A STUDY ON AUGMENTED GRADED RINGS[1]

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Abstract. In this paper, we study some properties of augmented graded rings and give the relationships between augmented graded rings and other types of well known strongly graded rings.

Keywords: augmented graded rings, augmented graded modules, strongly graded rings.

2010 AMS Subject Classifications: 13A02.

1. Introduction

Let G be a group with identity e. A ring R is said to be G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The G-graded ring is denoted by (R, G). We denote by $\text{supp}(R, G)$ the support of G which is defined to be $\{g \in G : R_g \neq 0\}$. The elements of R_g are called homogeneous of degree g and R_e (the identity component of R) is a subring of R and $1 \in R_e$ (see [4]). For $x \in R$, x can be written uniquely as $\sum_{g \in G} x_g$ where x_g is the component of x in R_g. Also we write $h(R) = \bigcup_{g \in G} R_g$.

One of the most important problems in graded ring theory is to study the link between a certain property for R or (R, G) and R_e (the identity component of R). One can think about grading R_e by a group G. These types of rings appear naturally when we deal with the group ring of G over a G-graded ring R.

In [5], we studied the G-graded rings in which the identity component is itself a G-graded ring satisfying some related conditions with the graduation of G. We called these rings augmented G-graded rings. Also, we gave the relationship between these new rings and the stronger properties of G-graded rings given in [6].

[1]This research was supported by Yarmouk University.

This simple effort is useful for studying those rings in detail and that is a good tool to solve many problems in Graded Ring Theory. In [6], we defined three successively stronger properties that a grading may have and investigated the relationship between these new strongly gradings and the stronger nondegenerate and faithful properties which are motivated by the work of Cohen and Rowen.

In this paper, we follow the work done in [5], [6] to give further properties of augmented graded rings. Also, we give some relationships between these rings and other types of graded rings like nondegenerate, faithful, strong, first strong, etc.

2. Preliminaries

In this section, we give some preliminaries concerning graded rings. For more details, one can look in [4], [5], [6], [8].

Definition 2.1. Let R be a G-graded ring. Then (R,G) is said to be left (resp., right) nondegenerate if for every $r \in R - \{0\}$, $(rR)_e = \{(rx)_e : x \in R\} \neq 0$ (resp., if for every $r \in R - \{0\}$, $(Rr)_e = \{(xr)_e : x \in R\} \neq 0$. Otherwise, (R,G) is left degenerate (resp., right degenerate). Also, (R,G) is nondegenerate if it is both left and right nondegenerate.

Lemma 2.2. *Let R be a G-graded ring and $a_g \in R_g$ with $g \in G$. Then*

1. $(a_g R)_e = a_g R_{g^{-1}}$.
2. $(R a_g)_e = R_{g^{-1}} a_g$.

Proposition 2.3. *Let R be a G-graded ring. Then (R,G) is nondegenerate if and only if $R_{g^{-1}} a_g \neq 0$ and $a_g R_{g^{-1}} \neq 0$ for all $a_g \in R_g - \{0\}$.*

Corollary 2.4. *Suppose (R,G) is nondegenerate. If $g \in supp(R,G)$ then $g^{-1} \in supp(R,G)$.*

Definition 2.5. Let R be a G-graded ring. Then (R,G) is said to be left (resp., right) faithful if for each $a_g \in R_g - \{0\}$, $a_g R_h \neq 0$ for all $g,h \in G$ (resp., $R_h a_g \neq 0$ for all $g,h \in G$). We say that (R,G) is faithful if it is both left and right faithful.

Definition 2.6. A G-graded ring (R,G) is said to be strongly graded if $R_g R_h = R_{gh}$ for all $g,h \in G$.

Proposition 2.7. *Let R be a G-graded ring. Then, (R,G) is strong if and only if $R_g R_{g^{-1}} = R_e$ for all $g \in G$. This is equivalent to $1 \in R_g R_{g^{-1}}$ for all $g \in G$.*

Definition 2.8. Let R be a G-graded ring. Then, (R,G) is first strong if $1 \in R_g R_{g^{-1}}$, for all $g \in \mathrm{supp}(R,G)$, or, equivalently, if $R_g R_{g^{-1}} = R_e$, for all $g \in \mathrm{supp}(R,G)$.

Remark 2.9. Every strongly graded is first strong and faithful but the converse is not true (see [6]). Indeed, (R,G) is strong if and only if it is first strong and $\mathrm{supp}(R,G) = G$.

Proposition 2.10. *If (R, G) is first strong then supp(R, G) is a subgroup of G.*

Proposition 2.11. *Let R be a first strongly graded ring. Suppose R_g is simple R_e-submodule of R for all $g \in G$. Then R is gr-simple.*

Proof. Suppose R_g is simple R_e-module for all $g \in G$. Let $0 \neq Y \subseteq R$ be a graded R-submodule of R. Then there exists $g \in G$ such that $Y_g = Y \bigcap R_g \neq 0$. But $Y_g \subseteq R_g$ is a nonzero R_e-submodule. Hence $Y_g = Y \bigcap R_g = R_g$ or $R_g \subseteq Y$ and then $R_e = R_{g^{-1}} R_g \subseteq R_{g^{-1}} Y \subseteq Y$. Thus $1 \in Y$. Since 1 is invertible we get $Y = R$. Therefore, R is gr-simple. ∎

Definition 2.12. Let R be a G-graded ring. Then (R, G) is said to be second strong if supp(R, G) is a monoid in G and $R_g R_h = R_{gh}$ for all $g, h \in$ supp(R, G).

Remark 2.13. Every first strongly graded ring is second strong but the converse is not true in general (see [6]).

Definition 2.14. A G-graded ring R is said to be left semifaithful if

1. supp(R, G) is a subgroup of G.
2. For all $g, h \in$ supp(R, G) and $a_g \in R_g - \{0\}$ we have $a_g R_h \neq 0$.

   Similarly we define right semifaithful graded rings. So, (R, G) is called semifaithful if it is both left and right semifaithful.

Proposition 2.15. *Every semifaithful graded ring is nondegenerate.*

Proof. Suppose (R, G) is semifaithful. Let $g \in$ supp(R, G). Then $g^{-1} \in$ supp(R, G) and hence $a_g R_{g^{-1}} \neq 0$ for all $a_g \in R_g - \{0\}$. Thus (R, G) is left nondegenerate. Similarly we prove the right nondegeneracy. ∎

Corollary 2.16. *Every first strongly graded ring is semifaithful.*

Proof. Assume (R, G) is first strong. By Proposition 2.10, supp(R, G) is a subgroup of G. Let $g, h \in$ supp(R, G) and $a_g \in R_g - \{0\}$. Suppose for the contrary $a_g R_h = 0$. Thus $0 = a_g R_h R_{h^{-1}} = a_g R_e$. Hence $a_g = 0$ which contradicts the fact $a_g \neq 0$ and hence R is left semifaithful. Similarly, we show R is right semifaithful. ∎

Proposition 2.17. *Let R be a G-graded ring. Then (R, G) is faithful if and only if (R, G) is semifaithful and supp$(R, G) = G$.*

Definition 2.18. A ring R is said to be augmented G-graded ring if it satisfies the following conditions:

1. $R = \bigoplus_{g \in G} R_g$ where R_g is an additive subgroup of R and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.

2. If R_e is the identity component of the graduation, then $R_e = \bigoplus_{g\in G} R_{e-g}$ where R_{e-g} is an additive subgroup of R_e and $R_{e-g}R_{e-h} \subseteq R_{e-gh}$ for all $g, h \in G$.

3. For each $g \in G$, there exists $r_g \in R_g$ such that $R_g = \bigoplus_{h\in G} R_{e-h}r_g$. We assume $r_e = 1$.

4. If $g, h \in G$ and r_g, r_h are both nonzero, then $r_g r_h = r_{gh}$ and for all $x, y \in R_e$ we have $(xr_g)(yr_h) = xyr_{gh}$.

Remark 2.19. By the last definition we have:

1. Condition 3 of the definition implies $R_h = R_e r_h$ for all $h \in G$.

2. R_g is a G-graded R_e-module with the usual multiplication on R and with the graduation $R_{g-h} = R_{e-h}r_g$ for all $h \in G$.

3. $R_{g-h}R_{g'-h'} \subseteq R_{gg'-hh'}$ for all $g, g', h, h' \in G$.
   Also, $R_{g-h}R_{g'-h'} = R_{e-h}r_g R_{e-h'}r_{g'}$.

4. If $r_g, r_{g^{-1}}$ are both nonzero, then $r_g R_e = R_e r_g = R_g$.

Corollary 2.20. *Let R be an augmented G-graded ring where supp$(R, G) = G$. Then (R, G) is strong and hence faithful.*

3. Properties of augmented graded rings

In this section, we give some properties of the augmented graded rings as well as its relationships to other kinds of graded rings.

Proposition 3.1. *Let R be a G-graded ring such that supp(R, G) is a subgroup of G. Then (R, G) is first strong.*

Proof. Let $g \in \text{supp}(R, G)$. Then $g^{-1} \in \text{supp}(R, G)$, i.e., $R_g \neq 0$ and $R_{g^{-1}} \neq 0$. Since $R_g = R_e r_g$ and $R_{g^{-1}} = R_e r_{g^{-1}}$, we get $r_g \neq 0$ and $r_{g^{-1}} \neq 0$. Hence $r_g r_{g^{-1}} = r_{gg^{-1}} = r_e = 1$ and then $1 \in R_g R_{g^{-1}}$. Therefore, R is first strong. ∎

Proposition 3.2. *Suppose R is augmented G-graded ring such that supp(R, G) is a monoid in G. Then (R, G) is second strong.*

Proof. Let $g, h \in \text{supp}(R, G)$. Then $r_g \neq 0$ and $r_h \neq 0$. So, $r_g r_h = r_{gh}$ implies $r_{gh} \in R_g R_h$ and hence $R_{gh} = R_e r_{gh} \subseteq R_e R_g R_h = R_g R_h \subseteq R_{gh}$. Thus $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$. Since $\text{supp}(R, G)$ is a monoid in G we have $R, G)$ is second strong. ∎

Corollary 3.3. *If (R, G) is augmented graded ring then $R_g R_h = R_{gh}$ for all $g, h \in$ supp(R, G).*

Corollary 3.4. *Suppose (R, G) is augmented graded ring such that R has no zero divisors in $h(R)$. Then (R, G) is second strong.*

Proof. If $g, h \in \text{supp}(R, G)$ then $r_g \neq 0$ and $r_h \neq 0$ and hence $r_{gh} = r_g r_h \neq 0$ where $r_g, r_h \in h(R)$. Thus $gh \in \text{supp}(R, G)$ and hence $\text{supp}(R, G)$ is a monoid in G. By Proposition 3.2, (R, G) is second strong. ∎

Proposition 3.5. *Let R be an augmented G-graded ring. Then, the following are equivalent:*

1. *(R, G) is nondegenerate.*
2. *supp(R, G) is a subgroup of G.*
3. *(R, G) is first strong.*
4. *(R, G) is semifaithful.*

Proof. (1) implies (2): Assume (R, G) is nondegenerate graded ring. Let $g, h \in \text{supp}(R, G)$. Then $g^{-1} \in \text{supp}(R, G)$ and hence $r_g r_{g^{-1}} = r_e = 1$ where $R_g = R_e r_g$ for $g \in G$. Suppose for the contrary $gh \notin \text{supp}(R, G)$. Then $r_{gh} = 0$. But $r_{gh} = r_g r_h$ implies $r_g r_h = 0$ and hence $r_{g^{-1}} r_g r_h = 0$ or $r_h = 0$ and then $h \notin \text{supp}(R, G)$ which is a contradiction. Therefore, $\text{supp}(R, G)$ is a monoid in G. By Corollary 2.4, $\text{supp}(R, G)$ is a subgroup of G.

(2) implies (3): Follows from Proposition 3.1.

(3) implies (4): Follows from Proposition 2.15.

(4) implies (1): Follows from Proposition 2.14. ∎

Corollary 3.6. *Let (R, G) be an augmented G-graded ring. Then the following are equivalent:*

1. *supp$(R, G) = G$.*
2. *(R, G) is strongly graded.*
3. *(R, G) is faithful.*

Proof. Follows directly from Remark 2.9 and Proposition 3.5. ∎

Proposition 3.7. *Let R be an augmented G-graded ring such that supp$(R, G) = G$. Then the following are equivalent:*

1. *(R, G) is faithful.*
2. *(R, G) is semifaithful.*
3. *(R, G) is nondegenerate.*
4. *(R, G) is first strong.*
5. *(R, G) is second strong.*
6. *(R, G) is strong.*

Proof. (1) implies (2): Follows from Proposition 2.17.

(2) implies (3): Follows from Proposition 2.14.

(3) implies (4): Follows from Proposition 3.5.

(4) implies (5): Follows from Remark 2.13.

(5) implies (6): Follows from Corollary 2.20.

(6) implies (1): Follows from Remark 2.9. ∎

Corollary 3.8. *If (R, G) is augmented and supp$(R, G) = G$, then by Corollary 2.20, (R, G) is faithful, semifaithful, nondegenerate, strong, first strong and second strong at the same time.*

After giving the relationships between the augmented graded rings and other types of strongly graduations, in addition to, nondegeneracy, faithfulness and semifaithfulness, we present some properties of the augmented graded rings.

Proposition 3.9. *Let R be an augmented G-graded ring such that r_g is an R_e-torsion free element (i.e., $mr_g \neq 0$ for all $m \in R_e$) for all $g \in$ supp(R, G). Then R_g is cyclic projective R_e-submodule of R and hence R is projective R_e-module. In particular, R and R_g are free R_e-modules.*

Proof. Let $g \in G$. If $g \notin \text{supp}(R, G)$, then done. Suppose $g \in \text{supp}(R, G)$. Then $r_g \neq 0$ and $R_g = R_e r_g$. Since r_g is R_e-torsion free element, r_g is linearly independent over R_e and then R_g is cyclic projective R_e-module being free R_e-module. Since $R = \bigoplus_{g \in G} R_g$, the set $F = \{r_g : g \in \text{supp}(R, G)\}$ forms a basis of R as an R_e-module. Thus R is free and hence projective. ∎

Corollary 3.10. *By Proposition 3.9, if supp(R, G) is finite, then R is finitely generated free R_e-module. Clearly, $R \cong R_e^{|supp(R,G)|} = R_e^{|F|}$ where $F = \{r_g : g \in supp(R, G)\}$.*

Proposition 3.11. *Let R be an augmented G-graded ring such that supp$R, G)$ is a monoid in G. Then R is commutative if and only if R_e is commutative and supp(R, G) is abelian.*

Proof. Suppose R is commutative. Then clearly R_e is commutative. Let $g, h \in \text{supp}(R, G)$. Then $r_g \neq 0$ and $r_h \neq 0$. But $r_{gh} = r_g r_h = r_h r_g = r_{hg}$. Since $gh, hg \in \text{supp}(R, G)$, $r_{gh} = r_{hg} \neq 0$. Thus $gh = hg$ and supp(R, G) is abelian monoid in G. For the converse, suppose R_e is commutative and $gh = hg$ for all $g, h \in \text{supp}(R, G)$. We prove R is commutative step by step.

Step 1: We show $x_g y_h = y_h x_g$ for all $g, h \in G$. Let $g, h \in G$, $x_g \in R_g = R_e r_g$, $y_h \in R_h = R_e r_h$. If either $g \notin \text{supp}(R, G)$ or $h \notin \text{supp}(R, G)$, then done. Suppose $g, h \in \text{supp}(R, G)$. Then $gh = hg$ and hence $r_{gh} = r_{hg}$. Suppose $x_g = xr_g$ and $y_h = yr_h$ for some $x, y \in R_e$. Then
$x_g y_h = (xr_g)(yr_h) = xyr_{gh} = yxr_{hg} = (yr_h)(xr_g) = y_h x_g$.

Step 2: We claim that $xy_g = y_g x$ for all $x \in R$, $y_g \in R_g$ and $g \in G$. Fix $g \in G$. Suppose $x \in R$ and $y_g \in R_g$. If $g \notin \text{supp}(R, G)$, then $y_g = 0$ and done. Assume $g \in \text{supp}(R, G)$ and let $x = \sum_{h \in G} x_h$ where $x_h \in R_h$ for all $h \in G$. Then

$$xy_g = \left(\sum_{h \in G} x_h) y_g = \sum_{h \in G} (x_h y_g \right) = \sum_{h \in G} (y_g x_h) = y_g \left(\sum_{h \in G} x_h \right) = y_g x.$$

Step 3: We now show that $xy = yx$ for all $x, y \in R$. Let $x, y \in R$ and suppose $y = \sum_{g \in G} y_g$ where $y_g \in R_g$ for all $g \in G$. Then we have

$$xy = x\left(\sum_{g\in G} y_g\right) = \sum_{g\in G} xy_g = \sum_{g\in G} y_g x = \left(\sum_{g\in G} y_g\right) x = yx.$$

Therefore, R is commutative. ■

Proposition 3.12. *Let R be an augmented G-graded ring. Then R_g is simple left R_e-module for all $g \in G$ if and only if R_e is simple left R_e-module.*

Proof. Suppose R_e is simple left R_e-module and $g \in G$. If $g \notin \text{supp}(R, G)$, then $R_g = 0$ is simple R_e-module. Suppose $g \in \text{supp}(R, G)$ and let $r_g \in R_g$ such that $R_g = R_e r_g \neq 0$. Let $M \subseteq R_g$ be a nonzero R_e-submodule. We show $M = Xr_g$ where X is an R_e-submodule of R_e, i.e., X is a left ideal in R_e. Let $X = \{x \in R_e : xr_g \in M\}$. Then clearly, $0 \in X$, and hence $X \neq \emptyset$. Since $M \subseteq R_g$ and $M \neq 0$, there exists $a_g \in M - \{0\}$ such that $a_g = xr_g$ for some $x \in R_e$. So, $x \in X - \{0\}$, i.e., $X \neq 0$. Let $x, y \in X$. Then $xr_g, yr_g \in M$. Hence $(x - y)r_g = x_{rg} - yr_g \in M$. Since $x - y \in R_e$, $x - y \in X$, i.e., X is a subgroup of R_e. Let $x \in X$ and $y_e \in R_e$. Then $xr_g \in M$ and hence $(y_e x)r_g = y_e(xr_g) \in R_e M = M$. Since $y_e x \in R_e$, $y_e x \in X$. Thus X is an R_e-submodule of R_e. Now, if $m \in M \subseteq R_g$, then $m = yr_g$ for some $y \in R_e$. So, $y \in X$ and $m \in Xr_g$. If $m \in Xr_g$, then $m = xr_g$ for some $x \in X$. By the definition of X we have $m \in M$. Therefore, $M = Xr_g$. Since R_e is simple R_e-module, $X = R_e$ and hence $M = R_e r_g = R_g$, i.e., R_g is simple R_e-module for all $g \in G$. The converse is obvious. ■

Corollary 3.13. *Let R be an augmented G-graded ring. If R is gr-simple, then R_g is simple left R_e-module for all $g \in G$.*

Proof. Follows directly from Proposition 3.12. ■

Corollary 3.14. *Suppose R is an augmented G-graded ring and $N \subseteq R_g$ is R_e-submodule. Then $N = Xr_g$ where X is a left ideal in R_e for all $g \in G$.*

Corollary 3.15. *Suppose R is an augmented G-graded ring such that supp(R, G) is a subgroup of G. Then the following are equivalent:*

1. *(R, G) is gr-simple.*
2. *R_e is simple ring (R_e-module).*
3. *R_g is simple R_e-module for all $g \in G$.*

Proof. (1) implies (2): Trivial.

(2) implies (3): Follows from Proposition 3.12.

(3) implies (1): Follows from Proposition 2.11. ■

Remark 3.16. Suppose R is an augmented G-graded ring such that $\text{supp}(R, G)$ is a subgroup of G. Then we have the following:

R is simple ring $\Rightarrow$ R is gr-simple ring $\Leftrightarrow$ R_g is simple R_e-module for all $g \in G \Leftrightarrow R_e$ is simple ring.

Proposition 3.17. *Let R be an augmented G-graded ring such that $\text{supp}(R, G) = H$ is a subgroup of G and M be a G-graded left R-module. Let x be a homogeneous element in M. Then x is R-torsion free if and only if x is R_e-torsion free.*

Proof. Suppose $x \in h(M)$ and x is R-torsion free. Then x is R_e-torsion free. Conversely, suppose $x \in h(M)$ and x is R_e-torsion free. Let $\omega \in R$ such that $\omega x = 0$. Suppose $\omega = \sum_{g \in G} \omega_g = \sum_{g \in H} \omega_g$. Hence $\sum_{g \in H} \omega_g x = 0$ and so $\omega_g x = 0$ for all $g \in H$. Let $g \in H$ and assume $\omega_g = r_g \lambda^{(g)} = \lambda^{(g)} r_g$ where $\lambda^{(g)} \in R_e$ for all $g \in H$ (Notice that H is a subgroup of G). So, $(r_g \lambda^{(g)})x = 0$ or $r_g(\lambda^{(g)} x) = 0$ and hence $R_e r_g(\lambda^{(g)} x) = 0$, i.e., $R_g(\lambda^{(g)} x) = 0$. Thus $R_{g^{-1}} R_g(\lambda^{(g)} x) = 0$. So, $R_e(\lambda^{(g)} x) = 0$, i.e., $\lambda^{(g)} x = 0$. Hence $\lambda^{(g)} = 0$ for all $g \in H$. Therefore, $\omega_g = 0$ for all $g \in H$ and hence $\omega = 0$, i.e., x is R-torsion free. ∎

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Accepted: 3.10.2011