}\\
&+5807166992P^{42}Q^{28}-1895116544P^{48}Q^{28}-311296P^{60}Q^{28}\\
&-43661221076P^{36}Q^{36}+1249370688P^{48}Q^{24}+16316592Q^{16}P^{30}\\
&-228Q^{4}P^{36}-472858624P^{12}Q^{36}+727401472P^{18}Q^{32}+Q^{80}P^{36}\\
&-1479092392Q^{20}P^{36}+100704256P^{12}Q^{32}-311296Q^{28}P^{12}\\
&+419578368Q^{28}P^{18}-2674668(Q^{68}P^{36}+P^{36}Q^{12})-143792Q^{68}P^{30}\\
&+5603328Q^{36}P^{6}-143792Q^{68}P^{42}+16270080Q^{20}P^{48}\\
&+31548679200Q^{32}P^{42}+100704256Q^{32}P^{60}+727401472Q^{32}P^{54}\\
&+16316592Q^{64}P^{30}+1249370688Q^{56}P^{24}-3210240Q^{56}P^{54}\\
&-8618720720Q^{56}P^{42}+1249370688Q^{56}P^{48}-1895116544Q^{52}P^{24}\\
&+262144Q^{40}P^{72}+31548679200P^{30}Q^{32}+421689344Q^{20}P^{42}\\
&-183111246912P^{42}Q^{40}-472858624P^{12}Q^{44})=0.
\end{aligned}
\tag{3.34}
$$

By examining the factors near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the first factor vanish but the other factors does not. By the Identity Theorem, the first factor vanishes identically. This proves the theorem. ∎

Theorem 3.9. *If β is of degree 23 over α, then*

$$\begin{aligned}
&Q^{12}+\frac{1}{Q^{12}}+4223122 = 8\left[4\sqrt{2}\left(P^{11}+\frac{1}{P^{11}}\right)-92\left(P^{10}+\frac{1}{P^{10}}\right)\right.\\
&+506\sqrt{2}\left(P^{9}+\frac{1}{P^{9}}\right)-3588\left(P^{8}+\frac{1}{P^{8}}\right)+9292\sqrt{2}\left(P^{7}+\frac{1}{P^{7}}\right)\\
&-37605\left(P^{6}+\frac{1}{P^{6}}\right)+61916\sqrt{2}\left(P^{5}+\frac{1}{P^{5}}\right)-170292\left(P^{4}+\frac{1}{P^{4}}\right)\\
&\left.+199042\sqrt{2}\left(P^{3}+\frac{1}{P^{3}}\right)-400108\left(P^{2}+\frac{1}{P^{2}}\right)+348404\sqrt{2}\left(P+\frac{1}{P}\right)\right],
\end{aligned} \tag{3.35}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

The proof of the identity (3.35) is similar to the proof of the identity (3.19), except that in place of result (2.17), result (2.22) is used.

Theorem 3.10. *If β is of degree 25 over α, then*

$$\begin{aligned}
&64\left(P^{12}+\frac{1}{P^{12}}\right)\left[1+\left(Q+\frac{1}{Q}\right)+\left(Q^{2}+\frac{1}{Q^{2}}\right)\right]=\left(Q^{15}+\frac{1}{Q^{15}}\right)\\
&+26\left(Q^{14}+\frac{1}{Q^{14}}\right)+301\left(Q^{13}+\frac{1}{Q^{13}}\right)+2076\left(Q^{12}+\frac{1}{Q^{12}}\right)\\
&+9726\left(Q^{11}+\frac{1}{Q^{11}}\right)+33880\left(Q^{10}+\frac{1}{Q^{10}}\right)+94480\left(Q^{9}+\frac{1}{Q^{9}}\right)\\
&+222580\left(Q^{8}+\frac{1}{Q^{8}}\right)+456305\left(Q^{7}+\frac{1}{Q^{7}}\right)+827530\left(Q^{6}+\frac{1}{Q^{6}}\right)\\
&+1346255\left(Q^{5}+\frac{1}{Q^{5}}\right)+1985080\left(Q^{4}+\frac{1}{Q^{4}}\right)+2668655\left(Q^{3}+\frac{1}{Q^{3}}\right)\\
&+3285730\left(Q^{2}+\frac{1}{Q^{2}}\right)+3718805\left(Q+\frac{1}{Q}\right)+3875380,
\end{aligned} \tag{3.36}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

Proof. Using (3.2) in (2.23) and (2.24), we find that

$$\begin{aligned}
&2d_1^2Q^3+d_1^4-d_1^3-2d_1^3Q+Q^6-Q^3d_1-2Q^4d_1-Q^6d_1-Q^3d_1^3\\
&+2Q^4d_1^2-2Q^5d_1-2Q^2d_1^3+2Q^2d_1^2=0,
\end{aligned} \tag{3.37}$$

where $d_1 = \left(\frac{\beta}{\alpha}\right)^{1/8}$. Substituting $d_1^2 = d_2$ and isolating the terms involving d_1 on one side of equation, squaring both sides and again substituting $d_2^2 = d_3$ and $d_3^2 = d_4$, we find that

$$\begin{aligned}&-4d_3Q^2d_2-4d_3d_2Q-4d_3d_2Q^5-4d_3Q^4d_2-d_3d_2Q^6-2d_3Q^9-6d_2Q^3d_3\\&-4Q^8d_3-4Q^4d_3-2d_3Q^3-d_2d_3+d_4-6d_2Q^9-6d_3Q^6-4Q^{10}d_2-4Q^8d_2\\&-d_2Q^6-4d_2Q^7-d_2Q^{12}-4d_2Q^{11}+Q^{12}-8d_3Q^7-8d_3Q^5=0.\end{aligned}\tag{3.38}$$

Now isolating the terms involving d_2 on one side of the above equation, squaring both sides, we find that

$$\begin{aligned}&-d_4d_3-Q^{12}d_3-d_3Q^{24}-48d_3Q^{21}-104d_3Q^{17}-114d_3Q^{18}+2d_4Q^{18}\\&-80Q^{20}d_3+Q^{24}-80Q^{16}d_3+2d_4Q^6-48d_3Q^{15}+2d_4Q^{12}-104Q^{19}d_3\\&-24Q^{14}d_3-8Q^{13}d_3-24d_3Q^{22}-8d_3Q^{23}-8d_4d_3Q^{11}-104d_3Q^5d_4\\&-80Q^4d_3d_4-48d_3Q^9d_4-80Q^8d_3d_4-104d_4d_3Q^7-48d_3Q^3d_4\\&-114d_3Q^6d_4-24Q^2d_4d_3-8d_4d_3Q-d_4d_3Q^{12}-24d_4d_3Q^{10}+d_4^2=0,\end{aligned}\tag{3.39}$$

where $d_4=\dfrac{\beta}{\alpha}$. Isolating the terms involving d_3 on one side of the above equation and squaring on both sides and eliminating r and s, we find that

$$\begin{aligned}&(-64Q^{15}-64Q^{17}-64Q^{16}-64Q^{14}-64Q^{13}+2668655P^{12}Q^{12}+P^{12}\\&+94480P^{12}Q^{24}+2076Q^3P^{12}+26QP^{12}+222580Q^7P^{12}\\&+456305Q^8P^{12}+9726Q^4P^{12}+301Q^2P^{12}+1346255Q^{10}P^{12}\\&+827530Q^9P^{12}+94480Q^6P^{12}+1985080Q^{11}P^{12}+33880Q^5P^{12}\\&+Q^{30}P^{12}+2668655Q^{18}P^{12}+26Q^{29}P^{12}+33880Q^{25}P^{12}+2076Q^{27}P^{12}\\&+9726Q^{26}P^{12}+222580Q^{23}P^{12}+827530Q^{21}P^{12}+456305Q^{22}P^{12}\\&+3718805Q^{14}P^{12}+3718805Q^{16}P^{12}+1346255Q^{20}P^{12}+1985080Q^{19}P^{12}\\&+3285730Q^{13}P^{12}+3875380Q^{15}P^{12}+301Q^{28}P^{12}+3285730Q^{17}P^{12}\\&-64P^{24}Q^{14}-64P^{24}Q^{16}-64P^{24}Q^{13}-64P^{24}Q^{15}-64Q^{17}P^{24})\\&(Q^2+Q+1)^4(Q^2+1)^4(Q^8-Q^4+1)^2(Q^4+1)^2(Q+1)^2=0.\end{aligned}\tag{3.40}$$

By examining the factors near $q=0$, it can be seen that there is a neighbourhood about the origin, where the first factor vanish but the other factors does not. By the Identity Theorem, the first factor vanishes identically. This proves the theorem. ∎

Theorem 3.11. *If β is of degree 29 over α, then*

$$\begin{aligned}&Q^{15}+\frac{1}{Q^{15}}+128\left(P^{14}-\frac{1}{P^{14}}\right)-58\left(P^2-\frac{1}{P^2}\right)\left(Q^{12}+\frac{1}{Q^{12}}\right)\\&+29\left[36\left(P^4+\frac{1}{P^4}\right)+185\right]\left(Q^9+\frac{1}{Q^9}\right)-6264\left(P^6-\frac{1}{P^6}\right)\left(Q^6+\frac{1}{Q^6}\right)\\&+58\left[184\left(P^8+\frac{1}{P^8}\right)-394\left(P^4+\frac{1}{P^4}\right)+649\right]\left(Q^3+\frac{1}{Q^3}\right)\\&=24940\left(P^2-\frac{1}{P^2}\right)-10672\left(P^6-\frac{1}{P^6}\right)+6496\left(P^{10}-\frac{1}{P^{10}}\right),\end{aligned}\tag{3.41}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

Theorem 3.12. *If β is of degree 31 over α, then*

$$\begin{aligned}
&Q^{16} + \frac{1}{Q^{16}} + 775\left(Q^{12} + \frac{1}{Q^{12}}\right) + 38657\left(Q^{8} + \frac{1}{Q^{8}}\right)\\
&- 252526\left(Q^{4} + \frac{1}{Q^{4}}\right) + 537726 = 128\sqrt{2}\left(P^{15} + \frac{1}{P^{15}}\right)\\
&- 5952\left(P^{12} + \frac{1}{P^{12}}\right) - \left(P^{9} + \frac{1}{P^{9}}\right)\left[6448\sqrt{2}\left(Q^{4} + \frac{1}{Q^{4}}\right) - 31744\sqrt{2}\right]\\
&- \left(P^{6} + \frac{1}{P^{6}}\right)\left[1984\left(Q^{8} + \frac{1}{Q^{8}}\right) - 50840\left(Q^{4} + \frac{1}{Q^{4}}\right) + 157976\right]\\
&- \left(P^{3} + \frac{1}{P^{3}}\right)\left[62\sqrt{2}\left(Q^{12} + \frac{1}{Q^{12}}\right) - 15872\sqrt{2}\left(Q^{8} + \frac{1}{Q^{8}}\right)\right.\\
&\left. +119536\sqrt{2}\left(Q^{4} + \frac{1}{Q^{4}}\right) - 274660\sqrt{2}\right],
\end{aligned} \tag{3.42}$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

Theorem 3.13. *If β is of degree 47 over α, then*

$$\begin{aligned}
&Q^{24} + \frac{1}{Q^{24}} + 188\left(Q^{12} - \frac{1}{Q^{12}}\right)\left[8\sqrt{2}\left(P^{11} + \frac{1}{P^{11}}\right)\right.\\
&+ 744\left(P^{10} + \frac{1}{P^{10}}\right) + 11048\sqrt{2}\left(P^{9} + \frac{1}{P^{9}}\right) + 168664\left(P^{8} + \frac{1}{P^{8}}\right)\\
&+ 805016\sqrt{2}\left(P^{7} + \frac{1}{P^{7}}\right) + 5327756\left(P^{6} + \frac{1}{P^{6}}\right)\\
&+ 13001656\sqrt{2}\left(P^{5} + \frac{1}{P^{5}}\right) + 48688888\left(P^{4} + \frac{1}{P^{4}}\right)\\
&+ 71813704\sqrt{2}\left(P^{3} + \frac{1}{P^{3}}\right) + 169828168\left(P^{2} + \frac{1}{P^{2}}\right)\\
&\left. +162800104\sqrt{2}\left(P + \frac{1}{P}\right) + 254642055\right]\\
&= 2048\sqrt{2}\left(P^{23} + \frac{1}{P^{23}}\right) - 96256\sqrt{2}\left(P^{21} + \frac{1}{P^{21}}\right)\\
&- 240640\left[P^{20} + \frac{1}{P^{20}}\right] + 1756672\sqrt{2}\left[P^{19} + \frac{1}{P^{19}}\right]\\
&+ 9072128\left[P^{18} + \frac{1}{P^{18}}\right] - 9709824\sqrt{2}\left[P^{17} + \frac{1}{P^{17}}\right]\\
&- 125866752\left[P^{16} + \frac{1}{P^{16}}\right] - 101562112\sqrt{2}\left[P^{15} + \frac{1}{P^{15}}\right]\\
&+ 625579776\left[P^{14} + \frac{1}{P^{14}}\right] + 1536739072\sqrt{2}\left[P^{13} + \frac{1}{P^{13}}\right]
\end{aligned} \tag{3.43}$$

$$+ 1149034944 \left[P^{12} + \frac{1}{P^{12}}\right] - 5082091200\sqrt{2}\left[P^{11} + \frac{1}{P^{11}}\right]$$
$$- 17928597440 \left[P^{10} + \frac{1}{P^{10}}\right] - 8133057472\sqrt{2}\left[P^{9} + \frac{1}{P^{9}}\right]$$
$$+ 23387154880 \left[P^{8} + \frac{1}{P^{8}}\right] + 43138741696\sqrt{2}\left[P^{7} + \frac{1}{P^{7}}\right]$$
$$+ 61416509280 \left[P^{6} + \frac{1}{P^{6}}\right] + 2130217778\sqrt{2}\left[P^{5} + \frac{1}{P^{5}}\right]$$
$$+ 11813501888 \left[P^{4} + \frac{1}{P^{4}}\right] + 4974868032\sqrt{2}\left[P^{3} + \frac{1}{P^{3}}\right]$$
$$- 40687005120 \left[P^{2} + \frac{1}{P^{2}}\right] - 96512465088\sqrt{2}\left[P + \frac{1}{P}\right]$$
$$- 189300816838,$$

where P *and* Q *are defined as in* (3.1) *and* (3.2) *respectively.*

Theorem 3.14. *If* β *is of degree* 71 *over* α, *then*

$$Q^{36}+\frac{1}{Q^{36}} + 142\left(Q^{24}+\frac{1}{Q^{24}}\right)\Bigg(401715285197-250672328128\sqrt{2}\left[P+\frac{1}{P}\right]$$
$$+ 243012755392 \left[P^{2} + \frac{1}{P^{2}}\right] - 90783237160\sqrt{2}\left[P^{3} + \frac{1}{P^{3}}\right]$$
$$+ 51543826752 \left[P^{4} + \frac{1}{P^{4}}\right] - 10877163840\sqrt{2}\left[P^{5} + \frac{1}{P^{5}}\right]$$
$$+ 3301519972 \left[P^{6} + \frac{1}{P^{6}}\right] - 342806080\sqrt{2}\left[P^{7} + \frac{1}{P^{7}}\right]$$
$$+ 45053504 \left[P^{8} + \frac{1}{P^{8}}\right] - 1643912\sqrt{2}\left[P^{9} + \frac{1}{P^{9}}\right]$$
$$+ 51914 \left[P^{10} + \frac{1}{P^{10}}\right] -192\sqrt{2}\left[P^{11} + \frac{1}{P^{11}}\right]\Bigg) \tag{3.44}$$
$$+ 71\left(Q^{12} + \frac{1}{Q^{12}}\right)\Bigg(2048\sqrt{2}\left[P^{23} + \frac{1}{P^{23}}\right] + 942080\left[P^{22} + \frac{1}{P^{22}}\right]$$
$$+ 5676032\sqrt{2}\left[P^{21} + \frac{1}{P^{21}}\right] - 187541504\left[P^{20} + \frac{1}{P^{20}}\right]$$
$$+ 1097148416\sqrt{2}\left[P^{19} + \frac{1}{P^{19}}\right] - 9176226304\left[P^{18} + \frac{1}{P^{18}}\right]$$
$$+ 32585199616\sqrt{2}\left[P^{17} + \frac{1}{P^{17}}\right] - 204517744640\left[P^{16} + \frac{1}{P^{16}}\right]$$
$$+ 566617288448\sqrt{2}\left[P^{15} + \frac{1}{P^{15}}\right] - 2751003146240\left[P^{14} + \frac{1}{P^{14}}\right]$$

$$+ 5807932620800\sqrt{2}\left[P^{13}+\frac{1}{P^{13}}\right] - 21186828070080\left[P^{12}+\frac{1}{P^{12}}\right]$$
$$+ 33186228353536\sqrt{2}\left[P^{11}+\frac{1}{P^{11}}\right] - 88563043373568\left[P^{10}+\frac{1}{P^{10}}\right]$$
$$+ 98954852869312\sqrt{2}\left[P^{9}+\frac{1}{P^{9}}\right] - 176963128626688\left[P^{8}\right.$$
$$\left.+\frac{1}{P^{8}}\right] + 108271940623872\sqrt{2}\left[P^{7}+\frac{1}{P^{7}}\right] - 7129338452448\left[P^{6}+\frac{1}{P^{6}}\right]$$
$$- 221694467125760\sqrt{2}\left[P^{5}+\frac{1}{P^{5}}\right] + 819155626103296\left[P^{4}+\frac{1}{P^{4}}\right]$$
$$- 1026292194878784\sqrt{2}\left[P^{3}+\frac{1}{P^{3}}\right] + 2083094118350336\left[P^{2}+\frac{1}{P^{2}}\right]$$
$$\left.-1805589616637440\sqrt{2}\left[P+\frac{1}{P}\right] + 2727816983432377\right)$$
$$= 131072\sqrt{2}\left[P^{35}+\frac{1}{P^{35}}\right] - 9306112\left[P^{34}+\frac{1}{P^{34}}\right] + 153550848\sqrt{2}\left[P^{33}+\frac{1}{P^{33}}\right]$$
$$- 3150118912\left[P^{32}+\frac{1}{P^{32}}\right] + 22730178560\sqrt{2}\left[P^{31}+\frac{1}{P^{31}}\right] - 248314986496\left[P^{30}\right.$$
$$\left.+\frac{1}{P^{30}}\right] + 1082027458560\sqrt{2}\left[P^{29}+\frac{1}{P^{29}}\right] - 7839822381056\left[P^{28}+\frac{1}{P^{28}}\right]$$
$$+ 24407413309440\sqrt{2}\left[P^{27}+\frac{1}{P^{27}}\right] - 133963714543616\left[P^{26}+\frac{1}{P^{26}}\right]$$
$$+ 330092418940928\sqrt{2}\left[P^{25}+\frac{1}{P^{25}}\right] - 1479732762742784\left[P^{24}+\frac{1}{P^{24}}\right]$$
$$+ 3046058228477952\sqrt{2}\left[P^{23}+\frac{1}{P^{23}}\right] - 11604272147963904\left[P^{22}+\frac{1}{P^{22}}\right]$$
$$+ 20576178256737280\sqrt{2}\left[P^{21}+\frac{1}{P^{21}}\right] - 68251974000062464\left[P^{20}+\frac{1}{P^{20}}\right]$$
$$+ 106291599722672128\sqrt{2}\left[P^{19}+\frac{1}{P^{19}}\right] - 311894723269365248\left[P^{18}+\frac{1}{P^{18}}\right]$$
$$+ 432349292593594368\sqrt{2}\left[P^{17}+\frac{1}{P^{17}}\right] - 1135412123693654016\left[P^{16}+\frac{1}{P^{16}}\right]$$
$$+ 1415543996178500096\sqrt{2}\left[P^{15}+\frac{1}{P^{15}}\right] - 3358715145903804416\left[P^{14}+\frac{1}{P^{14}}\right]$$
$$+ 3800077608784535552\sqrt{2}\left[P^{13}+\frac{1}{P^{13}}\right] - 8218354561861466496\left[P^{12}+\frac{1}{P^{12}}\right]$$
$$+ 8512046427768020736\sqrt{2}\left[P^{11}+\frac{1}{P^{11}}\right] - 16925549916428891904\left[P^{10}+\frac{1}{P^{10}}\right]$$
$$+ 16187505728909967008\sqrt{2}\left[P^{9}+\frac{1}{P^{9}}\right] - 29846446514225130752\left[P^{8}+\frac{1}{P^{8}}\right]$$

$$+ 265726172803569205 76\sqrt{2}\left[P^7+\frac{1}{P^7}\right] - 45769170606955635920\left[P^6+\frac{1}{P^6}\right]$$
$$+ 38178953157178087680\sqrt{2}\left[P^5+\frac{1}{P^5}\right] - 61755339772352116992\left[P^4+\frac{1}{P^4}\right]$$
$$+ 48456136115051360544\sqrt{2}\left[P^3+\frac{1}{P^3}\right] - 73802624665517984512\left[P^2+\frac{1}{P^2}\right]$$
$$+ 54558586002509042432\sqrt{2}\left[P+\frac{1}{P}\right] - 78308975348446520116.$$

where P and Q are defined as in (3.1) *and* (3.2) *respectively.*

Proofs of the identities (3.41)–(3.44) are similar to the proof of (3.24), except that in place of result (2.7), result (2.9) is used to prove (3.41); result (2.14) is used to prove (3.42); result (2.15) is used to prove (3.43) and the result (2.16) is used to prove (3.44).

Remark 3.1. The identities (3.19), (3.24), (3.28) and (3.29) have been verified using modular forms in [3, Entry 41, pp. 378, 379] and other identities (3.5), (3.11), (3.12), (3.27), (3.35), (3.36), (3.41)–(3.44) are appears to be new.

4. Explicit evaluations of class invariants

In this section, we establish several explicit evaluations of Ramanujan-Weber class invariants G_n using modular equations obtained in Section as an application.

Theorem 4.1. *We have*

$$G_2^8 = \frac{\sqrt{2}+1}{2}, \tag{4.1}$$

$$G_4^8 = \sqrt{2}\left(\frac{1+\sqrt{2}}{2}\right)^2, \tag{4.2}$$

$$G_{16}^8 = \left(\frac{\sqrt{2}+1}{4}\right)\left(8+4\sqrt{2}+\sqrt[4]{2}(6\sqrt{2}+3)\right), \tag{4.3}$$

$$G_9^3 = \frac{\sqrt{3}+1}{\sqrt{2}}, \tag{4.4}$$

$$G_{11}^2 = \frac{1}{2^{1/6}3^{1/2}}\left[(3\sqrt{3}-\sqrt{11})^{1/3}+(3\sqrt{3}+\sqrt{11})^{1/3}\right], \tag{4.5}$$

$$G_{13}^4 = \frac{3+\sqrt{13}}{2}, \tag{4.6}$$

$$G_{15}^6 = \sqrt{2}\left(\frac{1+\sqrt{5}}{\sqrt{2}}\right)^2, \tag{4.7}$$

$$G_{17} = \left(\frac{1}{4}+\frac{\sqrt{17}}{4}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\sqrt{17}}{2}}\right)^{1/2}, \tag{4.8}$$

$$(4.9)\quad G_{19}=\frac{1}{\sqrt{3}}\left(2\sqrt{2}+\left(\frac{23}{\sqrt{2}}-3\sqrt{\frac{57}{2}}\right)^{1/3}+\left(\frac{23}{\sqrt{2}}+3\sqrt{\frac{57}{2}}\right)^{1/3}\right)^{1/2},$$

$$(4.10)\quad G_{23}=\left(\frac{2\sqrt{2}+\left(25\sqrt{2}-3\sqrt{138}\right)^{1/3}+\left(25\sqrt{2}+3\sqrt{138}\right)^{1/3}}{3}\right)^{1/2},$$

$$(4.11)\quad G_{25}=\frac{\sqrt{5}+1}{2},$$

$$(4.12)\quad \begin{aligned}G_{29}^4&=\frac{1}{6}\Big\{9+3^{1/3}\left[(207-16\sqrt{87})^{1/3}+(207+16\sqrt{87})^{1/3}\right]\\&+\sqrt{36+\left(-9-3^{1/3}\left[(207-16\sqrt{87})^{1/3}-(207+16\sqrt{87})^{1/3}\right]\right)^2}\Big\},\end{aligned}$$

$$(4.13)\quad G_{31}^2=\frac{1}{3}\left\{\sqrt{2}+(47\sqrt{2}-3\sqrt{186})^{1/3}+(47\sqrt{2}+3\sqrt{186})^{1/3}\right\},$$

$$(4.14)\quad \begin{aligned}G_{81}^3&=\left(47-2\sqrt{3}\right)^{2/3}\left(\frac{43+27\sqrt{3}}{169}\right)\\&+\left(47-2\sqrt{3}\right)^{1/3}\left(\frac{15+7\sqrt{3}}{13}\right)+3+2\sqrt{3},\end{aligned}$$

$$(4.15)\quad \begin{aligned}G_{121}&=\frac{1}{12}\Big[4\sqrt{2}+2^{5/6}11^{1/3}\left((7-3\sqrt{3})^{1/3}+(7+3\sqrt{3})^{1/3}\right)\\&+\sqrt{144+\left[-4\sqrt{2}-2^{5/6}11^{1/3}\left((7-3\sqrt{3})^{1/3}-(7+3\sqrt{3})^{1/3}\right)\right]^2}\Big].\end{aligned}$$

Proof of (4.1)**.** Putting $n=\frac{1}{2}$, using the fact that $G_n=G_{1/n}$ and by the definition of G_n in (3.5), we find that

$$(4.16)\quad \left(4G^{16}-4G^8-1\right)\left(2G^8+1\right)^2\left(2G^4-1\right)^2\left(2G^4+1\right)^2=0,$$

where $G:=G_2$. But

$$(4.17)\quad 4G^{16}-4G^8-1=0.$$

Solving the above equation and $G_2>1$, we obtain the required result (4.1). ∎

Proofs of the identities (4.2)–(4.15) are similar to the proof of (4.1), except that in place of result (3.5), result (3.11) is used to evaluate (4.2) and (4.3); result (3.12) is used to prove (4.4) and (4.14); result (3.19) is used to prove (4.5) and (4.15); result (3.24) is used to prove (4.6); result (3.27) is used to prove (4.7); result (3.28) is used to prove (4.8); result (3.29) is used to prove (4.9); result (3.35) is used to prove (4.10); result (3.36) is used to prove (4.11); result (3.41) is used to prove (4.12) and result (3.42) is used to prove (4.13).

Acknowledgement. The authors are thankful to referee and Prof. Bruce C. Berndt for their useful comments.

References

[1] Adiga, C., Mahadeva Naika, M.S. and Shivashankara, K., *On some P-Q eta-function identities of Ramanujan,* Indian J. Math., 44 (2) (2002), 253–267.

[2] Berndt, B.C., *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[3] Berndt, B.C., *Ramanujan's Notebooks, Part V*, Springer-Verlag, New York, 1998.

[4] Bhargava, S., Adiga, C., and Mahadeva Naika, M.S., *A new class of modular equations akin to Ramanujan's P-Q eta-function identities and some evaluations there from,* Adv. Stud. Contemp. Math., 5 (1) (2002), 37-48.

[5] Bhargava, S., Adiga, C., and Mahadeva Naika, M.S., *A new class of modular equations in Ramanujan's alternative theory of elliptic function of signature 4 and some new P-Q eta-function identities*, Indian J. Math., 45 (1) (2003), 23-39.

[6] Hart, W., *Schläfli modular equations for generalised Weber functions,* Ramanujan J., 15 (2008), 435-468.

[7] Mahadeva Naika, M.S., *P-Q eta-function identities and computation of Ramanujan-Weber class invariants,* J. Indian Math. Soc., 70(1-4) (2003), 121-134.

[8] Mahadeva Naika, M.S., *Some new explicit values for Ramanujan class invariants,* Adv. Stud. Contemp. Math., 20 (4) (2010), 557-568.

[9] Mahadeva Naika, M.S. and Sushan Bairy, K., *On some new explicit evaluations of class invariants,* Vietnam J. Math., 36 (1) (2008), 103-124.

[10] Mahadeva Naika, M.S. and Sushan Bairy, K., *On some new Schläfli−type mixed modular equations*, Adv. Stud. Contemp. Math., 21 (2) (2011), 189-206.

[11] Ramanathan, K.G., *Ramanujan's Modular Equations*, Acta. Arith., 53 (1990), 403-420.

[12] Ramanujan, S., *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[13] Schläfli, L., *Beweis der Hermiteschen Verwandlungstafeln für elliptischen Modularfunctionen*, J. Reine Angrew. Math., 72 (1870), 360-369.

[14] Watson, G.N., *Some Singular moduli.* II, Quart.J.Math., 3(1932), 189-212.

[15] Watson, G.N., *Über die Schläflischen Modulargleichungen,* J.Reine Angew. Math., 19(1933), 238-251.

Accepted: 27.05.2012