ON ρ-HOMEOMORPHISMS IN TOPOLOGICAL SPACES

C. Devamanoharan

S. Pious Missier

Post Graduate and Research Department of Mathematics
V.O. Chidambaram College
Thoothukudi 628 008
Tamil Nadu
India
e-mail: kanchidev@gmail.com
spmissier@yahoo.com

S. Jafari

College of Vestsjaelland South
Herrestraede 11
4200 Slagelse
Denmark
e-mail: jafari@stofanet.dk

Abstract. In this paper, we first introduce a new class of closed map called ρ-closed map. Moreover, we introduce a new class of homeomorphism called a ρ-homeomorphism. We also introduce another new class of closed map called ρ^*-closed map and introduce a new class of homeomorphism called a ρ^*-homeomorphism and prove that the set of all ρ^*-homeomorphisms forms a group under the operation of composition of maps.

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1. Introduction

The notion homeomorphism plays a very important role in topology. By definition, a homeomorphism between two topological spaces X and Y is a bijection map $f : X \to Y$ when both f and f^{-1} are continuous. In the course of generalizations of the notion of homeomorphism, Maki et al. [35] introduced g-homeomorphisms and gc-homeomorphisms in topological spaces. Devi et al. [13], [14] studied semi-generalized homeomorphisms and generalized semi-homeomorphisms and also they have introduced α-homeomorphisms in topological spaces. Jafari et al. [28] introduced $\tilde{g}$-homeomorphisms in topological spaces.

In this chapter, we first introduce ρ-closed maps in topological spaces, ρ-open maps and then we introduce and study ρ-homeomorphisms. We prove that the concepts of ρ-homeomorphism and of homeomorphism (resp. g-homeomorphism, semihomeomorphism, prehomeomorphism) are independent. We also introduce ρ^*-closed map and ρ^*-homeomorphism. It turns out that the set of all ρ^*-homeomorphisms forms a group under the operation of composition of maps.

2. Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) will always denote topological spaces on which no separation axioms are assumed,unless otherwise mentioned. When A is a subset of (X, τ), cl(A) and int(A) denote the closure and the interior of the set A, respectively.

We recall the following definitions and some results, which are used in the sequel.

Definition 2.1. Let (X, τ) be a topological space. A subset A of a space (X, τ) is called:

1. preopen [20] if $A \subseteq$ int(cl(A)) and preclosed if cl(int(A))$\subseteq A$.
2. semiopen [18] if $A \subseteq$ cl(int(A)) and semiclosed if int(cl(A))$\subseteq A$.
3. semipreopen [1] if $A \subseteq$ cl(int(cl(A))) and semipreclosed if int(cl(int(A))) $\subseteq A$.
4. regular open if A=int(cl(A)) and regular closed if A=cl(int(A)).

Definition 2.2. Let (X, τ) be a topological space. A subset A of a space (X, τ) is called:

1. generalized closed (briefly g-closed) [19] if cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
2. generalized preclosed (briefly gp-closed) [25] if pcl(A)⊆U whenever $A \subseteq U$ and U is open in (X, τ).
3. generalized preregular closed (briefly gpr-closed) [13] if pcl$(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ).
4. πgp-closed [27] if pcl$(A) \subseteq U$ whenever $A \subseteq U$ and U is π-open in X.
5. ω-closed [32] if cl(A)⊆U whenever $A \subseteq U$ and U is semi open in X.
6. $\hat{}g$-closed [33] if cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.

7. $*g$-closed [36] if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{}g$-open in X.

8. $\#g$-semi closed(briefly $\#gs$-closed) [35] if $\mathrm{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $*g$-open in X.

9. $\tilde{g}$-closed set [16] if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\#gs$-open in X.

10. ρ-closed set [6] if $\mathrm{pcl}(A) \subseteq \mathrm{Int}(U)$ whenever $A \subseteq U$ and U is $\tilde{g}$-open in (X, τ).

11. ρ_s-closed set [6] if $\mathrm{pcl}(A) \subseteq \mathrm{Int}(\mathrm{cl}(U))$ whenever $A \subseteq U$ and U is $\tilde{g}$-open in (X, τ).

12. π-open [37] if it is a finite union of regular open sets. The complement of a π-open set is said to be π-closed.

The complements of the above mentioned sets are called their respective open set.

Definition 2.3. *A function* $f : (X, \tau) \to (Y, \sigma)$ is called :

1. semi-continuous [18] if $f^{-1}(V)$ is semiopen in (X, τ) for every open set V in (Y, σ);

2. pre-continuous [20] (resp. g-continuous [4], ω-continuous [32], gp-continuous [2], gpr-continuous [12], πgp-continuous [28], $\#g$-semicontinuous [35], $\tilde{g}$-continuous [30]), if $f^{-1}(F)$ is Preclosed (resp. g-closed, ω-closed, gp-closed, gpr-closed, πgp-closed, $\#gs$-closed, $\tilde{g}$-closed) in (X, τ) for every closed set F in (Y, σ);

3. contra-continuous [9] if $f^{-1}(V)$ is closed in (X, τ) for every open set V in (Y, σ);

4. M-precontinuous [38] if $f^{-1}(V)$ is preclosed in (X, τ) for every preclosed set V in (Y, σ);

5. RC-continuous [12] if $f^{-1}(V)$ is regular closed in (X, τ) for every open set V in (Y, σ);

6. ρ-continuous [7] (resp. ρ_s-continuous [7]) if $f^{-1}(V)$ is ρ-closed (resp. ρ_s-closed) in (X, τ) for every closed set V in (Y, σ);

7. ρ-irresolute [7] (resp. $\tilde{g}$-irresolute [30]), if $f^{-1}(A)$ is ρ-closed (resp.$\tilde{g}$-closed) in (X, τ) for every ρ-closed (resp. $\tilde{g}$-closed) set A in (Y, σ);

8. contra-open [4] if $f(V)$ is closed in (Y, σ) for every open set V in (X, τ);

9. M-preclosed [22] if $f(V)$ is preclosed in (Y, σ) for every preclosed set V in (X, τ);

10. preclosed [46] (resp. ω-closed [59], g-closed [37], gp-closed [46], gpr-closed [47], πgp-closed [50], gs-closed, $\tilde{g}$-closed [27]), if $f(F_1)$ is preclosed (resp. ω-closed, g-closed, gp-closed, gpr-closed, πgp-closed, gs-closed, $\widetilde{g}$-closed), in (Y, σ) for every closed set F_1 in (X, τ).

Definition 2.4. A space (X, τ) is called: a $T_{1/2}$ space [32] (resp. T_ω space [59], $gsT^{\#}1/2$ space [71], $T\tilde{g}$-space [56], ρ-T_s space [10]), if every g-closed (resp. ω-closed, $\#g$-semi-closed, $\tilde{}g$ -closed, ρ_s-closed) set is closed in (X, τ).

Definition 2.5. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called a

1. homeomorphism if f is both open and continuous.
2. generalized homeomorphism (briefly g-homeomorphism) [35] if f is both g-open and g-continuous.
3. semi-homeomorphism [13] if f is both continuous and semi-open.
4. pre-homeomorphism [40] if f is both M-precontinuous and M-preopen.
5. gp-homeomorphism if f is both gp-continuous and gp-open.
6. gpr-homeomorphism if f is both gpr-continuous and gpr-open.
7. πgp-homeomorphism if f is both πgp-continuous and πgp-open.

Definition 2.6.

(i) Let (X, τ) be a topological space and $A \subseteq X$. We define the ρ-closure of A [10] (briefly ρ-cl(A)) to be the intersection of all ρ-closed sets containing A.

(ii) Let (X, τ) be a topological space and $A \subseteq X$. We define the ρ-interior of A [10] (briefly ρ-int(A)) to be the union of all ρ-open sets contained in A.

(iii) A topological space (X, τ) is ρ-compact [11] if every ρ-open cover of X has a finite subcover.

(iv) Let (X, τ) be a topological space. Let x be a point of (X, τ) and V be a subset of X. Then V is called a ρ-open neighbourhood (simply ρ-neighbourhood) [11] of x in (X, τ) if there exists a ρ-open set U of (X, τ) such that $x \in U \subseteq V$.

Proposition 2.7. [10] *Let (X, τ) be a topological space and $A \subseteq X$. The following properties are hold:*

(i) *If A is ρ-closed then ρ-cl$(A) = A$. The converse is not true.*

(ii) *If $A \subset B$ then ρ-cl$(A) \subset \rho$-cl(B).*

Theorem 2.8.

(i) [10] *Every open and preclosed subset of (X, τ) is ρ-closed.*

(ii) [10] *Every ρ-closed set is gp-closed (resp. gpr-closed, πgp-closed, ρ_s-closed) set.*

(iii) [11] *If $f : (X, \tau) \to (Y, \sigma)$ is $\tilde{g}$-irresolute and M-preclosed function, then $f(A)$ is ρ-closed in (Y, σ) for every ρ-closed set A of (X, τ).*

(iv) [11] *Every ρ-continuous function is gp-continuous (resp. gpr-continuous, πgp-continuous, ρ_s-continuous) function.*

(v) [11] *Every ρ-closed subset of a ρ-compact space X is ρ-compact relative to X.*

(vi) [11] *If $f : (X, \tau) \to (Y, \sigma)$ is ρ-irresolute and a subset A of X is ρ-compact relative to X then its image $f(A)$ is ρ-compact relative to Y.*

(vii) [53] *If $A \subseteq Y \subseteq X$ where A is $\tilde{g}$–open in Y and Y is $\tilde{g}$-open in X then A is $\tilde{g}$-open in X.*

3. ρ-closed maps

Definition 3.1. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be ρ-closed if the image of every closed set in (X, τ) is ρ-closed in (Y, σ).

Example 3.2.

(i) Let $X = Y = \{a, b, c, d, e\}$, $\tau = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$, $\sigma = \{\phi, \{b\}, \{d, e\}, \{b, d, e\}, \{a, c, d, e\}, Y\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = d$, $f(b) = e$, $f(c) = b$, $f(d) = c$, $f(e) = a$. Then f is a ρ-closed map.

(ii) Let $X{=}Y{=}\{a, b, c, d, e\}$, $\tau{=}\{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$, $\sigma = \{\phi, \{b\}, \{d, e\}, \{b, d, e\}, \{a, c, d, e\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is not a ρ-closed map. Since for the closed set $V = \{e\}$ in (X, τ), $f(V) = \{e\}$, which is not a ρ-closed set in (Y, σ).

Theorem 3.3. *Every Contra-closed map and Preclosed map $f : (X, \tau) \to (Y, \sigma)$ is ρ-closed map.*

Proof. Let V be a closed set in (X, τ). Then $f(V)$ is open and preclosed in (Y, σ). Hence by Theorem 2.4(i), $f(V)$ is ρ-closed in (Y, σ). Therefore f is a ρ-closed map. ∎

The converse of this theorem need not be true as seen from the following example.

Example 3.4. As in Example 3.2(i), f is a ρ-closed map but neither contra-closed map nor preclosed map. Since for the closed set $V = \{a, b, e\}$ in (X, τ), $f(V) = \{a, d, e\}$ is neither preclosed nor open in (Y, σ).

Theorem 3.5. *Every ρ-closed map $f : (X, \tau) \to (Y, \sigma)$ is a gp-closed (resp. gpr-closed, πgp-closed) map.*

Proof. Let V be a closed set in (X,τ). Then $f(V)$ is a ρ-closed set in (Y,σ). By Theorem 2.4(ii), $f(V)$ is *gp*-closed in (Y,σ) (resp. By Theorem 2.4(ii), $f(V)$ is *gpr*-closed in (Y,σ), by Theorem 2.4(ii), $f(V)$ is πgp-closed in (Y,σ)). Hence f is a *gp*-closed (resp. *gpr*-closed, πgp-closed) map. ■

The converse of this theorem need not be true as seen from the following examples.

Example 3.6.

(i) Let $X = Y = \{a,b,c,d,e\}$, $\tau = \{\phi, \{a,b\}, \{a,b,d\}, \{a,b,c,d\}, \{a,b,d,e\}, X\}$, $\sigma = \{\phi, \{b,c,d\}, \{a,b,c,d\}, \{b,c,d,e\}, Y\}$. Define $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = c$; $f(b) = e$; $f(c) = a$; $f(d) = b$; $f(e) = d$. Then the function f is a *gp*-closed map but not ρ-closed map. Since for the closed set $V = \{e\}$ in (X,τ), $f(V) = \{d\}$, is a *gp*-closed set but not a ρ-closed set in (Y,σ).

(ii) Let $X = Y = \{a,b,c,d,e\}$, $\tau = \{\phi, \{a,b\}, \{a,b,d\}, \{a,b,c,d\}, \{a,b,d,e\}, X\}$, $\sigma = \{\phi, \{a,b\}, \{c,d\}, \{a,b,c,d\}, Y\}$. Define f as in Example 3.6(i), the function f is *gpr*-closed map but not ρ-closed map. Since for all the closed sets in (X,τ), its images are all *gpr*-closed sets in (X,σ) but no one is ρ-closed set in (Y,σ).

(iii) As in Example 3.6(i), define $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = c$; $f(b) = b$; $f(c) = a$; $f(d) = e$; $f(e) = d$. Then the function f is a πgp-closed map but not ρ-closed map. Since for the closed set $V = \{a,d\}$ is πgp-closed set but not a ρ-closed set in (Y,σ).

The following examples show that the concepts of closed map and of ρ-closed map are independent.

Example 3.7.

(i) As in Example 3.2(i), f is a ρ-closed map but not a closed map. Since for the closed set $v = \{e\}$ in (X,τ), $f(V) = \{a\}$ is ρ-closed but not closed in (Y,σ).

(ii) Let τ be the collection of subsets of N consisting of ϕ and all subsets of N of the form $\{n, n+1, n+2, ...\}$, $n \in N$. Then τ is a topology for N. Let (Z,κ) be the digital topology. Let $f : (N,\tau) \to (Z,\kappa)$ be the constant map defined by $f(x) = \{4\}$ for each $x \in N$. The image of each closed set in (N,τ) is $\{4\}$ which is closed in (X,κ), but not ρ-closed in (X,κ), because there is a $\tilde{g}$-open set $U = \{1,2,3,4\}$ containing $\{4\}$, is not open in (X,κ) such that $\mathrm{pcl}(\{4\}) = \{4\} \nsubseteq \mathrm{int}(U) = \{1,2,3\}$. Therefore f is a closed map but not a ρ-closed map.

The following examples show that the concepts of g-closed map and of ρ-closed map are independent.

Example 3.8.

(i) Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a, c\}, X\}$, $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$; $f(b) = b$; $f(c) = a$. Then f is a ρ-closed map but not g-closed map. since for the closed set $V = \{b\}$ in (X, τ), $f(V) = \{b\}$ is ρ-closed but not g-closed in (Y, σ).

(ii) As in Example 3.7(ii), f is a g-closed map but not a ρ-closed map.

The following example shows that the composition of two ρ-closed maps need not be ρ-closed.

Example 3.9. Let $X=Y=Z=\{a, b, c\}$, $\tau=\{\phi, \{a\}, \{b, c\}, X\}$, $\sigma=\{\phi, \{a, c\}, Y\}$, $\eta=\{\phi, \{a\}, \{a, b\}, Z\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = f(b) = b$; $f(c) = a$ and define $g : (Y, \sigma) \to (Z, \eta)$ by $g(a) = c$; $g(b) = b$; $g(c) = a$. Then both f and g are ρ-closed maps but their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is not a ρ-closed map, since for the closed set $V = \{b, c\}$ in (X, τ), $g \circ f(V) = \{a, b\}$, which is not a ρ-closed set in (Z, η).

Theorem 3.10. *If $f : (X, \tau) \to (Y, \sigma)$ is ρ-closed, $g : (Y, \sigma) \to (Z, \eta)$ is ρ-closed and (Y, σ) is a ρ-$T_s$space then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is ρ-closed.*

Proof. Let V be a closed set in (X, τ). Then $f(V)$ is a ρ-closed set in (Y, σ). Since (Y, σ) is ρ-T_s and by Theorem 2.4(ii), $f(V)$ is a closed set in (Y, σ). Hence $g(f(V)) = (g \circ f)(V)$ is a ρ-closed in (Z, η). Therefore $g \circ f$ is a ρ-closed map. ∎

Theorem 3.11. *If $f : (X, \tau) \to (Y, \sigma)$ is a $\tilde{g}$-closed (resp. g-closed, ω-closed, gs-closed) map, $g : (Y, \sigma) \to (Z, \eta)$ is a ρ-closed map and Y is $T\tilde{g}$-space (resp. $T_{1/2}$ space, T_ω space, $gsT^{\#}_{1/2}$ space) then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is a ρ-closed map.*

Proof. Let V be a closed set in (X, τ). Then $f(V)$ is a $\tilde{g}$-closed (resp. g-closed, ω-closed, gs-closed) set in(Y, σ). Since (Y, σ) is a $T\tilde{g}$-space (resp. $T_{1/2}$ space, T_ω space, $gsT^{\#}_{1/2}$ space), therefore $f(V)$ is a closed set in (Y, σ). Since g is ρ-closed, $g(f(V)) = (g \circ f)(V)$ is ρ-closed in (Z, η). Therefore $g \circ f$ is a ρ-closed map. ∎

Theorem 3.12. *If $f : (X, \tau) \to (Y, \sigma)$ is a $\tilde{g}$-closed and contra-closed map, $g : (Y, \sigma) \to (Z, \eta)$ is an M-preclosed and open map then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is ρ-closed map.*

Proof. Let V be a closed set in (X, τ). Then $f(V)$ is $\tilde{g}$-closed and open in (Y, σ). Since every $\tilde{g}$-closed is preclosed and g is M-preclosed and open, hence $g(f(V)) = (g \circ f)(V)$ is preclosed and open in (Z, η). By Theorem 2.4(i), $(g \circ f)(V)$ is ρ-closed in (Z, η). Therefore $g \circ f$ is a ρ-closed map. ∎

Theorem 3.13. *Let $f : (X, \tau) \to (Y, \sigma)$ be a closed map and $g : (Y, \sigma) \to (Z, \eta)$ be a ρ-closed map then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is ρ-closed.*

Proof. Let V be a closed set in (X,τ). Then $f(V)$ is a closed set in (Y,σ). Hence $g(f(V)) = (g\circ f)(V)$ is ρ-closed set in (Z,η). Therefore $g\circ f$ is a ρ-closed map. ∎

If f is ρ-closed map and g is closed, then their composition need not be a ρ-closed map as seen from the following example.

Example 3.14. Let $X{=}Y{=}Z{=}\{a,b,c\}$, $\tau{=}\{\phi,\{a\},\{b,c\},X\}$, $\sigma{=}\{\phi,\{a,c\},Y\}$, $\eta{=}\{\phi,\{c\},\{a,c\},Z\}$. Define $f:(X,\tau)\to(Y,\sigma)$ be $f(a)=f(b)=c$; $f(c)=b$ and define $g:(Y,\sigma)\to(Z,\eta)$ be the identity map. Then f is a ρ-closed map and g is a closed map. But their composition $g\circ f:(X,\tau)\to(Z,\eta)$ is not a ρ-closed map. Since for the closed set $V=\{a\}$ in (X,τ), $(g\circ f)(V) = g(f(V)) = g(c) = \{c\}$, which is not is ρ-closed set in (Z,η).

Theorem 3.15. *If $f:(X,\tau)\to(Y,\sigma)$ is a ρ-closed, $g:(Y,\sigma)\to(Z,\eta)$ is M-preclosed and $\tilde{g}$-irresolute map then $g\circ f:(X,\tau)\to(Z,\eta)$ is ρ-closed.*

Proof. Let V be a closed set in (X,τ). Then $f(V)$ is a ρ-closed set in (Y,σ). Hence by Theorem 2.4(iii), $g(f(V)) = (g\circ f)(V)$ ρ-closed in (Z,η). Therefore $g\circ f$ is a ρ-closed map. ∎

Theorem 3.16. *Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two mappings such that their composition $g\circ f:(X,\tau)\to(Z,\eta)$ be a ρ-closed mapping. Then the following statements are true if:*

1. *f is continuous and surjective then g is ρ-closed.*
2. *g is ρ-irresolute, injective then f is ρ-closed.*
3. *f is ˜g-continuous, surjective and (X,τ) is a $T\tilde{g}$-space, then g is ρ-closed.*
4. *f is g-continuous, surjective and (X,τ) is a $T_{1/2}$ space then g is ρ-closed.*
5. *f is ρ-continuous, surjective and (X,τ) is a ρ-T_s space then g is ρ-closed.*

Proof. 1. Let A be a closed set in (Y,σ). Since f is continuous, $f^{-1}(A)$ is closed in (X,τ) and since $g\circ f$ is ρ-closed, $(g\circ f)(f^{-1}(A)) = g(A)$ is a ρ-closed in (Z,η), since f is surjective. Therefore, g is a ρ-closed map.

2. Let A be a closed set in (X,τ). Since $g\circ f$ is ρ-closed, then $(g\circ f)(A)$ is ρ-closed in (Z,η). Since g is ρ-irresolute, then $g^{-1}(g\circ f)(A)$ is ρ-closed in (Y,σ), since g is injective. Thus, f is a ρ-closed map.

3. Let A be a closed set of (Y,σ). Since f is $\tilde{g}$-continuous, $f^{-1}(A)$ is $\tilde{g}$-closed in (X,τ). Since (X,τ) is a $T\tilde{g}$-space, $f^{-1}(A)$ is closed in (X,τ), since $g\circ f$ is ρ-closed, $(g\circ f)(f^{-1}(A)) = g(A)$ is ρ-closed in (Z,η), since f is surjective. Thus g is a ρ-closed map.

4. Let A be a closed set of (Y,σ). Since f is g-continuous, $f^{-1}(A)$ is g-closed in (X,τ). Since (X,τ) is a $T_{1/2}$-space, $f^{-1}(A)$ is closed in (X,τ), since $g\circ f$ is ρ-closed, $(g\circ f)(f^{-1}(A)) = g(A)$ is ρ-closed in (Z,η), since f is surjective. Thus g is a ρ-closed map.

5. Let A be a closed set (Y, σ). Since f is ρ-continuous, $f^{-1}(A)$ is ρ-closed in (X, τ). Since (X, τ) is a ρ-T_s space and by Theorem 2.4(ii), $f^{-1}(A)$ is closed in (X, τ). Since $g \circ f$ is ρ-closed, $(g \circ f)\left(f^{-1}(A)\right) = g(A)$ is ρ-closed in (Z, η). Since f is surjective. Thus, g is a ρ-closed map. ∎

As for the restriction f_A of a map $f : (X, \tau) \to (Y, \sigma)$ to a subset A of (X, τ), we have the following.

Theorem 3.17. *Let (X, τ) and (Y, σ) be any topological spaces, Then if:*

1. *$f : (X, \tau) \to (Y, \sigma)$ is ρ-closed and A is a closed subset of (X, τ) then $f_A : (A, \tau_A) \to (Y, \sigma)$ is ρ-closed.*
2. *$f : (X, \tau) \to (Y, \sigma)$ is ρ-closed and $A = f^{-1}(B)$, for some closed set B of (Y, σ), then $f_A : (A, \tau_A) \to (Y, \sigma)$ is ρ-closed.*

Proof. 1. Let B be a closed set of (A, τ_A). Then $B = A \cap F$ for some closed set F of (X, τ) and so B is closed in (X, τ). Since f is ρ-closed, then $f(B)$ is ρ-closed in (Y, σ). But $f(B) = f_A(B)$ and therefore f_A is a ρ-closed map.

2. Let F be a closed set of (A, τ_A). Then $F = A \cap H$ for some closed set H of (X, τ).Now $f_A(F) = f(F) = f(A \cap H) = f(f^{-1}(B) \cap H) = B \cap f(H)$. Since f is ρ-closed, $f(H)$ is ρ-closed in (Y, σ) and so $B \cap f(H)$ is ρ-closed in (Y, σ). Therefore f_A is a ρ-closed map. ∎

Theorem 3.18. *A map $f : (X, \tau) \to (Y, \sigma)$ is ρ-closed if and only if for each subset S of (Y, σ) and for each open set U containing $f^{-1}(S)$ there is a ρ-open set V of (Y, σ) such that $S \subset V$ and $f^{-1}(V) \subset U$.*

Proof. Suppose that f is a ρ-closed map. Let $S \subset Y$ and U be an open subset of (X, τ) such that $f^{-1}(S) \subset U$. Then $V = (f(U^c))^c$ is a ρ-open set containing S such that $f^{-1}(V) \subset U$. For the converse. Let S be a closed set of (X, τ). Then $f^{-1}\left((f(s))^c\right) \subset S^c$ and S^c is open. By assumption, there exists a ρ-open set V of (Y, σ) such that $(f(S)^c) \subset V$ and $f^{-1}(V) \subset S^c$ and so $S \subset \left(f^{-1}(V)\right)^c$. Hence $V^c \subset f(S) \subset f\left(f^{-1}(V)^c\right) \subset V^c$ which implies $f(S) = V^c$. Since V^c is ρ-closed in (Y, σ), $f(S)$ is ρ-closed in (Y, σ) and therefore f is ρ-closed. ∎

Theorem 3.19. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is ρ-closed then ρ-$cl(f(A)) \subseteq f(cl(A))$ for every subset A of (X, τ).*

Proof. Suppose that f is ρ-closed and $A \subseteq X$, then $f(\mathrm{cl}(A))$ is ρ-closed in (Y, σ). Hence by Proposition 2.7(i), ρ-$\mathrm{cl}(f(\mathrm{cl}(A))) = f(\mathrm{cl}(A))$. Also $(A) \subseteq f(cl(A))$, and by Proposition 2.7(ii), we have, ρ-$\mathrm{cl}(f(A)) \subseteq \rho$-$\mathrm{cl}(f(\mathrm{cl}(A)) = f(\mathrm{cl}(A))$. ∎

The converse of this theorem need not be true as seen from the following example.

Example 3.20. Let $X = \{a, b, c\} = Y$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$, $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$; $f(b) = a$; $f(c) = b$. For every subset A of X, we have ρ-$\mathrm{cl}(f(A)) \subseteq f(\mathrm{cl}(A))$. But f is not a ρ-closed map. Since for the closed set $V = \{b, c\}$ in (X, τ), $f(V) = \{a, b\}$ is not a ρ-closed set in (Y, σ).

4. ρ-Open maps

Definition 4.1. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be ρ-open map if the image $f(A)$ is ρ-open in (Y, σ) for every open set A in (X, τ).

Theorem 4.2. *For any bijection $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent.*

1. $f^{-1} : (Y, \sigma) \to (X, \tau)$ *is ρ-continuous*
2. f *is a ρ-open map and*
3. f *is a ρ-closed map.*

Proof. (1)→(2). Let U be an open set of (X, τ). By assumption, $\left(f^{-1}\right)^{-1}(U) = f(U)$ is ρ-open in (Y, σ) and so f is a ρ-open map.

(2)→(3). Let V be a closed set of (X, τ). Then V^c is open in (X, τ). By assumption $f(V^c) = (f(V))^c$ is ρ-open in (Y, σ) and therefore $f(V)$ is ρ-closed in (Y, σ). Hence f is a ρ-closed map.

(3)→(1) Let V be a closed set of (X, τ). By assumption $f(V)$ is ρ-closed in (Y, σ). But $f(V) = \left(f^{-1}\right)^{-1}(V)$ and therefore f^{-1} is ρ-continuous on (Y, σ). ∎

Theorem 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be mapping. If f is a ρ-open mapping, then for each $x \epsilon X$ and for each neighbourhood U of x in (X, τ), there exists a ρ-neighbourhood W of $f(x)$ in (Y, σ) such that $W \subset f(U)$.*

Proof. Let $x \epsilon X$ and U be an arbitrary neighbourhood of x. Then there exists an open set V in (X, τ) such that $x \epsilon V \subseteq U$. By assumption, $f(V)$ is a ρ-open set in (Y, σ). Further, $f(x) \epsilon f(V) \subseteq f(U)$, clearly $f(U)$ is a ρ-neighbourhood of f(x) in (Y, σ) and so the theorem holds, by taking $W = f(V)$. ∎

The converse of this theorem need not be true as seen from the following example.

Example 4.4. As in example 4.4, Let $U = \{a, b, c, d\}$ be an open set in (X, τ) and $f(a) = a$. Then $a \epsilon U$ and for each $a = f(a) \epsilon f(U) = \{a, c, d, e\}$, by assumption, there exists a ρ-neighbourhood $W_a = \{a, c, d, e\}$ of a in (Y, σ) such that $W_a \subseteq f(U)$. But $f(U)$ is not a ρ-open set in (Y, σ).

Theorem 4.5. *A function $f : (X, \tau) \to (Y, \sigma)$ is ρ-open if and only if for any subset B of (Y, σ) and for any closed set S containing $f^{-1}(B)$, there exists a ρ-closed set A of (Y, σ) containing B such that $f^{-1}(A) \subseteq S$.*

Proof. Similar to Theorem 3.18. ∎

5. ρ-Homeomorphisms

Definition 5.1. A bijection $f:(X,\tau)\to(Y,\sigma)$ is called ρ-homeomorphism if f is both ρ-continuous and ρ-open.

Example 5.2. Let $X{=}Y{=}\{a,b,c\}$, $\tau{=}\{\phi,\{a\},\{a,b\},X\}$, $\sigma{=}\{\phi,\{b\},\{b,c\},Y\}$. Define a map $f:(X,\tau)\to(Y,\sigma)$ by $f(a)=c,\ f(b)=b,\ f(c)=a$. Then f is a ρ-homeomorphism.

Theorem 5.3. *Every ρ-homeomorphism is a gp-homeomorphism (resp. gpr-homeomorphism, πgp-homeomorphism).*

Proof. By Theorem 2.4(iv), every ρ-continuous map is gp-continuous (resp. *gpr*-continuous,πgp-continuous) and also by Theorem 2.4(ii), every ρ-open map is *gp*-open (resp. *gpr*-open,πgp-open), the proof follows. ∎

The converse of the above theorem need not be true as seen from the following example.

Example 5.4.

(i) Let $X=Y=\{a,b,c,d,e\}$, $\tau=\{\phi,\{b,c,d\},\{a,b,c,d\},\{b,c,d,e\},X\}$ and $\sigma=\{\phi,\{a,b,e\},X\}$. Define $f:(X,\tau)\to(X,\sigma)$ by $f(a)=e,\ f(b)=c,\ f(c)=d,\ f(d)=a,\ f(e)=b$. Then f is *gp*-homeomorphism but not ρ-homeomorphism. Observe that for the closed set $V=\{c,d\}$ in (X,σ), $f^{-1}(V)=\{b,c\}$ is *gp*-closed but not ρ-closed in (X,τ), that is f is not ρ-continuous.

(ii) Let X=Y={a,b,c}, τ ={ϕ,{a},{a,b},{c,a},X}, $\sigma=\{\phi$,{b},{c,a},Y}. Let f :$(X,\tau)\to(Y,\sigma)$ be an identity map. Then f is πgp-homeomorphism but not ρ-homeomorphism. Since for the closed set $V=\{c.a\}$ in (Y,σ), f$^{-1}(V)$ $=\{c,a\}$ is not ρ-closed in (X,τ), that is f is not ρ-continuous.

(iii) Let $X=\{a,b,c,d,e\}, Y=\{a,b,c,d\}, \tau=\{\phi,\{c\},\{e\},\{a,b\},\{c,e\},\{a,b,c\},$ $\{a,b,e\},\{a,b,c,e\},X\}$, $\sigma=\{\phi,\{c.d\},Y\}$. Define $f:(X,\tau)\to(Y,\sigma)$ by $f(a)=f(c)=a,\ f(b)=f(e)=b,\ f(d)=c$. Then f is *gpr*-homeomorphism but not ρ-homeomorphism. Since for the closed set $V=\{a,b\}$ in (Y,σ), $f^{-1}(V)=\{a,b,c,e\}$ is not ρ-closed in (X,τ), that is f is not ρ-continuous.

Theorem 5.5. *Let $f:(X,\tau)\to(Y,\sigma)$ be both a contra-open and contra-continuous function; further let f be a gp-homeomorphism.Then f is a ρ-homeomorphism.*

Proof. Let U be open in (X,τ). Then $f(U)$ is *gp*-open in (Y,σ). Hence $Y-f(U)$ is *gp*-closed in (Y,σ). Since f is contra-open, then $f(U)$ is closed in (Y,σ) and so $Y-f(U)$ is open in (Y,σ). By Theorem 2.2 [29], $Y-f(U)$ is preclosed in (Y,σ) and by Theorem 2.4(i), $Y-f(U)$ is ρ-closed in (Y,σ), that is $f(U)$ is ρ-open in (Y,σ). Hence f is ρ-open. Let V be closed in (Y,σ). Then $f^{-1}(V)$ is *gp*-closed in (X,τ). Since f is contra-continuous, then $f^{-1}(V)$ is open in (X,τ). By Theorem 2.2 [29] and by Theorem 2.4(i), $f^{-1}(V)$ is ρ-closed in (X,τ). Hence f is ρ-continuous. Since f is ρ-continuous and ρ-open, therefore f is ρ-homeomorphism. ∎

Definition 5.6. A function $f : (X, \tau) \to (Y, \sigma)$ is called

1. contra-π-open (resp.regular-contra-open), if f(U) is π-closed (resp. regular closed) in (Y, σ) for every open set U in (X, τ).
2. contra-π-continuous, if $f^{-1}(V)$ is π-open in (X, τ) for every closed set V in (Y, σ).

Theorem 5.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be both a contra-π-open and contra-π-continuous function; further, let f be a πgp-homeomorphism. Then f is a ρ-homeomorphism.*

Proof. Let U be open in (X, τ). Then $f(U)$ is πgp-open in (Y, σ). Hence $Y - f(U)$ is πgp-closed in (Y, σ). Since f is contra-π-open, then $f(U)$ is π-closed in (Y, σ) and so $Y - f(U)$ is π-open in (Y, σ). By Theorem 2.4 [53], $Y - f(U)$ is preclosed in (Y, σ) and since every π-open is open and by Theorem 2.4(i), $Y - f(U)$ is ρ-closed in (Y, σ), that is $f(U)$ is ρ-open in (Y, σ). Hence f is ρ-open. Let V be closed in (Y, σ). Then $f^{-1}(V)$ is πgp-closed in (X, τ). Since f is contra-π-continuous, then $f^{-1}(V)$ is π-open in (X, τ). By Theorem 2.4 [53] and since every π-open is open and by Theorem 2.4(i), $f^{-1}(V)$ is ρ-closed in (X, τ). Hence f is ρ-continuous. Since f is ρ-continuous and ρ-open, therefore f is ρ-homeomorphism. ∎

Theorem 5.8. *Let $f : (X, \tau) \to (Y, \sigma)$ be both a contra-regular open and RC-continuous function; further, let f be a gpr-homeomorphism. Then f is a ρ-homeomorphism.*

Proof. Let U be open in (X, τ). Then $f(U)$ is *gpr*-open in (Y, σ). Hence $Y - f(U)$ is *gpr*-closed in (Y, σ). Since f is contra-regular open, then $f(U)$ is regular closed in (Y, σ) and so $Y - f(U)$ is regular open in (Y, σ). By Theorem 3.10 [22], $Y - f(U)$ is preclosed in (Y, σ) and since every regular open is open and by Theorem 2.4(i), $Y - f(U)$ is ρ-closed in (Y, σ), that is $f(U)$ is ρ-open in (Y, σ). Hence f is ρ-open. Let V be closed in (Y, σ). Then $f^{-1}(V)$ is *gpr*-closed in (X, τ). Since f is completely contra-continuous, then $f^{-1}(V)$ is regular open in (X, τ). By Theorem 3.10 [22] and since every regular open is open and by Theorem 2.4(i), $f^{-1}(V)$ is ρ-closed in (X, τ). Hence f is ρ-continuous. Since f is ρ-continuous and ρ-open, therefore f is ρ-homeomorphism. ∎

Theorem 5.9. *Let $f : (X, \tau) \to (Y, \sigma)$ be both a contra-open and contra-continuous function; let f be a pre-homeomorphism. Then f is a ρ-homeomorphism.*

Proof. Let U be open in (X, τ). Then f(U) is preopen in (Y, σ). Hence Y-f(U) is preclosed in (Y, σ). Since f is contra-open, then f(U) is closed in (Y, σ) and so Y-f(U) is open in (Y, σ). By Theorem 2.4(i), Y-f(U) is ρ-closed in (Y, σ), that is f(U) is ρ-open in (Y, σ). Hence f is ρ-open. Let V be closed in (Y, σ). Then $f^{-1}(V)$ is preclosed in (X, τ). Since f is contra-continuous, then $f^{-1}(V)$ is open in (X, τ). By Theorem 2.4(i), $f^{-1}(V)$is ρ-closed in (X, τ). Hence f is ρ-continuous. Since f is ρ-continuous and ρ-open, therefore f is ρ-homeomorphism. ∎

The concepts of ρ-homeomorphism and of homeomorphism are independent as can be seen from the following examples.

Example 5.10.

(i) Let X = {a,b,c} = Y, $\tau = \{\phi, \{a,b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a,b\}, Y\}$. The identity map f on X is ρ-homeomorphism but not a homeomorphism, because it is not continuous.

(ii) Let (R,τ) be a usual topology and ((0,1),τ^*), where τ^* denotes that relativized usual topology on (0,1). Let $f : (R, \tau) \to ((0,1), \tau^*)$ be a map defined by f(x) = 2x-1/x(x-1). Then f is a homeomorphism but not a ρ-homeomorphism, because it is not ρ-continuous.

The concepts of ρ-homeomorphism and of g-homeomorphism are independent as can be seen from the following examples.

Example 5.11.

(i) Let $X = \{a,b,c\} = Y$, $\tau = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$, $\sigma = \{\phi, \{b\}, \{a,b\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b,\ f(b) = a,\ f(c) = c$. Then f is a ρ-homeomorphism but not g-homeomorphism. Since for the open set $V = \{a,c\}$ in (X,τ), $f(V) = \{b,c\}$ is not g-open in (Y,σ).

(ii) As in Example 5.10(ii), f is a g-homeomorphism but not ρ-homeomorphism, because it is not ρ-continuous.

The concepts of ρ-homeomorphism and of semi-homeomorphism are independent as can be seen from the following examples.

Example 5.12.

(i) Let $X = Y = \{a,b,c\}$,$\tau = \{\phi, \{a,b\}, X\}$, $\sigma = \{\phi, \{a\}, \{a,b\}, Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be an identity map. Then f is a ρ-homeomorphism. But f is not a semi-homeomorphism. Since for the closed set V=$\{b,c\}$ in (Y,σ), f^{-1}(V)=$\{b,c\}$, which is not closed in (X,τ). Therefore f is not a continuous map.

(ii) Let $X = Y = \{a,b,c\}$, $\tau = \{\phi, \{a\}, \{b,c\}, X\}$, $\sigma = \{\phi, \{c\}, \{a,b\}, Y\}$. Define $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = c,\ f(b) = a,\ f(c) = b$. Then f is a semihomeomorphism. But f is not a ρ-homeomorphism. Since for the closed set V=$\{c\}$ in (Y,σ), f^{-1}(V)=$\{a\}$, which is not ρ-closed in (X,τ). Therefore f is not a ρ-continuous map.

The concepts of ρ-homeomorphism and of pre-homeomorphism are independent as can be seen from the following examples.

Example 5.13.

(i) Let $X = Y = \{a,b,c\}$, $\tau = \{\phi, \{a\}, \{a,b\}, X\}$, $\sigma = \{\phi, \{a\}, \{b\}, \{a,b\}, \{b,c\}, Y\}$. Define $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b,\ f(b) = a,\ f(c) = c$. Then f is a ρ-homeomorphism. But f is not a pre-homeomorphism. Since for the closed set $V = \{b,c\}$ in (Y,σ), $f^{-1}(V) = \{c,a\}$, which is not preclosed in (X,τ). Therefore f is not a pre-continuous map.

(ii) Let $X = \{a, b, c, d, e\}$ with the topology $\tau = \{\phi, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$ and $Y = \{a, b\}$ with the discrete topology σ. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(c) = f(d) = a$, $f(a) = f(b) = f(e) = b$. Then f is pre-homeomorphism but not ρ-homeomorphism. Since for the closed set $V = \{a\}$ in (Y, σ), $f^{-1}(V) = \{c, d\}$ is not ρ-closed in (X, τ), that is f is not ρ-continuous.

Theorem 5.14. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection ρ-continuous map. Then the following statements are equivalent.*

1. *f is a ρ-open map.*
2. *f is a ρ-homeomorphism.*
3. *f is a ρ-closed map.*

Proof. (1)→(2) By hypothesis and by assumption, proof is obvious.

(2)→(3) Let V be a closed set in (X, τ). Then V^c is open in (X, τ). By hypothesis, $f(V^c) = (f(V))^c$ is ρ-open in (Y, σ). That is, $f(V)$ is ρ-closed in (Y, σ). Therefore f is a ρ-closed map.

(3)→(1) Let V be a open set in (X, τ). Then V^c is closed in (X, τ). By hypothesis, $f(V^c) = (f(V))^c$ is ρ-closed in (Y, σ). That is, $f(V)$ is ρ-open in (Y, σ). Therefore f is a ρ-open map. ■

The composition of two ρ-homeomorphism maps need not be a ρ- homeomorphism as can be seen from the following example.

Example 5.15. Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\phi, \{a, b\}, X\}$, $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$, $\eta = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Z\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be an identity map and define $g : (Y, \sigma) \to (Z, \eta)$ by $g(a) = b,\ g(b) = a,\ g(c) = c$. Then both f and g are ρ-homeomorphisms, but their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is not a ρ-homeomorphism. Since for the closed set $V=\{a\}$ in (Z, η), $(g \circ f)^{-1}(V) = \{b\}$, which is not a ρ-closed set in (X, τ). Therefore $g \circ f$ is not a ρ-continuous map and so $g \circ f$ is not a ρ-homeomorphism.

Theorem 5.16. *Let $f : (X, \tau) \to (Y, \sigma)$ be a ρ-homeomorphism. Let A be an open ρ-closed subset of X and let B be a closed subset of Y such that $f(A)=B$. Assume that $\rho C(X, \tau)$ (the class of all ρ-closed sets of (X, τ)) be closed under finite intersections. Then the restriction $f_A : (A, \tau_A) \to (B, \sigma_B)$ is a ρ-homeomorphism.*

Proof. We have to show that f_A is a bijection, f_A is a ρ-open map and f_A is a ρ-continuous map.

(i) Since f is one-one, f_A is also one-one. Also since $f(A) = B$ we have $f_A(A) = B$ so that f_A is onto and hence f_A is a bijection.

(ii) Let U be an open set of (A, τ_A). Then $U = A \bigcap H$, for some open set H in (X, τ). Since f is one-one, then $f(U) = f(A \bigcap H) = f(A) \bigcap f(H) = B \bigcap f(H)$. Since f is ρ-open and H is an open set in (X, τ), then $f(H)$ is a ρ-open set in (Y, σ). Therefore $f(U)$ is a ρ-open set in (B, σ_B). Hence f_A is a ρ-open map.

(iii) Let V be a closed set in (B, σ_B). Then $V = B \bigcap K$, for some closed set K in (Y, σ). Since B is a closed set in (Y, σ), then V is a closed set in (Y, σ). By hypothesis and assumption, $f^{-1}(V) \bigcap A = H_1$ (say) is a ρ-closed set in (X, τ). Since $f_A^{-1}(V) = H_1$, it is sufficient to show that H_1 is a ρ-closed set in (A, τ_A). Let G_1 be $\tilde{g}$-open in (A, τ_A) such that $H_1 \subseteq G_1$. Then by hypothesis and by Lemma 2.4(vii), G_1 is $\tilde{g}$-open in X. Since H_1 is a ρ-closed set in (X, τ), we have $\mathrm{Pcl}_X(H_1) \subseteq \mathrm{Int}(G_1)$. Since A is open and by Lemma 2.10 [23], $\mathrm{Pcl}_A(H_1) = \mathrm{Pcl}_X(H_1) \bigcap A \subseteq \mathrm{Int}(G_1) \bigcap A = \mathrm{Int}(G_1) \bigcap \mathrm{Int}(A) = \mathrm{Int}(G_1 \bigcap A) \subseteq \mathrm{Int}(G_1)$ and so $H_1 = f_A^{-1}(V)$ is ρ-closed set in (A, τ_A). Therefore f_A is a ρ-continuous map. Hence f_A is a ρ-homeomorphism. ∎

Definition 5.17. A topological space (X, τ) is called a ρ-Hausdorff if for each pair x, y of distinct points of X, there exists ρ-open neighbourhoods U_1 and U_2 of x and y, respectively, that are disjoint.

Theorem 5.18. *Let (X, τ) be a topological space and let (Y, σ) be a ρ-Hausdorff space. Let $f : (X, \tau) \to (Y, \sigma)$ be a one-one ρ-irresolute map. Then (X, τ) is also a ρ-Hausdorff space.*

Proof. Let x_1, x_2 be any two distinct points of X. Since f is one-one, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. Let $y_1 = f(x_1)$, $y_2 = f(x_2)$ so that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Then $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since (Y, σ) is ρ-Hausdorff, then there exists ρ-open sets U_1 and U_2 of (Y, σ) such that $y_1 \in U_1$, $y_2 \in U_2$ and $U_1 \cap U_2 = \phi$. Since f is ρ-irresolute, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are ρ-open sets of (X, τ). Now $f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2) = f^{-1}(\phi) = \phi$, and $y_1 \in U_1$ implies $f^{-1}(y_1) \in f^{-1}(U_1)$ implies $x_1 \in f^{-1}(U_1)$, $y_2 \in U_2$ implies $f^{-1}(y_2) \in f^{-1}(U_2)$ implies $x_2 \in f^{-1}(U_2)$. Thus it is shown that for every pair of distinct points x_1, x_2 of X, there exists disjoint ρ-open sets $f^{-1}(U_1)$ and $f^{-1}(U_2)$ such that $x_1 \in f^{-1}(U_1)$ and $x_2 \in f^{-1}(U_2)$. Accordingly, the space (X, τ) is a ρ-Hausdorff space. ∎

Theorem 5.19. *Every ρ-compact subset A of a ρ-Hausdorff space X is ρ-closed. Assume that $\rho O(X, \tau)$ (the class of all ρ-open sets of (X, τ)) be closed under finite intersections.*

Proof. We shall show that $X - A$ is ρ-open. Let $x \in X - A$. Since X is ρ-Hausdorff, for every $y \in A$, there exists disjoint ρ-open neighbourhoods U_y and V_y of x and y such that $U_y \bigcap V_y = \phi$. Now the collection $\{V_y / y \in A\}$ is a ρ-open cover of A, since A is compact, there exists a finite subcover $\{y_i,\ i = 1, ..., n\}$ such that $A \subset \cup\{V_{y_i},\ i = 1, ..., n\}$.

Let $U = \cap\{U_{y_i},\ i = 1, ..., n\}$ and $V = \cup\{V_{y_i},\ i = 1, ..., n\}$. Then, by assumption, U is an ρ-open neighbourhood of x. Clearly, $U \cap V = \phi$, hence $U \cap A = \phi$, thus $U \subset X - A$, which means $X - A$ is ρ-open, therefore A is ρ-closed. ∎

Theorem 5.20. *Let (X, τ) a topological space and let (Y, σ) be a ρ-Hausdorff space. Assume that $\rho O(X, \tau)$ (the class of all ρ-open sets of (X, τ)) be closed under finite intersections. If f, g are ρ-irresolute maps of X into Y, then the set $A = \{x \in X : f(x) = g(x)\}$ is a ρ-closed subset of (X, τ).*

Proof. We shall show that $X-A$ is an ρ-open subset of (X,τ). Now $X-A=\{x\in X : f(x)\neq g(x)\}$. Let $p\in X-A$. Set $y_1=f(p)$, $y_2=g(p)$. By the definition of $X-A$, we have $y_1\neq y_2$. Thus y_1,y_2 are two distinct points of Y. Since (Y,σ) is a ρ-Hausdorff space, there exists ρ-open sets U_1,U_2 of (Y,σ) such that $y_1=f(p)\in U_1$, $y_2=g(p)\in U_2$ and $U_1\cap U_2=\phi$. Therefore $p\in f^{-1}(U_1)$, $p\in g^{-1}(U_2)$, so that $p\in f^{-1}(U_1)\cap g^{-1}(U_2)=W$(say). Since f and g are ρ-irresolute maps, $f^{-1}(U_1)$ and $g^{-1}(U_2)$ are ρ-open sets of (X,τ) and by assumption W is a ρ-open set containing p. We will now show that $W\subset X-A$. Let $y\in W$, since $U_1\cap U_2=\phi$, then $f(y)\neq g(y)$ and hence from the definition of $X-A$, $y\in X-A$. Therefore $W\subset X-A$, which means $X-A$ is a ρ-open set. It follows that A is a ρ-closed subset of (X,τ). ∎

We define another new class of maps called ρ^*-closed maps.

Definition 5.21. A map $f:(X,\tau)\to(Y,\sigma)$ is said to be a ρ^*-closed map if the image $f(A)$ is ρ-closed in (Y,σ) for every ρ-closed set A in (X,τ).

Example 5.22. As in Example 3.2(i), f is a ρ^*-closed map.

Theorem 5.23. *If $f:(X,\tau)\to(Y,\sigma)$ is $\tilde{g}$-irresolute and M-preclosed functions then f is a ρ^*-closed map.*

Proof. By Theorem 2.4(iii), the theorem follows. ∎

Theorem 5.24. *Every ρ-closed map is a ρ^*-closed map if (X,τ) is ρ-T_S space.*

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a ρ-closed map and V be a ρ-closed set in (X,τ). Since (X,τ) is a ρ-T_S space and by Theorem 2.4(ii), V is a closed set in (X,τ) and since f is ρ-closed, then $f(V)$ is a ρ-closed set in (Y,σ). Hence f is a ρ^*-closed map. ∎

We next introduce a new class of maps called ρ^*-homeomorphisms. This class of maps is closed under composition of maps.

Definition 5.25. A bijection $f:(X,\tau)\to(Y,\sigma)$ is said to be ρ^*- homeomorphism if both f and f^{-1} are ρ-irresolute.

Example 5.26. Let $X=Y=\{a,b,c\}$, $\tau=\{\phi,\{b\},\{c\},\{a,b\},\{b,c\},X\}$, $\sigma=\{\phi,\{b\},Y\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by $f(a)=f(b)=a, f(c)=c$. Then f is a ρ^*-homeomorphism.

Theorem 5.27. *A bijective ρ-irresolute map of a ρ-compact space X onto a ρ-Hausdorff space Y is a ρ^*-homeomorphism.*

Proof. Let (X,τ) be a ρ-compact space and (Y,σ) be a ρ-Hausdorff space. Let $f:(X,\tau)\to(Y,\sigma)$ be a bijective ρ-irresolute map. We have to show that f is a ρ^*-homeomorphism. We need only to show that f^{-1} is a ρ-irresolute map. Let F be a ρ-closed set in (X,τ). Since (X,τ) is a ρ-compact space, then by Theorem 2.4(v), F is a ρ-compact subset of (X,τ). Since f is ρ-irresolute and by Theorem 2.4(vi), $f(F)$ is a ρ-compact subset of (Y,σ). Since (Y,σ) is a ρ-Hausdorff space and assume that $\rho O(X,\tau)$ be closed under finite intersections, then by Theorem 5.19, $f(F)$ is a ρ-closed set in (Y,σ). Hence f is a ρ^*-homeomorphism. ∎

Theorem 5.28. *If $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$ are ρ^*-homeomorphisms then their composition $g \circ f : (X,\tau) \to (Z,\eta)$ is also ρ^*-homeomorphism.*

Proof. Let V be a ρ-closed set in (Z,η). Now $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$. Since g is a ρ^*-homeomorphism, then $g^{-1}(V)$ is a ρ-closed set in (Y,σ) and since f is a ρ^*-homeomorphism, then $f^{-1}(g^{-1}(V))$ is a ρ-closed set in (X,τ). Therefore $g \circ f$ is ρ-irresolute. Also for a ρ-closed set F in (X,τ), we have $(g \circ f)(F) = g(f(F))$. Since f is a ρ^*-homeomorphism, then $f(F)$ is a ρ-closed set in (Y,σ) and since g is a ρ^*-homeomorphism, then $g(f(F))$ is a ρ-closed set in (Z,η). Therefore $(g \circ f)^{-1}$ is ρ-irresolute. Hence $g \circ f$ is a ρ^*-homeomorphism. ∎

Let Γ be a collection of all topological spaces. We introduce a relation, say "$\equiv_{\rho *}$", into the family Γ as follows: for two elements (X,τ) and (Y,σ) of Γ, (X,τ) is ρ^*-homeomorphic to (Y,σ), say $(X,\tau) \equiv_{\rho *} (Y,\sigma)$, if there exists a ρ^*-homeomorphism $f : (X,\tau) \to (Y,\sigma)$. Then, we have the following theorem on the relation "$\equiv_{\rho^*}$".

Theorem 5.29. *The relation $\equiv_{\rho^*}$ above is an equivalence relation in the collection of all topological spaces Γ.*

Proof. (i) For any element $(X,\tau) \in \Gamma$, $(X,\tau) \equiv_{\rho^*} (X,\tau)$ holds. Indeed, the identity function $I_x : (X,\tau) \to (X,\tau)$ is a ρ^*-homeomorphism.

(ii) Suppose $(X,\tau) \equiv_{\rho *} (Y,\sigma)$, where (X,τ) and $(Y,\sigma) \in \Gamma$. Then, there exists a ρ^*-homeomorphism $f : (X,\tau) \to (Y,\sigma)$. By definition it is seen that $f^{-1} : (Y,\sigma) \to (X,\tau)$ is a ρ^*-homeomorphism and $(Y,\sigma) \equiv_{\rho^*} (X,\tau)$.

(iii) Suppose that $(X,\tau) \equiv_{\rho *} (Y,\sigma)$ and $(Y,\sigma) \equiv_{\rho *} (Z,\eta)$, where $(X,\tau), (Y,\sigma)$ and $(Z,\eta) \in \Gamma$. By Theorem 5.28, it is shown that $(X,\tau) \equiv_{\rho *} (Z,\eta)$. ∎

We denote the family of all ρ^*-homeomorphism of a topological space (X,τ) onto itself by ρ^*-$h(X,\tau)$.

Theorem 5.30. *The set ρ^*-$h(X,\tau)$ is a group under the composition of maps.*

Proof. Define a binary operation $\Upsilon : \rho^*\text{-}h(X,\tau) \times \rho^* - h(X,\tau) \to \rho^* - h(X,\tau)$ by $\Upsilon(f,g) = g \circ f$ (the composition of f and g) for all $f, g \in \rho^*\text{-}h(X,\tau)$. Then by Theorem 5.28, $g \circ f \in \rho^*\text{-}h(X,\tau)$. We know that the composition of maps is associative and the identity map $I : (X,\tau) \to (X,\tau)$ belonging to ρ^*-$h(X,\tau)$ serves as the identity element. If $f \in \rho^*\text{-}h(X,\tau)$ then $f^{-1} \in \rho^*\text{-}h(X,\tau)$ such that $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exists for each element of ρ^*-$h(X,\tau)$. Therefore $(\rho^*\text{-}h(X,\tau), \circ)$ is a group under the operation of composition of maps.∎

Theorem 5.31. *Let $f : (X,\tau) \to (Y,\sigma)$ be a ρ^*-homeomorphism. Then f induces an isomorphisms from the group ρ^*-$h(X,\tau)$ onto the group ρ^*-$h(Y,\sigma)$.*

Proof. We define a map $\kappa_f : \rho^*\text{-}h(X,\tau) \to \rho^*\text{-}h(Y,\sigma)$ by $\kappa_f(\theta) = f \circ \theta \circ f^{-1}$, for every $\theta \in \rho^*\text{-}h(X,\tau)$, where f is a given map. By Theorem 5.28, κ_f is well defined in general, because $f \circ \theta \circ f^{-1}$ is a ρ^*-homeomorphism for every ρ^*-homeomorphism $\theta : (X,\tau) \to (Y,\sigma)$. We have to show that κ_f is a bijective homomorphism. Bijection of κ_f is clear. Further, for all $\theta_1, \theta_2 \in \rho^*\text{-}h(X,\tau)$, $\kappa_f(\theta_1 \circ \theta_2) = f \circ (\theta_1 \circ \theta_2) \circ f^{-1} = (f \circ \theta_1 \circ f^{-1}) \circ (f \circ \theta_2 \circ f^{-1}) = \kappa_f(\theta_1) \circ \kappa_f(\theta_2)$. Therefore, κ_f is a homomorphism and so it is an isomorphism induced by f. ∎

Converse of this theorem need not be true as seen from the following example. That is, there exists a map $f : (X,\tau) \to (Y,\sigma)$ which induces an isomorphism $\kappa_f : \rho^*\text{-}h(X,\tau) \to \rho^*\text{-}h(Y,\sigma)$, but not ρ^*-homeomorphism.

Example 5.32. Let $X{=}Y{=}\{a,b,c\}$ with $\tau{=}\{\phi, \{a,b\}, X\}$ and $\sigma = \{\phi, \{b,c\}, Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity function. Then f is not a ρ^*-homeomorphism. But the induced homeomorphism $\kappa_f : \rho^*\text{-}h(X,\tau) \to \rho^*\text{-}h(Y,\sigma)$ is an isomorphism, because $\rho^*\text{-}h(X,\tau) = \{I_x, \theta_c\} \cong \rho^*\text{-}h(Y,\sigma) = \{I_y, \theta_a\}$.

Corollary 5.33.

(i) *If* $\rho^*\text{-}h(X,\tau) \ncong \rho^*\text{-}h(Y,\sigma)$, *then* $(X,\tau) \ncong (Y,\sigma)$.

(ii) *Let* $f : (X,\tau) \to (Y,\sigma)$ *and* $g : (Y,\sigma) \to (Z,\eta)$ *be two* ρ^**-homeomorphisms. Then,* $\kappa_{g\circ f} = \kappa_g \circ \kappa_f$ *holds and* $\kappa_{I_x} = 1$ *is the identity isomorphism.*

Proof. (i) (resp.(ii)) It is obvious from Theorem 5.31 (resp. the definition of κ_f etc). ∎

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