DISTRIBUTIONAL AND TEMPERED DISTRIBUTIONAL DIFFRACTION FRESNEL TRANSFORMS AND THEIR EXTENSION TO BOEHMIAN SPACES

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Abstract. In [22], authors investigate the diffraction Fresnel transform on certain space of tempered distributions. Further, they extend their results to a context of Boehmian spaces. In this paper, we discuss various spaces of Boehmians. Spaces, so obtained, can handle the Fresnel transform in some approach. The extended transform and its inverse are therefore considered satisfactory and, are well recognized. Further theorems are also established in some detail.

Keywords: diffraction Fresnel transform; tempered distribution; Boehmian space; generalized function.

1991 Mathematics Subject Classification: Primary 54C40, 14E20; Secondary 46E25, 20C20.

1. Window on diffraction Fresnel transform

The diffraction Fresnel transform (optical Fresnel transform) is introduced as a four parameter class of linear integral transforms [22]. The diffraction Fresnel transform is adopted to express mathematically the Fraunhofer diffraction, when the kernel is $\exp i\varsigma\tau$ and, it describes a Fresnel diffraction, when the kernel is $exp(\varsigma - \tau)^2$ [10] . Moreover, the transform under consideration is of great importance in electromagnetic, acoustic, and other wave propagation problems which represent the solution of the wave equation under a variety of circumstances.

At optical frequencies, the diffraction Fresnel transform can model a broad class of optical systems including thin lenses, sections of free space, in the Fresnel approximation, and arbitrary concatenations, which, sometimes, referred to as first order optical systems; see, for example, [12], [16], [20], [21].

It is necessary to know that the transform under consideration has been referred to by various names such as: quadratic-phase integrals; see, for example, [16], fractional Fourier transform; see, for example [5], [6], [11], [12], [13] and [17], generalized Fresnel transforms; see [10], and some others.

The classical diffraction Fresnel transform of a function $f(\varsigma)$ is defined by

$$\mathcal{F}_d f(\tau) = \int_{\mathcal{R}} \mathcal{E}(\alpha_1, \gamma_1, \gamma_2, \alpha_2; \varsigma, \tau) f(\varsigma) d\varsigma. \tag{1}$$

where

$$\mathcal{E}(\alpha_1, \gamma_1, \gamma_2, \alpha_2; \varsigma, \tau) := \mathcal{E} := \frac{1}{\sqrt{2\pi i \gamma_1}} \exp\left(\frac{i}{2\gamma_1}\left(\alpha_1 \varsigma^2 - 2\varsigma\tau + \alpha_2 \tau^2\right)\right) \tag{2}$$

is the transform kernel with real parameters, $\alpha_1, \gamma_1, \alpha_2$ and γ_2. The parameters $(\alpha_1, \gamma_1, \alpha_2 \text{and } \gamma_2)$ are elements of more ray transfer matrix M describing optical systems, $\alpha_1\alpha_2 - \gamma_1\gamma_2 = 1$. For more details, if the parameters $(\alpha_1, \gamma_1, \alpha_2 \text{and } \gamma_2)$ are written in matrix form

$$\begin{pmatrix} \alpha_1 & \gamma_1 \\ \gamma_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

then the diffraction Fresnel transform becomes a fractional Fourier transform [24].

In an earlier paper [22], authors have established that the diffraction Fresnel transform maps the space of rapidly decreasing functions into itself and further applied it to a space of Boehmians. The present work represents an effort of us to obtain more general spaces of Boehmians that can adequately handle the four parameters with a different approach. We build upon analysis of [22]. We describe four spaces of Boehmians for the diffraction Fresnel transform with careful attention. An avenue towards this end is to first define the diffraction Fresnel transform for Boehmians and derive its properties. Definition of the inverse transform, when it exists, is also established.

Let $\mathcal{S}$ be the space of rapidly decreasing functions over $\mathcal{R}$ (the space of rapid descent) [7]. Then the diffraction Fresnel transform of $\psi \in \mathcal{S}(\mathcal{R})$ is also in $\mathcal{S}(\mathcal{R})$ [22, Theorem 2.1]. The Parseval relation for the diffraction Fresnel transform, compared to Fourier transforms, is interpreted to mean

$$\langle \mathcal{F}_d f, \psi \rangle = \langle f, \mathcal{F}_d \psi \rangle, \tag{3}$$

where $\psi \in \mathcal{S}, f \in \mathcal{S}'(R), \mathcal{S}'(\mathcal{R})$ being the dual space of $\mathcal{S}(\mathcal{R})$ of distributions of slow growth over $\mathcal{R}$. Hence, from equation (3), it may be noted that $\mathcal{F}_d f \in \mathcal{S}'(\mathcal{R})$ for every $f \in \mathcal{S}'(\mathcal{R})$.

In brief details, we spread the paper over six sections. The diffraction Fresnel transform is reviewed in Section 1. Section 2 presents a general construction of Boehmians. The first Boehmian space is presented in Section 3. The tempered distributional diffraction space of Boehmians is given in Section 4. The extended diffraction Fresnel transform of a Boehmian and its properties are established in Section 5. In Section 6, we give another approach of the cited transform to

Boehmian spaces. With regret to have some repetition in the analysis due to their necessity.

2. Boehmian Spaces and General Construction

To make this work self contained as much as possible we recall the following: Let $\mathcal{G}$ be a linear space and $\mathcal{H}$ be a subspace of $\mathcal{G}$. Assume, to each pair (f, ϕ) of elements, $f \in \mathcal{G}$ and $\phi \in \mathcal{H}$, is assigned the product $f * \phi$ such that the following conditions are satisfied:

(1) $\phi, \psi \in \mathcal{H} \Rightarrow \phi * \psi \in \mathcal{H}$ *and* $\phi * \psi = \psi * \phi$.

(2) $f \in \mathcal{G},\ \phi, \psi \in \mathcal{H} \Rightarrow (f * \phi) * \psi = f * (\phi * \psi)$.

(3) *If* $f, g \in \mathcal{G}, \phi \in \mathcal{H}$, *then* $(f + g) * \phi = f * \phi + g * \phi$

(4) *If* $k \in \mathcal{R}$, then $k (f * \phi) = (kf) * \phi = f * (k\phi)$.

Let Δ be a family of sequences from $\mathcal{H}$ such that:

(1) *If* $f, g \in \mathcal{G}, (\epsilon_n) \in \Delta$ *and* $f * \epsilon_n = g * \epsilon_n$ $(n = 1, 2, ...)$, *then* $f = g$.

(2) $(\epsilon_n), (\tau_n) \in \Delta \Rightarrow (\epsilon_n * \tau_n) \in \Delta$.

Each element of Δ will be called *delta sequence*.
Consider the class $\mathcal{A}$ of pairs of sequences defined by

$$\mathcal{A} = \left\{((f_n), (\epsilon_n)) : (f_n) \subseteq \mathcal{G}^{\mathbf{N}}, (\epsilon_n) \in \Delta\right\},$$

for each $n \in \mathbf{N}$, the set of natural numbers. The pair $((f_n), (\epsilon_n)) \in \mathcal{A}$ is said to be a *quotient of sequences*, denoted by $\frac{f_n}{\epsilon_n}$, if

$$f_n * \epsilon_m = f_m * \epsilon_n, \forall n, m \in \mathbf{N}. \tag{4}$$

Two quotients of sequences $\frac{f_n}{\epsilon_n}$ and $\frac{g_n}{\tau_n}$ are said to be *equivalent*, $\frac{f_n}{\epsilon_n} \sim \frac{g_n}{\tau_n}$, if

$$f_n * \epsilon_m = g_m * \tau_n, \forall n, m \in \mathbf{N}. \tag{5}$$

The relation $\sim$ is an equivalent relation on $\mathcal{A}$ and hence, splits $\mathcal{A}$ into equivalence classes. The equivalence class containing $\frac{f_n}{\epsilon_n}$ is denoted by $\left[\frac{f_n}{\epsilon_n}\right]$. These equivalence classes are called *Boehmians* and the *space of all Boehmians* is denoted by $\mathcal{H}_\beta$.

The sum and multiplication by a scalar of two Boehmians can be defined in a natural way

$$\left[\frac{f_n}{\epsilon_n}\right] + \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n * \tau_n + g_n * \epsilon_n}{\epsilon_n * \tau_n}\right]$$

and

$$a\left[\frac{f_n}{\epsilon_n}\right]=\left[\frac{af_n}{\epsilon_n}\right], a\in C,\ \text{the field of complex numbers.}$$

The operation $*$ and differentiation are defined by $\left[\frac{f_n}{\epsilon_n}\right]*\left[\frac{g_n}{\tau_n}\right]=\left[\frac{f_n*g_n}{\epsilon_n*\tau_n}\right]$ and $\mathcal{D}^\alpha\left[\frac{f_n}{\epsilon_n}\right]=\left[\frac{\mathcal{D}^\alpha f_n}{\epsilon_n}\right]$. Many a time, $\mathcal{G}$ is equipped with a notion of convergence. The intrinsic relationship between the notion of convergence and the product $*$ are given by:

(1) *If* $f_n\to f$ *as* $n\to\infty$ *in* $\mathcal{G}$ *and,* $\phi\in\mathcal{H}$ *is any fixed element, then* $f_n*\phi\to f*\phi$ in $\mathcal{G}$ (as $n\to\infty$).

(2) *If* $f_n\to f$ *as* $n\to\infty$ *in* $\mathcal{G}$ *and* $(\epsilon_n)\in\Delta$, *then* $f_n*\epsilon_n\to f$ *in* $\mathcal{G}$ (as $n\to\infty$).

The operation $*$ can be extended to $\mathcal{H}_\beta\times\mathcal{H}$ by the following definition.

Definition 2.1. If $\left[\frac{f_n}{\epsilon_n}\right]\in\mathcal{H}_\beta$ and $\phi\in\mathcal{H}$,then $\left[\frac{f_n}{\epsilon_n}\right]*\phi=\left[\frac{f_n*\phi}{\epsilon_n}\right]$.

In $\mathcal{H}_\beta$, two types of convergence, $\delta-$convergence and $\Delta-$convergence, are defined as follows:

Definition 2.2. A sequence of Boehmians (β_n) in $\mathcal{H}_\beta$ is said to be δ-*convergent* to a Boehmian β in $\mathcal{H}_\beta$, denoted by $\beta_n\overset{\delta}{\to}\beta$, if there exists a delta sequence (ϵ_n) such that $(\beta_n*\epsilon_n),(\beta*\epsilon_n)\in\mathcal{G},\ \forall k,n\in\mathbf{N}$, and

$$(\beta_n*\epsilon_k)\to(\beta*\epsilon_k)\ \text{ as } n\to\infty, in\ \mathcal{G},\ \text{ for every } k\in\mathbf{N}.$$

The following lemma is equivalent for the statement of δ-*convergence.*

Lemma 2.3. $\beta_n\overset{\delta}{\to}\beta$ $(n\to\infty)$ *in* $\mathcal{H}_\beta$ *if and only if there is* $f_{n,k},f_k\in\mathcal{G}$ *and* $\epsilon_k\in\Delta$ *such that* $\beta_n=\left[\frac{f_{n,k}}{\epsilon_k}\right],\beta=\left[\frac{f_k}{\epsilon_k}\right]$ *and for each* $k\in\mathbf{N}$,

$$f_{n,k}\to f_k \text{ as } n\to\infty \text{ in } \mathcal{G}.$$

Definition 2.4. A sequence of Boehmians (β_n) in $\mathcal{H}_\beta$ is said to be Δ-*convergent to a Boehmian* β in $\mathcal{H}_\beta$, denoted by $\beta_n\overset{\Delta}{\to}\beta$, if there exists a $(\epsilon_n)\in\Delta$ such that $(\beta_n-\beta)*\epsilon_n\in\mathcal{G},\forall n\in\mathbf{N}$, and $(\beta_n-\beta)*\epsilon_n\to 0$as $n\to\infty$ in $\mathcal{G}$.

See, for examples, [1]–[4], [8], [14]–[15] and [22]–[23].

3. The tempered space $\mathcal{H}_{\beta_1}\left(\mathcal{S}',\mathcal{D},\Delta,\bullet\right)$ of Boehmians

To start our mission of extending diffraction Fresnel transforms to Boehmian spaces we first discuss the space $\mathcal{H}_{\beta_1}\left(\mathcal{S}',\mathcal{D},\bullet,\Delta\right)$ of Boehmians with the notation $\bullet$ described by [23, Definition 2.2]

$$\langle f\bullet\sigma,\psi\rangle=\langle f,\psi\star\check{\sigma}\rangle \tag{6}$$

for every $f \in \mathcal{S}', \sigma \in \mathcal{D}$, the Schwartz space of test functions of bounded supports, $\check{\sigma}(t) = \sigma(-t), t \in \mathcal{R}, \psi \in \mathcal{S}$ is arbitrary and $\star$ acts for the usual convolution product. The righthand side of Equ.(6) is clearly well-defined, since $\psi \star \check{\sigma} \in \mathcal{S}$. Hence, $f \bullet \sigma \in \mathcal{S}'$ is justified.

Following, are results proved for defining $\mathcal{H}_{\beta_1}$.

Theorem 3.1. *Let $f \in \mathcal{S}'$ and $\sigma \in \mathcal{D}$ then $f \star \check{\sigma} \in \mathcal{S}'$.*

Proof. By aid of [18, Theorem 1.14.1 (iii), p. 26], $f \star \check{\sigma} \in \mathcal{O}_M \subset \mathcal{S}', \mathcal{O}_M$ is the space of multipliers of $\mathcal{S}'$. Hence the theorem.

By Δ we denote the subset of $\mathcal{D}$ of all sequences with the following properties:

(1) $\int_{\mathcal{R}} \epsilon_n(\varsigma)\, d\varsigma = 1, n \in \mathbf{N}$.

(2) $\int_{\mathcal{R}} |\epsilon_n(\varsigma)|\, d\varsigma \le M, 0 < M \in \mathcal{R}$.

(3) $\mathrm{supp}\epsilon_n \to 0$ as $n \to \infty, \mathrm{supp}\epsilon_n = \{\varsigma : \epsilon_n(\varsigma) \neq 0, \forall n \in \mathbf{N}\}$.

Each member (ϵ_n) in Δ is said to be delta sequence.

Following are needful:

(1) If $(\epsilon_n), (\alpha_n) \in \Delta$, then

$$\int_{\mathcal{R}} (\epsilon_n \star \check{\alpha}_n)(\varsigma)\, d\varsigma = \int_{\mathcal{R}} \epsilon_n(\varsigma)\, d\varsigma \int_{\mathcal{R}} \alpha_n(y)\, dy = 1;$$

(2) If $(\epsilon_n), (\alpha_n) \in \Delta$ then

$$\int_{\mathcal{R}} |(\epsilon_n \star \check{\alpha}_n)(\varsigma)|\, d\varsigma \le \int_{\mathcal{R}} |\epsilon_n(\varsigma)|\, d\varsigma \int_{\mathcal{R}} |\check{\alpha}_n(\tau)|\, d\tau \le M_1 M_2,$$

for some constants M_1 and M_2 where $\int_{\mathcal{R}} |\epsilon_n| \le M_1, \int_{\mathcal{R}} |\alpha_n| \le M_2$.

(3) $\mathrm{supp}(\epsilon_n \star \check{\alpha}_n) \subseteq \mathrm{supp}\epsilon_n + \mathrm{supp}\check{\alpha}_n \to 0$ as $n \to \infty$.

Moreover, it is easy to see that for each $(\epsilon_n) \in \Delta$, $(\check{\epsilon}_n) \in \Delta$ as well.

Lemma 3.2. *If $f \in \mathcal{S}'$ and $\sigma_1, \sigma_2 \in \mathcal{D}$ then the following hold:*
(1) $\sigma_1 \bullet \sigma_2 = \sigma_1 \star \sigma_2 = \sigma_2 \bullet \sigma_1$; (2) $f \bullet (\sigma_1 \bullet \sigma_2) = (f \bullet \sigma_1) \bullet \sigma_2$.

For proof see [23, Lemma 2.5].

Lemma 3.3. *Let $f, g \in \mathcal{S}', (\epsilon_n) \in \Delta$ and $f \bullet \epsilon_n = g \bullet \epsilon_n$ then $f = g$ in $\mathcal{S}'$, for every $n \in \mathbf{N}$.*

Proof. At first, certainly $\psi \star \check{\epsilon}_n \to \psi$ as $n \to \infty$ for each $(\epsilon_n) \in \Delta$ and $\psi \in \mathcal{S}$. Hence, from the hypothesis of this lemma, $\langle f \bullet \epsilon_n, \psi \rangle = \langle g \bullet \epsilon_n, \psi \rangle, \psi \in \mathcal{S}$. Therefore, by equation (6), we get

$$\langle f, \psi \star \check{\epsilon}_n \rangle = \langle g, \psi \star \check{\epsilon}_n \rangle .$$

Thus

$$\langle f, \psi \rangle = \langle g, \psi \rangle .$$

for all $\psi \in \mathcal{S}$. This completes the proof of the lemma.

The desired space $\mathcal{H}_{\beta_1}\left(\mathcal{S}', \mathcal{D}, \Delta, \bullet\right)$, briefly $\mathcal{H}_{\beta_1}$, of Boehmians is described. Addition, multiplication by a scalar and the operation $\bullet$, in $\mathcal{H}_{\beta_1}\left(\mathcal{S}', \mathcal{D}, \Delta, \bullet\right)$, are respectively defined in the usual way:

$$\left[\frac{f_n}{\epsilon_n}\right] + \left[\frac{g_n}{\alpha_n}\right] = \left[\frac{f_n \bullet \alpha_n + g_n \bullet \epsilon_n}{\epsilon_n \bullet \alpha_n}\right],$$

$$k\left[\frac{f_n}{\epsilon_n}\right] = \left[\frac{k f_n}{\epsilon_n}\right], \text{ for all } k \in \mathcal{R},$$

and

$$\left[\frac{f_n}{\epsilon_n}\right] \bullet \left[\frac{g_n}{\alpha_n}\right] = \left[\frac{f_n \bullet g_n}{\epsilon_n \bullet \alpha_n}\right].$$

Lemma 3.4. *Let $f_n \to f \in \mathcal{S}', \sigma \in \mathcal{D}$ then $f_n \bullet \sigma \to f \bullet \sigma$ as $n \to \infty$.*

Proof. Let $\psi \in \mathcal{S}$ be arbitrary then the fact that $\psi \star \check{\sigma} \in \mathcal{S}$ suggests to write

$$\begin{aligned} \langle f_n \bullet \sigma - f \bullet \sigma, \psi \rangle &= \langle (f_n - f) \bullet \sigma, \psi \rangle \\ &= \langle f_n - f, \psi \star \check{\sigma} \rangle . \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This completes the proof.

Lemma 3.5. *Let $f_n \to f$ in $\mathcal{S}', (\epsilon_n) \in \Delta$ then $f_n \bullet \epsilon_n \to f$ as $n \to \infty$.*

Proof. Using properties of delta sequences and the integral operator $\int$, we get

$$\begin{aligned} \langle f_n \bullet \epsilon_n - f, \psi \rangle &= \langle f_n \bullet \epsilon_n - f \bullet \epsilon_n, \psi \rangle \\ &= \langle (f_n - f) \bullet \epsilon_n, \psi \rangle \\ &= \langle f_n - f, \psi \star \check{\epsilon}_n \rangle \\ &\to \langle f_n - f, \psi \rangle \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This completes the proof of the lemma.

Theorem 3.6. *The mapping $\mathcal{S}' \to \mathcal{H}_{\beta_1}\left(\mathcal{S}', \mathcal{D}, \Delta, \bullet\right)$ defined by $f \to \left[\frac{f \bullet \epsilon_n}{\epsilon_n}\right]$ is a continuous imbedding with respect to δ convergence.*

Proof. Let $\left[\frac{f \bullet \epsilon_n}{\epsilon_n}\right] = \left[\frac{g \bullet \alpha_n}{\alpha_n}\right]$ then $(f \bullet \epsilon_n) \bullet \alpha_m = (g \bullet \alpha_m) \bullet \epsilon_m$. In particular, $(f \bullet \epsilon_n) \bullet \alpha_n = (g \bullet \alpha_n) \bullet \epsilon_n$. Allowing $n \to \infty$ and using Lemma 3.5 implies that the mapping is one-to-one. Next, let $f_n \to 0$ as $n \to \infty$ in $\mathcal{S}'$, then, $\langle f_n, \psi \rangle \to 0$ for all $\psi \in \mathcal{S}$. Hence, $\langle f_n \bullet \epsilon_n, \psi \rangle = \langle f_n, \psi \star \check{\epsilon}_n \rangle \to \langle f_n, \psi \rangle \to 0$ as $n \to \infty$ in $\mathcal{S}'$. Therefore, $\left[\frac{f \bullet \epsilon_n}{\epsilon_n}\right] \to 0$ as $n \to \infty$ in $\mathcal{H}_{\beta_1}\left(\mathcal{S}', \mathcal{D}, \Delta, \bullet\right)$. This establishes that the mapping is continuous with respect to δ convergence. The proof is completed.

4. The Diffraction Space $\mathcal{H}_{\beta_2}\left(\mathcal{S}', \mathcal{D}, \Delta, \curlywedge\right)$ of Boehmians

In this section we present a space of Boehmians by a different operation.

Between $\mathcal{S}'$ and $\mathcal{D}$ we define a mapping $\curlywedge$ by

$$(f \curlywedge \sigma)(\tau) = \int_{\mathcal{R}} \mathcal{F}_d f(\tau - y)\, \sigma(y)\, dy$$

where $f \in \mathcal{S}'$ and $\sigma \in \mathcal{D}$ are arbitrary.

Lemma 4.1. *Let $f \in \mathcal{S}'$ and $\sigma \in \mathcal{D}$ then $\mathcal{F}_d(f \bullet \sigma)(\tau) = (f \curlywedge \sigma)(\tau)$.*

Proof. Let $f \in \mathcal{S}'$. Employing equation (3) and equation (6) yield

$$\begin{aligned} \langle \mathcal{F}_d(f \bullet \sigma)(\tau), \psi(\tau)\rangle &= \langle f \bullet \sigma(\tau), \mathcal{F}_d \psi(\tau)\rangle \\ &= \langle f(\tau), (\mathcal{F}_d \psi \star \check{\sigma})(\tau)\rangle, \\ &= \left\langle f(\tau), \int_{\mathcal{R}} \mathcal{F}_d \psi(\varsigma)\, \sigma(\varsigma - \tau)\, d\varsigma \right\rangle \end{aligned}$$

The substitution $\varsigma - \tau = y$ and, the translation property of distributions, through y, jointly with equation (3), yield

$$\begin{aligned} \langle \mathcal{F}_d(f \bullet \sigma)(\tau), \psi(\tau)\rangle &= \left\langle f(\tau), \int_{\mathcal{R}} \mathcal{F}_d \psi(\tau + y)\, \sigma(y)\, dy \right\rangle \\ &= \int_{\mathcal{R}} \langle f(\tau - y), \mathcal{F}_d \psi(\tau)\rangle\, \sigma(y)\, dy \\ &= \int_{\mathcal{R}} \langle \mathcal{F}_d f(\tau - y), \psi(\tau)\rangle\, \sigma(y)\, dy \\ &= \left\langle \int_{\mathcal{R}} \mathcal{F}_d f(\tau - y)\, \sigma(y)\, dy, \psi(\tau) \right\rangle \\ &= \langle (f \curlywedge \sigma)(\tau), \psi(\tau)\rangle, \end{aligned}$$

where $\psi \in \mathcal{S}$ is arbitrary. The proof of this theorem is completed.

Lemma 4.2. *$f \curlywedge \sigma \in \mathcal{S}'$, for every $f \in \mathcal{S}, \sigma \in \mathcal{D}$.*

Proof. Let $f \in \mathcal{S}'$, $\sigma \in \mathcal{D}$ then $f \bullet \sigma \in \mathcal{S}'$, by equation (6). Hence, $f \curlywedge \sigma = \mathcal{F}_d(f \bullet \sigma) \in \mathcal{S}'$. This completes the proof of the lemma.

Lemma 4.3. *If* $f \in \mathcal{S}', \sigma_1, \sigma_2 \in \mathcal{D}$ *then the following are true*:
(1) $\sigma_1 \curlywedge \sigma_2 = \sigma_2 \curlywedge \sigma_1$; (2) $f \curlywedge (\sigma_2 \curlywedge \sigma_1) = (f \curlywedge \sigma_1) \curlywedge \sigma_2$.

Proof. We attempt to proof the first part of this lemma since the proof of second part is similar. For, let $\sigma_1, \sigma_2 \in \mathcal{D}$. Using Lemma 3.2, $\sigma_1 \curlywedge \sigma_2 = \mathcal{F}_d (\sigma_1 \bullet \sigma_2) = \mathcal{F}_d (\sigma_2 \bullet \sigma_1) = \sigma_2 \curlywedge \sigma_1$. Hence the lemma.

Lemma 4.4. *If* $f_1, f_2 \in \mathcal{S}', (\epsilon_n) \in \Delta$ *and* $f_1 \curlywedge \epsilon_n = f_2 \curlywedge \epsilon_n$ *then* $f_1 = f_2$.

Proof of this lemma follows from Theorem 4.1 and the fact that

$$\mathcal{F}_d \epsilon_n \to \frac{i}{2\gamma_1 \sqrt{2\pi i \gamma_1}} \alpha_2 \tau^2 \text{ as } n \to \infty.$$

The relationship between convergence and the operation $\curlywedge$ is given by:

Lemma 4.5. *If* $f_n \to f \in \mathcal{S}', \sigma \in \mathcal{D}$ *then* $f_n \curlywedge \sigma \to f_n \curlywedge \sigma$ *as* $n \to \infty$.

Lemma 4.6. *If* $f_n \to f \in \mathcal{S}', (\epsilon_n) \in \Delta$ *then* $f_n \curlywedge \epsilon_n \to f$ *as* $n \to \infty$.

Proof of Lemma 4.4 and Lemma 4.5 is straightforward from Lemma 4.1, detailed proof thus avoided.

Theorem 4.7. *The mapping* $\mathcal{S}' \to \mathcal{H}_{\beta_2}\left(\mathcal{S}', \mathcal{D}, \Delta, \bullet\right)$ *defined by* $f \to \left[\frac{f \bullet \epsilon_n}{\epsilon_n}\right]$ *is a continuous imbedding with respect to* δ *convergence.*

Proof. (See Theorem 3.6.)

By the above conclusions, the desired space $\mathcal{H}_{\beta_2}\left(\mathcal{S}', \mathcal{D}, \Delta, \curlywedge\right)$ is considered. We preserve the operations of addition, multiplication by a scalar and the operation $\curlywedge$ between Boehmians.

The operation $\curlywedge$ can be extended to $\mathcal{H}_{\beta_2} \times \mathcal{D}$ by the following definition:

Definition 4.7. If $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_2}$ and $\phi \in \mathcal{D}$, then $\left[\frac{f_n}{\epsilon_n}\right] \curlywedge \phi = \left[\frac{f_n \curlywedge \phi}{\epsilon_n}\right]$.

In $\mathcal{H}_{\beta_2}$, convergence is defined as:

Definition 4.8. A sequence of Boehmians (β_n) in $\mathcal{H}_{\beta_2}$ is δ *convergent* to a Boehmian β in H_{β_2}, if there exists (ϵ_n) *such that* $(\beta_n \curlywedge \epsilon_n), (\beta \curlywedge \epsilon_n) \in \mathcal{H}_{\beta_2}$, $\forall k$, $n \in \mathbf{N}$, and $(\beta_n \curlywedge \epsilon_k) \to (\beta \curlywedge \epsilon_k)$ *as* $n \to \infty$, in $\mathcal{H}_{\beta_2}$, for every $k \in \mathbf{N}$.

The following lemma is equivalent to δ convergence.

Lemma 4.9. $\beta_n \xrightarrow{\delta} \beta$ *in* $\mathcal{H}_{\beta_2}$ *if and only if there is* $f_{n,k}, f_k \in \mathcal{S}'$ *and* $(\epsilon_k) \in \Delta$ *such that* $\beta_n = \left[\frac{f_{n,k}}{\epsilon_k}\right], \beta = \left[\frac{f_k}{\epsilon_k}\right]$ *and* $\forall k \in \mathbf{N}, f_{n,k} \to f_k$ as $n \to \infty$ in $\mathcal{S}'$.

Definition 4.10. (β_n) in $\mathcal{H}_{\beta_2}$ is Δ *convergent* to β in $\mathcal{H}_{\beta_2}$ if there is a $(\epsilon_n) \in \Delta$ such that $(\beta_n - \beta) \curlywedge \epsilon_n \in \mathcal{H}_{\beta_2}, \forall n \in \mathbf{N}$, and $(\beta_n - \beta) \curlywedge \epsilon_n \to 0$ as $n \to \infty$ in $\mathcal{H}_{\beta_2}$.

5. The Diffraction transform of $\mathcal{H}_{\beta_1}\left(\mathcal{S}', \mathcal{D}, \Delta, \bullet\right)$

Let $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_1}\left(\mathcal{S}', \mathcal{D}, \Delta, \bullet\right)$. Since operations we introduced on $\mathcal{H}_{\beta_1}$ and $\mathcal{H}_{\beta_2}$ ($\bullet$ and $\curlywedge$, respectively), are mathematically not equal, the typical Boehmian $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_1}$, is denoted by $\left[\frac{f_n^*}{\epsilon_n}\right]$ in the space $\mathcal{H}_{\beta_2}$. We may agree this will make no confusion in notations. Restrictions on parameters here are that $\alpha_1 = \alpha_2$.

Definition 5.1. *The diffraction Fresnel transform* $\hat{\mathcal{F}}_d^* : \mathcal{H}_{\beta_1} \to \mathcal{H}_{\beta_2}$ of $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_1}\left(\mathcal{S}', \mathcal{D}, \Delta, \bullet\right)$ is defined by

$$\hat{\mathcal{F}}_d^*\left(\left[\frac{f_n}{\epsilon_n}\right]\right) = \left[\frac{f_n^*}{\epsilon_n}\right], \tag{7}$$

in $\mathcal{H}_{\beta_2}$.

Theorem 5.2. $\hat{\mathcal{F}}_d^* : \mathcal{H}_{\beta_1} \to \mathcal{H}_{\beta_2}$ *is well-defined, linear and independent of the representative.*

Proof. Let $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_1}$ then $f_n \bullet \epsilon_m = f_m \bullet \epsilon_n, \forall n, m \in \mathbf{N}$. Applying the diffraction Fresnel transform and using Theorem 4.1 yield $f_n^* \curlywedge \epsilon_m = f_m^* \curlywedge \epsilon_n$, for every $m, n \in \mathbf{N}$. Therefore $\left[\frac{f_n^*}{\epsilon_n}\right] \in \mathcal{H}_{\beta_2}$. To show $\hat{\mathcal{F}}_d^*$ is well defined, let $\left[\frac{g_n}{\alpha_n}\right] = \left[\frac{f_n}{\epsilon_n}\right]$ in $\mathcal{H}_{\beta_1}$ then, $f_n \bullet \alpha_m = g_m \bullet \epsilon_n$. Once again, applying the diffraction Fresnel transform and Theorem 4.1 yields $f_n^* \curlywedge \alpha_m = g_m^* \curlywedge \epsilon_n, \forall n, m \in \mathbf{N}$. Hence,

$$\left[\frac{f_n^*}{\epsilon_n}\right] = \left[\frac{g_m^*}{\alpha_n}\right].$$

Let $k_1, k_2 \in \mathcal{R}$. Theorem 4.1. implies

$$\hat{\mathcal{F}}_d^*\left(k_1\left[\frac{f_n}{\epsilon_n}\right] + k_2\left[\frac{g_n}{\alpha_n}\right]\right) = \left[\frac{k_1 f_n^* \curlywedge \epsilon_n + k_2 g_m^* \curlywedge \alpha_n}{\epsilon_n \curlywedge \alpha_n}\right].$$

That is

$$\hat{\mathcal{F}}_d^*\left(k_1\left[\frac{f_n}{\epsilon_n}\right] + k_2\left[\frac{g_n}{\alpha_n}\right]\right) = k_1\left[\frac{f_n^*}{\epsilon_n}\right] + k_2\left[\frac{g_m^*}{\alpha_n}\right].$$

This completes the proof of the theorem.

Theorem 5.3. $\hat{\mathcal{F}}_d^* : \mathcal{H}_{\beta_1} \to \mathcal{H}_{\beta_2}$ *is a bijection map.*

Proof. Let $\left[\frac{f_n^*}{\epsilon_n}\right] = \left[\frac{g_n^*}{\alpha_n}\right] \in \mathcal{H}_{\beta_2}$, then $f_n^* \curlywedge \alpha_m = g_n^* \curlywedge \epsilon_n, \forall n$. From Theorem 4.1, $\mathcal{F}_d\left(f_n \bullet \alpha_m\right) = \mathcal{F}_d\left(g_n \bullet \epsilon_n\right)$. Since $\mathcal{F}_d$ is one-one, on $\mathcal{S}'$, we get $f_n \bullet \alpha_m = g_n \bullet \epsilon_n$. Hence, $\left[\frac{f_n}{\epsilon_n}\right] = \left[\frac{g_n}{\alpha_n}\right]$. Finally, let $\left[\frac{f_n^*}{\epsilon_n}\right] \in \mathcal{H}_{\beta_2}$ then the Boehmian $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_1}$ satisfies $\hat{\mathcal{F}}_d^*\left(\left[\frac{f_n}{\epsilon_n}\right]\right) = \left[\frac{f_n^*}{\epsilon_n}\right]$.

Proof is, therefore, completed.

The inversion formula $\left(\hat{\mathcal{F}}_d^*\right)^{-1}$ for $\hat{\mathcal{F}}_d^*$ is defined by the following definition:

Definition 5.4. Let $\left[\frac{f_n^*}{\epsilon_n}\right] \in \mathcal{H}_{\beta_2}$ then the inverse transform of $\hat{\mathcal{F}}_d^*$ is defined by

$$\left(\hat{\mathcal{F}}_d^*\right)^{-1}\left(\left[\frac{f_n^*}{\epsilon_n}\right]\right) = \left[\frac{f_n}{\epsilon_n}\right], \tag{8}$$

in $\mathcal{H}_{\beta_1}$.

Theorem 5.5. $\left(\hat{\mathcal{F}}_d^*\right)^{-1} : \mathcal{H}_{\beta_2} \to \mathcal{H}_{\beta_1}$ *is well-defined, linear and bijection.*

This theorem can be proved by analysis which is alike to that employed for Theorem 5.2.

Theorem 5.6. $\hat{\mathcal{F}}_d^* : \mathcal{H}_{\beta_1} \to \mathcal{H}_{\beta_2}, \left(\hat{\mathcal{F}}_d^*\right)^{-1} : \mathcal{H}_{\beta_2} \to \mathcal{H}_{\beta_1}$ *are continuous with respect to* δ *convergence.*

Proof. Let $(\beta_n) \in \mathcal{H}_{\beta_1}, \beta \in \mathcal{H}_{\beta_1}$ be such that $\beta_n \xrightarrow{\delta} \beta$ then, using Lemma 2.3, we can find $f_{n,k}, f_k \in \mathcal{S}', (\epsilon_n) \in \Delta$, such that $\beta_n = \left[\frac{f_{n,k}}{\epsilon_n}\right], \beta = \left[\frac{f_k}{\epsilon_n}\right]$ and $f_{n,k} \to f_k$ for every $k \in \mathbf{N}$ as $n \to \infty$ in $\mathcal{S}'$. Since $\mathcal{F}_d$ is continuous from $\mathcal{S}'$ into $\mathcal{S}'$, it follows $\mathcal{F}_d\left(f_{n,k}\right) \to \mathcal{F}_d\left(f_k\right)$ as $n \to \infty$ in $\mathcal{S}'$. Hence, $\left[\frac{f_{n,k}^*}{\epsilon_n}\right] \to \left[\frac{f_k^*}{\epsilon_n}\right]$ as $n \to \infty$.

To prove the second part of the theorem, let $\tilde{\beta}, \left(\tilde{\beta}_n\right) \in \mathcal{H}_{\beta_2}$ then, as above, there are $f_{n,k}^*, f_k^* \in \mathcal{S}'$ such that $\tilde{\beta}_n = \left[\frac{f_{n,k}^*}{\epsilon_n}\right], \tilde{\beta} = \left[\frac{f_k^*}{\epsilon_n}\right]$ and, $f_{n,k}^* \to f_k^*$ for every k, as $n \to \infty$. Hence $f_{n,k} \to f_k$ as $n \to \infty$ for every $k \in \mathbf{N}$ in $\mathcal{S}'$. Thus,

$$\left[\frac{f_{n,k}}{\epsilon_n}\right] = \left(\hat{\mathcal{F}}_d^*\right)^{-1}\left(\left[\frac{f_{n,k}^*}{\epsilon_n}\right]\right) \to \left[\frac{f_k}{\epsilon_n}\right] = \left(\hat{\mathcal{F}}_d^*\right)^{-1}\left(\left[\frac{f_k^*}{\epsilon_n}\right]\right)$$

as $n \to \infty$ for every k.

Hence, the theorem.

Theorem 5.7. $\hat{\mathcal{F}}_d^* : \mathcal{H}_{\beta_1} \to \mathcal{H}_{\beta_2}, \left(\hat{\mathcal{F}}_d^*\right)^{-1} : \mathcal{H}_{\beta_2} \to \mathcal{H}_{\beta_1}$ *are continuous with respect to* Δ *convergence.*

Proof. We prove the first part of the theorem since the second part of the theorem can be proved similarly. Let $\beta_n \overset{\Delta}{\to} \beta \in \mathcal{H}_{\beta_1}$ as $n \to \infty$ then there is $(f_n) \in \mathcal{S}'$ such that

$$(\beta_n - \beta) \bullet \epsilon_n = \left[\frac{f_n \bullet \epsilon_k}{\epsilon_k} \right] \to 0 \text{ and } f_n \to 0 \text{ as } n \to \infty, (\epsilon_n) \in \Delta. \text{ Thus,}$$

$$\begin{aligned} \left(\hat{\mathcal{F}}_d^* \beta_n - \hat{\mathcal{F}}_d^* \beta \right) \curlywedge \epsilon_n &= \hat{\mathcal{F}}_d^* (\beta_n - \beta) \curlywedge \epsilon_n \\ &= \left[\frac{f_n^* \curlywedge \epsilon_k}{\epsilon_k} \right] \\ &= f_n^* \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This completes the proof of the theorem.

6. The diffraction Fresnel transform of $\mathcal{H}_{\beta_3}(\mathcal{E}', \mathcal{D}, \Delta, \bullet)$

By $\mathcal{E}$ we denote the space of smooth functions of arbitrary support on $\mathcal{R}$ and, $\mathcal{E}'$ its strong dual of distributions of compact support. By aid of the kernel method, the diffraction Fresnel transform of $f \in \mathcal{E}'$ has been extended to [22]

$$\mathcal{F}_d f(\tau) = \tfrac{1}{\sqrt{2\pi i \gamma_1}} \left\langle f(t), \exp \frac{i(\alpha_1 \varsigma^2 - 2\tau\varsigma + \alpha_2 \tau^2)}{2\gamma_1} \right\rangle, \tag{9}$$

where $f \in \mathcal{E}'$ is arbitrary.

Let $\mathcal{H}_{\beta_3}(\mathcal{E}', \mathcal{D}, \Delta, \bullet)$ be the Boehmian space with $\mathcal{E}'$, as linear space, $\mathcal{D}$, as a subspace of $\mathcal{E}'$, and $\bullet$, as an operation between $\mathcal{E}'$ and $\mathcal{D}$. Denote by $\mathcal{D}_F$ the set of analytic functions that are diffraction Fresnel transforms of compactly supported distributions in $\mathcal{E}'$.

Convergence in $\mathcal{D}_F$ is defined as follows:

$\mathcal{T}_n \to \mathcal{T}$ in $\mathcal{D}_F$ if there are distributions $\upsilon_n, \upsilon \in \mathcal{E}'$ such that $\mathcal{T}_n = \mathcal{F}_d \upsilon_n$, $\mathcal{T} = \mathcal{F}_d \upsilon$ and $\upsilon_n \to \upsilon$ in $\mathcal{E}'$.

Between $\mathcal{D}_F$ and $\mathcal{D}$ we introduce a mapping defined by

$$(\mathcal{T} \curlyvee \sigma)(\tau) = \int_{\mathcal{R}} \mathcal{T}(\tau - y)\, \sigma(y)\, dy, \tag{10}$$

where $\mathcal{T} \in \mathcal{D}_F$ and $\sigma \in \mathcal{D}$ are arbitrary.

Theorem 6.1. *Let $\mathcal{T} \in \mathcal{D}_F, \mathcal{T} = \mathcal{F}_d \upsilon, \upsilon \in \mathcal{E}'$, and $\sigma \in \mathcal{D}$ then*

$$\mathcal{F}_d (f \bullet \sigma)(\tau) = (\mathcal{T} \curlyvee \sigma)(\tau).$$

Proof. Let $\mathcal{T} \in \mathcal{D}_F, \mathcal{T} = \mathcal{F}_d \upsilon, \upsilon \in \mathcal{E}'$ then, employing Equ.(10) yields

$$\begin{aligned}
\mathcal{F}_d (\upsilon \bullet \sigma)(\tau) &= \left\langle (\upsilon \bullet \sigma)(\varsigma), \tfrac{1}{\sqrt{2\pi i \gamma_1}} \exp\left(\tfrac{i}{2\gamma_1}\left(\varsigma^2 - 2\varsigma\tau + \tau^2\right)\right)\right\rangle \\
&= \tfrac{1}{\sqrt{2\pi i \gamma_1}} \left\langle \upsilon(\varsigma), \left(\exp\left(\tfrac{i}{2\gamma_1}\left(\varsigma^2 - 2\varsigma\tau + \tau^2\right)\right) \star \check{\sigma}\right)(\varsigma)\right\rangle \\
&= \tfrac{1}{\sqrt{2\pi i \gamma_1}} \left\langle \upsilon(\varsigma), \int_{\mathcal{R}} \exp\left(\tfrac{i}{2\gamma_1}\left(t^2 - 2t\tau + \tau^2\right)\right) \sigma(t - \varsigma)\, dt \right\rangle, \\
&= \tfrac{1}{\sqrt{2\pi i \gamma_1}} \left\langle \upsilon(\varsigma), \int_{\mathcal{R}} \exp\left(\tfrac{i}{2\gamma_1}\left((y+\varsigma) - 2(y+\varsigma)\tau + \tau^2\right)\right) \sigma(y)\, dt \right\rangle \\
&= \tfrac{1}{\sqrt{2\pi i \gamma_1}} \int_{\mathcal{R}} \left\langle \upsilon(\varsigma), \exp\left(\tfrac{i}{2\gamma_1}\left(\varsigma^2 - 2(\tau - y)\varsigma + (\tau - y)^2\right)\right)\right\rangle \sigma(y)\, dt \\
&= \int_{R} \mathcal{T}(\tau - y)\, \sigma(y)\, dy \\
&= (\mathcal{T} \curlyvee \sigma)(\tau),
\end{aligned}$$

where $\mathcal{T} = \mathcal{F}_d \upsilon, \sigma \in \mathcal{D}$ and $\star$ is the convolution product.

The proof of this theorem is therefore completed.

Lemma 6.2. $\mathcal{T} \curlyvee \sigma \in \mathcal{D}_F$, *for every* $\mathcal{T} \in \mathcal{D}_F$ and $\sigma \in \mathcal{D}$.

Proof. Let $\upsilon \in \mathcal{E}', \sigma \in \mathcal{D}$, then $\upsilon \bullet \sigma \in \mathcal{E}'$. From Theorem 6.1 we get $\mathcal{T} \curlyvee \sigma = \mathcal{F}_d (\upsilon \bullet \sigma) \in \mathcal{D}_F, \mathcal{T} = \mathcal{F}_d \upsilon$. Hence, we proved the lemma.

Lemma 6.3. *If* $\mathcal{T} \in \mathcal{D}_F, \sigma_1, \sigma_2 \in \mathcal{D}$ *then the following are true*:
(1) $\sigma_1 \curlyvee \sigma_2 = \sigma_2 \curlyvee \sigma_1$; (2) $\mathcal{T} \curlyvee (\sigma_2 \curlyvee \sigma_1) = (\mathcal{T} \curlyvee \sigma_1) \curlyvee \sigma_2$.

Proof of this theorem is a straightforward consequence of Theorem 6.1 and properties of the integral operator $\int$.

Lemma 6.4. *If* $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{D}_F, (\epsilon_n) \in \Delta$ *and* $\mathcal{T}_1 \curlyvee \epsilon_n = \mathcal{T}_2 \curlyvee \epsilon_n$ *then* $\mathcal{T}_1 = \mathcal{T}_2$.

Lemma 6.5. *If* $\mathcal{T}_n \to \mathcal{T} \in \mathcal{D}_F, \sigma \in \mathcal{D}$ *then* $\mathcal{T}_n \curlyvee \sigma \to \mathcal{T}_n \curlyvee \sigma$ *as* $n \to \infty$.

Lemma 6.6. *If* $\mathcal{T}_n \to \mathcal{T} \in \mathcal{D}_F, (\epsilon_n) \in \Delta$ *then* $\mathcal{T}_n \curlyvee \epsilon_n \to \mathcal{T}$ *as* $n \to \infty$.

Proofs of Lemma 6.4-Lemma 6.6 are straightforward.

From the above conclusions, the space $\mathcal{H}_{\beta_4}(\mathcal{D}_F, \mathcal{D}, \Delta, \curlyvee)$, $\mathcal{H}_{\beta_4}$, can be considered as a Boehmian space. In $\mathcal{H}_{\beta_4}$, as earlier, addition, multiplication by a scalar, the operation $\curlyvee$ and differentiation are normally defined.

The operation $\curlyvee : \mathcal{H}_{\beta_4} \times \mathcal{D} \to \mathcal{H}_{\beta_4}$ is given by the following definition:

Definition 6.7. $\left[\dfrac{\mathcal{T}_n}{\epsilon_n}\right] \curlyvee \phi = \left[\dfrac{\mathcal{T}_n \curlyvee \phi}{\epsilon_n}\right], \left[\dfrac{\mathcal{T}_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_4}, \ \phi \in \mathcal{D}$.

Definition 6.8. A sequence $(\beta_n) \in \mathcal{H}_{\beta_4}$ is δ-*convergent* to $\beta \in \mathcal{H}_{\beta_4}$, if there exists (ϵ_n) such that $(\beta_n \curlyvee \epsilon_n), (\beta \curlyvee \epsilon_n) \in \mathcal{H}_{\beta_4}$, $\forall k, n \in \mathbf{N}$, and $(\beta_n \curlyvee \epsilon_k) \to (\beta \curlyvee \epsilon_k) \in \mathcal{H}_{\beta_4}$ as $n \to \infty$, for every $k \in \mathbf{N}$.

In other words,

Lemma 6.9. *$\beta_n \xrightarrow{\delta} \beta$ in $\mathcal{H}_{\beta_4}$ if and only if there is $\mathcal{T}_{n,k}, \mathcal{T}_k \in \mathcal{D}_F$ and $(\epsilon_k) \in \Delta$ such that $\beta_n = \left[\frac{\mathcal{T}_{n,k}}{\epsilon_k}\right], \beta = \left[\frac{\mathcal{T}_k}{\epsilon_k}\right]$ and $\forall k \in \mathbf{N}, \mathcal{T}_{n,k} \to \mathcal{T}_k \in \mathcal{D}_F$* as $n \to \infty$.

Definition 6.10. $(\beta_n) \in \mathcal{H}_{\beta_4}$ is Δ-*convergent* to $\beta \in \mathcal{H}_{\beta_4}$ if there is a $(\epsilon_n) \in \Delta$ such that $(\beta_n - \beta) \curlyvee \epsilon_n \in \mathcal{H}_{\beta_4}, \forall n \in \mathbf{N}$, and $(\beta_n - \beta) \curlyvee \epsilon_n \to 0$ as $n \to \infty$ in $\mathcal{H}_{\beta_4}$.

Definition 6.11. *The extended diffraction Fresnel transform* of $\left[\frac{f_n}{\epsilon_n}\right]$ is defined by

$$\hat{\mathcal{F}}_d\left(\left[\frac{\upsilon_n}{\epsilon_n}\right]\right) = \left[\frac{\mathcal{T}_n}{\epsilon_n}\right], \tag{11}$$

where $\mathcal{T}_n = \mathcal{F}_d \upsilon_n$.

Theorem 6.12. *$\hat{\mathcal{F}}_d : \mathcal{H}_{\beta_3} \to \mathcal{H}_{\beta_4}$ is well-defined, linear and independent of the representative.*

Proof. Let $\left[\frac{\upsilon_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_3}$ then $\upsilon_n \bullet \epsilon_m = \upsilon_m \bullet \epsilon_n, \forall n, m \in \mathbf{N}$. Applying the diffraction Fresnel transform and using Theorem 6.1 yield

$$\mathcal{T}_n \curlyvee \epsilon_m = \mathcal{T}_m \curlyvee \epsilon_n,$$

where $\mathcal{T}_n = \mathcal{F}_d \upsilon_n, \mathcal{T}_m = \mathcal{F}_d \upsilon_m$, for every $m, n \in \mathbf{N}$. Therefore $\left[\frac{\mathcal{T}_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_4}$. To show $\hat{\mathcal{F}}_d$ is well defined, let $\left[\frac{g_n}{\alpha_n}\right] = \left[\frac{\upsilon_n}{\epsilon_n}\right]$ in $\mathcal{H}_{\beta_3}$ then $\upsilon_n \bullet \alpha_m = g_m \bullet \epsilon_n$.

Once again, applying the diffraction Fresnel transform and using Theorem 6.1 yield $\mathcal{T}_n \curlyvee \alpha_m = H_m \curlyvee \epsilon_n$ where, $\mathcal{T}_n = \mathcal{F}_d \upsilon_n$ and $H_n = \mathcal{F}_d g_n, \forall n \in \mathbf{N}$. Hence, $\left[\frac{\mathcal{T}_n}{\epsilon_n}\right] = \left[\frac{H_n}{\alpha_n}\right]$.

Next, let $k_1, k_2 \in \mathcal{R}$ then, using Theorem 6.1, we have

$$\hat{\mathcal{F}}_d\left(k_1\left[\frac{\upsilon_n}{\epsilon_n}\right] + k_2\left[\frac{g_n}{\alpha_n}\right]\right) = \left[\frac{k_1 \mathcal{T}_n \curlyvee \epsilon_n + k_2 H_n \curlyvee \alpha_n}{\epsilon_n \curlyvee \alpha_n}\right].$$

That is $\hat{\mathcal{F}}_d\left(k_1\left[\frac{\upsilon_n}{\epsilon_n}\right] + k_2\left[\frac{g_n}{\alpha_n}\right]\right) = k_1\left[\frac{\mathcal{T}_n}{\epsilon_n}\right] + k_2\left[\frac{H_n}{\alpha_n}\right]$, $\mathcal{T}_n = \mathcal{F}_d \upsilon_n, H_n = \mathcal{F}_d g_n$. This completes the proof of the theorem.

Theorem 6.13. *The mapping $\hat{\mathcal{F}}_d : \mathcal{H}_{\beta_3} \to \mathcal{H}_{\beta_4}$ is bijection.*

Proof. Assume that $\left[\frac{\mathcal{T}_n}{\epsilon_n}\right] = \left[\frac{H_n}{\alpha_n}\right]$, where $\mathcal{T}_n = \mathcal{F}_d \upsilon_n, H_n = \mathcal{F}_d g_n, \forall n$, then $\mathcal{T}_n \curlyvee \alpha_m = H_m \curlyvee \epsilon_n$. By Theorem 6.1, $\mathcal{F}_d(\upsilon_n \bullet \alpha_m) = \mathcal{F}_d(g_n \bullet \epsilon_n)$ and hence $\upsilon_n \bullet \alpha_m = g_n \bullet \epsilon_n$. Hence, $\left[\frac{\upsilon_n}{\epsilon_n}\right] = \left[\frac{g_n}{\alpha_n}\right]$. Finally, let $\left[\frac{\mathcal{T}_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_4}$ then $\mathcal{T}_n = \mathcal{F}_d \upsilon_n$, for some υ_n and all n.

Therefore, $\left[\frac{\upsilon_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_3}$ satisfies $\hat{\mathcal{F}}_d\left(\left[\frac{\upsilon_n}{\epsilon_n}\right]\right) = \left[\frac{\mathcal{T}_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_4}$.

The theorem is completely proved.

Definition 6.14. Let $\left[\frac{\mathcal{T}_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_4}$. The *inverse* of $\hat{\mathcal{F}}_d$ is defined by

$$\left(\hat{\mathcal{F}}_d\right)^{-1}\left(\left[\frac{\mathcal{T}_n}{\epsilon_n}\right]\right) = \left[\frac{\upsilon_n}{\epsilon_n}\right], \tag{12}$$

in $\mathcal{H}_{\beta_1}, \mathcal{T}_n = \hat{\mathcal{F}}_d \upsilon_n, \upsilon_n \in \mathcal{E}'$.

Theorem 6.15. $\left(\hat{\mathcal{F}}_d\right)^{-1} : \mathcal{H}_{\beta_4} \to \mathcal{H}_{\beta_3}$ *is well-defined, linear and bijection mapping.*

This theorem can be proved easily by using the same analysis employed for Theorem 6.12. Detailed proof is, thus, avoided.

Theorem 6.16. $\hat{\mathcal{F}}_d : \mathcal{H}_{\beta_3} \to \mathcal{H}_{\beta_4}, \left(\hat{\mathcal{F}}_d\right)^{-1} : \mathcal{H}_{\beta_4} \to \mathcal{H}_{\beta_3}$ *are continuous with respect to* δ *convergence.*

Proof. Let $(\beta_n) \in \mathcal{H}_{\beta_1}, \beta \in \mathcal{H}_{\beta_1}$ be such that $\beta_n \xrightarrow{\delta} \beta$. By using Lemma 2.3, there can be found $\upsilon_{n,k}, \upsilon_k \in \mathcal{E}', (\epsilon_n) \in \Delta$, such that $\beta_n = \left[\frac{\upsilon_{n,k}}{\epsilon_n}\right], \beta = \left[\frac{\upsilon_k}{\epsilon_n}\right]$ and $\upsilon_{n,k} \to \upsilon_k$ for every $k \in \mathbf{N}$ as $n \to \infty$ in $\mathcal{E}'$. Continuity of $\mathcal{F}_d$ implies $\mathcal{F}_d(\upsilon_{n,k}) \to \mathcal{F}_d(\upsilon_k)$ as $n \to \infty$ in $\mathcal{D}_F$. Hence, $\left[\frac{\mathcal{T}_{n,k}}{\epsilon_n}\right] \to \left[\frac{\mathcal{T}_k}{\epsilon_n}\right]$, where $\mathcal{T}_{n,k} = \mathcal{F}_d(\upsilon_{n,k}), \mathcal{T}_k = \mathcal{F}_d(\upsilon_k)$. Next, let $\tilde{\beta}, \left(\tilde{\beta}_n\right) \in \mathcal{H}_{\beta_4}$ then, there are $\mathcal{T}_{n,k}, \mathcal{T}_k \in \mathcal{D}_F$ such that $\tilde{\beta}_n = \left[\frac{\mathcal{T}_{n,k}}{\epsilon_n}\right], \tilde{\beta} = \left[\frac{\mathcal{T}_k}{\epsilon_n}\right], \mathcal{T}_{n,k} = \mathcal{F}_d(\upsilon_{n,k}), \mathcal{T}_k = \mathcal{F}_d(\upsilon_k)$, and $\mathcal{T}_{n,k} \to \mathcal{T}_k$ for every k, as $n \to \infty$. Hence $\upsilon_{n,k} \to \upsilon_k$ as $n \to \infty$ for every $k \in \mathbf{N}$ in $\mathcal{E}'$. Thus,

$$\left[\frac{\upsilon_{n,k}}{\epsilon_n}\right] = \left(\hat{\mathcal{F}}_d\right)^{-1}\left(\left[\frac{\mathcal{T}_{n,k}}{\epsilon_n}\right]\right) \to \left[\frac{\upsilon_k}{\epsilon_n}\right] = \left(\hat{\mathcal{F}}_d\right)^{-1}\left(\left[\frac{\mathcal{T}_k}{\epsilon_n}\right]\right)$$

as $n \to \infty$ for every k. Hence, the theorem.

Theorem 6.17. $\hat{\mathcal{F}}_d : \mathcal{H}_{\beta_3} \to \mathcal{H}_{\beta_4}, \left(\hat{\mathcal{F}}_d\right)^{-1} : \mathcal{H}_{\beta_4} \to \mathcal{H}_{\beta_3}$ *are continuous with respect to* Δ *convergence.*

Proof of this theorem follows from repeated analysis employed for Theorem 5.6.

Theorem 6.18. *The mappings* $\mathcal{D}_F \to \mathcal{H}_{\beta_4}, \mathcal{T} \to \left[\frac{T \curlyvee \epsilon_n}{\epsilon_n}\right]$ and, $\mathcal{E}' \to \mathcal{H}_{\beta_3}$, $f \to \left[\frac{f \bullet \epsilon_n}{\epsilon_n}\right]$ *are continuous imbedding mappings with respect to* δ *convergence.*

See Theorem 3.6 for similar proof.

Conclusion.

The diffraction Fresnel transform which is a generalization of the Fresnel transform is extended to spaces of generalized functions, namely, Boehmian spaces. The extended transform maintains most of its general properties which have in the classical sense.

Acknowledgment. The author would like to thank the referee for his valuable suggestions towards the improvement of the content of this article.

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Accepted: 20.03.2012