RARELY $b$-CONTINUOUS FUNCTIONS

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Abstract. In this paper we introduce a new class of functions called rarely $b$-continuous. Some characterizations and several properties concerning rare $b$-continuity are obtained.

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1. Introduction and preliminaries

In 1979, Popa [15] introduced the notion of rarely continuous functions as a generalization of weak continuity. The function has been further investigated by Long and Herrington in [12] and by various authors [5], [6], [7], [8], [9], [16]. The first author of this article introduced and investigated weak $b$-continuity [17] as a generalization of weak continuity. The purpose of the present paper is to introduce concept of rare $b$-continuity in topological spaces as a generalization of rare continuity and weak $b$-continuity. We investigate several properties of rarely $b$-continuous functions. Rare $b$-continuity implied by rare precontinuity and rare quasi continuity and implies rare $\beta$-continuity. The notion of $I.b$-continuity is also introduced which is weaker than $b$-continuity and stronger than $b$-continuity. It is shown that if $Y$ is a regular space, then the function $f : X \to Y$ is $I.b$-continuous on $X$ if and only if $f$ is rarely $b$-continuous on $X$. 
Throughout this paper, $X$ and $Y$ are topological spaces. Recall that a rare set $R$ is a set $R$ such that $\text{int}(R) = \emptyset$. A subset $S$ of a space $(X, \tau)$ is called regular open [18] (resp. regular closed [18]) if $S = \text{int}(\text{cl}(S))$ (resp. $S = \text{cl}(\text{int}(S))$).

A subset $S$ of a space $(X, \tau)$ is called semi-open [11] (resp. preopen [13], $\alpha$-open [14], semi-preopen [2] or $\beta$-open [1], $b$-open [3] or $\gamma$-open [4]) if $S \subset \text{cl}(\text{int}(S))$ (resp. $S \subset \text{int}(\text{cl}(S))$, $S \subset \text{cl}(\text{int}(S))$, $S \subset \text{cl}(\text{int}(S)) \cup \text{int}(\text{cl}(S))$) The complement of a semi-open (resp. preopen, $\alpha$-open, $\beta$-open, $b$-open) set is said to be semi-closed (resp. preclosed, $\alpha$-closed, $\beta$-closed, $b$-closed).

The family of all open (resp., regular open, semi-open, preopen, $\alpha$-open, $\beta$-open, $b$-open) sets of $X$ denoted by $O(X)$ (resp., $RO(X)$, $SO(X)$, $PO(X)$, $\alpha O(X)$, $\beta O(X)$, $BO(X)$).

The family of all $b$-closed sets of $X$ is denoted by $BC(X)$ and the family of all $b$-open (resp. open, regular open) sets of $X$ containing a point $x \in X$ is denoted by $BO(X, x)$ (resp., $O(X, x)$, $RO(X, x)$).

If $S$ is a subset of a space $X$, then the $b$-closure of $S$, denoted by $bcl(S)$, is the smallest $b$-closed set containing $S$. The $b$-interior of $S$, denoted by $bint(S)$ is the largest $b$-open set contained in $S$. Our next definition contains some types of functions used throughout this paper.

**Definition 1** A function $f : X \rightarrow Y$ is called:

(a) Weakly continuous [10] (resp. weakly $b$-continuous [17]) if for each $x \in X$ and each open set $G$ containing $f(x)$, there exists $U \subset O(X, x)$ (resp., $U \subset BO(X, x)$) such that $f(U) \subset \text{cl}(G)$.

(b) $b$-continuous [4] if $f^{-1}(V)$ is $b$-open in $X$ for every open set $V$ of $Y$;

(c) Rarely continuous [15] (resp., rarely precontinuous [8], rarely quasicontinuous [16], rarely $\beta$-continuous [7]) at $x \in X$ if for each $G \subset O(Y, f(x))$, there exists a rare set $R_G$ with $G \cap \text{cl}(R_G) = \emptyset$ and $U \in RO(X, x)$, $U \in SO(X, x)$, $U \in \beta O(X, x)$ such that $f(U) \subset G \cup R_G$.

2. Rarely $b$-continuous functions

**Definition 2** A function $f : X \rightarrow Y$ is called rarely $b$-continuous at $x \in X$ if for each open set $G \subset Y$ containing $f(x)$, there exists a rare set $R_G$ with $G \cap \text{cl}(R_G) = \emptyset$ and $U \in BO(X, x)$ such that $f(U) \subset G \cup R_G$.

**Theorem 3** The following statements are equivalent for a function $f : X \rightarrow Y$:

(a) The function is rarely $b$-continuous at $x \in X$.

(b) For each $G \in O(Y, f(x))$, there exists a rare set $R_G$ with $G \cap \text{cl}(R_G) = \emptyset$ such that $x \in bint(f^{-1}(G \cup R_G))$. 
For each \( G \in O(Y, f(x)) \), there exists a rare set \( R_G \) with \( \overline{cl}(G) \cap R_G = \emptyset \) such that
\[
x \in \text{bint}(f^{-1}(\overline{cl}(G) \cup R_G)).
\]

For each \( G \in RO(Y, f(x)) \), there exists a rare set \( R_G \) with \( G \cap \overline{cl}(R_G) = \emptyset \) such that
\[
x \in \text{bint}(f^{-1}(G \cup R_G)).
\]

For each \( G \in O(Y, f(x)) \), there exists \( U \in BO(X, x) \) such that
\[
\text{int}(f(U) \cap (Y - G)) = \emptyset.
\]

For each \( G \in O(Y, f(x)) \), there exists \( U \in BO(X, x) \) such that
\[
\text{int}(f(U)) \subset \overline{cl}(G).
\]

**Proof.** (a) \( \Rightarrow \) (b): Let \( x \in X \) and \( G \in O(Y, f(x)) \). Then, there exists a rare set \( R_G \) with \( \overline{cl}(R_G) = \emptyset \) and \( U \in BO(X, x) \) such that \( f(U) \subset G \cup R_G \). It follows that \( x \in U \subset f^{-1}(G \cup R_G) \), then we have \( x \in \text{bint}(f^{-1}(G \cup R_G)) \).

(b) \( \Rightarrow \) (c): Suppose that \( G \in O(Y, f(x)) \). Then, there exists a rare set \( R_G \) with \( G \cap \overline{cl}(R_G) = \emptyset \) such that \( x \in \text{bint}(f^{-1}(G \cup R_G)) \). Since \( G \cap \overline{cl}(R_G) = \emptyset \), \( R_G \subset Y - G \) where \( Y - G = (Y - \overline{cl}(G)) \cup (\overline{cl}(G) - G) \). Now, we have \( R_G \subset (R_G \cap (Y - \overline{cl}(G))) \cup (\overline{cl}(G) - G) \). Set \( R^* = R_G \cap (Y - \overline{cl}(G)) \). It follows that \( R^* \) is a rare set with \( \overline{cl}(G) \cap R^* = \emptyset \). Therefore, \( x \in \text{bint}(f^{-1}(G \cup R_G)) \subset \text{bint}(f^{-1}(\overline{cl}(G) \cup R^*)) \).

(c) \( \Rightarrow \) (d): Assume that \( x \in X \) and \( G \in RO(Y, f(x)) \). Then, there exists a rare set \( R_G \) with \( \overline{cl}(G) \cap R_G = \emptyset \) such that \( x \in \text{bint}(f^{-1}(\overline{cl}(G) \cup R_G)) \). Set \( R^* = R_G \cap (Y - \overline{cl}(G)) \). It follows that \( R^* \) is a rare set and \( G \cap \overline{cl}(R^*) = \emptyset \). Hence \( x \in \text{bint}(f^{-1}(\overline{cl}(G) \cup R_G)) = \text{int}(f(U) \cap (Y - G)) = \text{int}(f(U)) \cap (Y - G) \in \text{int}(\overline{cl}(G) \cup \overline{int}(R_G)) \subset ((\overline{cl}(G) \cup \overline{int}(R_G)) \cap (Y - \overline{cl}(G)) = \emptyset.

(d) \( \Rightarrow \) (e): Let \( G \in O(Y, f(x)) \). Then, using \( f(U) \subset G \subset \text{int}(\overline{cl}(G)) \) and the fact that \( \text{int}(\overline{cl}(G)) \in RO(Y, f(x)) \), there exists a rare set \( R_G \) with \( \text{int}(\overline{cl}(G)) \cap \overline{cl}(R_G) = \emptyset \) such that \( x \in \text{bint}(f^{-1}(\text{int}(\overline{cl}(G)) \cup R_G)) \). Suppose \( U = \text{bint}(f^{-1}(\text{int}(\overline{cl}(G)) \cup R_G)) \). Then, \( U \in BO(X, x) \) and, therefore, \( f(U) \subset \text{int}(\overline{cl}(G)) \cup R_G \). We have \( \text{int}(f(U) \cap (Y - G)) = \text{int}(f(U)) \cap (Y - G) \subset \text{int}((\overline{cl}(G) \cup \overline{cl}(R_G)) \cap (Y - \overline{cl}(G)) = \emptyset.

(e) \( \Rightarrow \) (f): Since \( \text{int}(f(U) \cap (Y - G)) = \text{int}(f(U)) \cap (Y - G) = \text{int}(f(U)) \subset \overline{cl}(G) \).

(f) \( \Rightarrow \) (a): \( G \in O(Y, f(x)) \). Then, by (f), there exists \( U \in BO(X, x) \) such that \( \text{int}(f(U)) \subset \overline{cl}(G) \). Then, \( f(U) = (f(U) - \text{int}(f(U))) \cup \text{int}(f(U)) \subset (f(U) - \text{int}(f(U))) \cup \overline{cl}(G) = (f(U) - \text{int}(f(U))) \cup \overline{cl}(G) \). Set \( R^* = (f(U) - \text{int}(f(U))) \cap (Y - G) \) and \( R^{**} = (\overline{cl}(G) - G) \). Then, \( R^* \) and \( R^{**} \) are rare sets. Moreover, \( R_G = R^* \cup R^{**} \) is a rare set and \( \overline{cl}(R_G) \cap G = \emptyset \) and \( f(U) \subset G \cup R_G \).

**Theorem 4** A function \( f : X \to Y \) is rarely \( b \)-continuous if and only if \( f^{-1}(G) \subset \text{bint}(f^{-1}(G \cup R_G)) \) for every open set \( G \) in \( Y \), where \( R_G \) is a rare set with \( G \cap \overline{cl}(R_G) = \emptyset \).
Proof. Clear from the Theorem 3. ■

Remark 5 Rare $b$-continuity is implied by rare quasi-continuity and rare precontinuity, and implies rare $\beta$-continuity, but the converse implications are not true in general as the following examples shows.

Example 6 Let $\tau$ be the usual topology for $\mathbb{R}$ and for $A = [0, 1] \cup (1, 2) \cap \mathbb{Q}$ define $\sigma = \{\emptyset, \mathbb{R}, A, \mathbb{R} - A\}$. Then, the identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ is rarely $b$-continuous but it is neither rarely quasi-continuous nor rarely precontinuous.

Example 7 Let $\tau$ be the usual topology for $\mathbb{R}$ and $\sigma = \{\emptyset, \mathbb{R}, [1, 2) \cap \mathbb{Q}\}$. Then, the identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ is rarely $\beta$-continuous but it is not rarely $b$-continuous.

Definition 8 A function $f : X \to Y$ is called $I:b$-continuous at $x \in X$ if for each open set $G \subset Y$ containing $f(x)$, there exists a $b$-open set $U$ containing $x$ such that $int[f(U)] \subset G$.

If $f$ has this property at each point $x \in X$, then we say that $f$ is $I:b$-continuous on $X$.

Remark 9 It is clear that $I:b$-continuity is weaker than $b$-continuity and stronger than rare $b$-continuity.

Theorem 10 Let $Y$ be a regular space. Then the function $f : X \to Y$ is $I:b$-continuous on $X$ if and only if $f$ is rarely $b$-continuous on $X$.

Proof. Necessity is clear.

Sufficiency. Let $f$ be rarely $b$-continuous on $X$. Suppose that $f(x) \in G$, where $G$ is an open set in $Y$ and $x \in X$. By the regularity of $Y$, there exists an open set $G_1$ in $Y$ such that $f(x) \in G_1$ and $cl(G_1) \subset G$. Since $f$ is rarely $b$-continuous, then there exists $U \in BO(X, x)$ such that $int[f(U)] \subset cl(G_1)$. This implies $int[f(U)] \subset G$ which means that $I:b$-continuous on $X$. ■

Definition 11 A function $f : X \to Y$ is called strongly $b$-open if for every $U \in BO(X)$, $f(U)$ is open.

Theorem 12 If a function $f : X \to Y$ is strongly $b$-open and rarely $b$-continuous then $f$ is weakly $b$-continuous.

Proof. Suppose that $x \in X$ and $G$ is any open set of $Y$ containing $f(x)$. Since $f$ is rarely $b$-continuous, there exists a rare set $R_G$ with $G \cap cl(R_G) = \emptyset$ and $U \in BO(X, x)$ such that $f(U) \subset G \cup R_G$. Then $f(U) \cap (Y - cl(G)) \subset R_G$. Since $f$ is strongly $b$-open $f(U) \cap (Y - cl(G))$ is open. But the rare set $R_G$ has no interior point. Then $f(U) \cap (Y - cl(G)) = \emptyset$. This implies that $f(U) \subset cl(G)$. Hence $f$ is weakly $b$-continuous. ■
Lemma 13 (Andrijevic [3]) The intersection of an open set and a b-open set is a b-open set.

Theorem 14 If a function $f : X \to Y$ is rarely b-continuous at $x$ and for each open set $G$ containing $f(x)$, $f^{-1}(cl(R_G))$ is closed in $X$, then $f$ is b-continuous at $x$ where $R_G$ is a rare set with $G \cap cl(R_G) = \emptyset$.

Proof. Let $G \in O(Y, f(x))$. Since $f$ is rarely b-continuous at $x$, there exist a rare set $R_G$ with $G \cap cl(R_G) = \emptyset$ and $U \in BO(X, x)$ such that $f(U) \subset G \cup R_G$. Since $G \cap cl(R_G) = \emptyset$, we have
\[ f(x) \notin cl(R_G) \text{ and } x \in X - f^{-1}(cl(R_G)). \]

Set $V = U \cap (X - f^{-1}(cl(R_G)))$ then, by Lemma 13,
\[ V \in BO(X, x) \text{ and } f(V) \subset f(U) \cap (Y - cl(R_G)) \subset G. \]

Therefore, $f$ is b-continuous at $x$. ■

Theorem 15 If a function $f : X \to Y$ is rarely b-continuous then the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for every $x \in X$ is rarely b-continuous.

Proof. Suppose that $x \in X$ and $W$ is any open set containing $g(x)$. Then there exist open sets $U$ and $V$ in $X$ and $Y$ respectively such that $(x, f(x)) \in U \times V \subset W$. Since $f$ is rarely b-continuous, there exists $G \in BO(X, x)$ such that $int[f(G)] \subset cl(V)$. Let $O = U \cap G$. By Lemma 13, $O \in BO(X, x)$ and we have
\[ int[g(O)] \subset int[U \times f(G)] \subset U \times cl(V) \subset cl(W). \]

Therefore, $g$ is rarely b-continuous. ■

Definition 16 A topological space $(X, \tau)$ is said to be b-compact [4] if every b-open cover of $X$ has a finite subcover.

Definition 17 Let $\mathcal{A} = \{G_i\}$ be a class of subsets of $X$. By rarely union sets [5] of $\mathcal{A}$ we mean $\{G_i \cup R_{G_i}\}$, where each $R_{G_i}$ is a rare set such that each of $\{G_i \cap cl(R_{G_i})\}$ is empty.

Definition 18 A topological space $(X, \tau)$ is called rarely almost compact [5] if each open cover of $X$ has a finite subfamily whose rarely union sets cover the space.

Definition 19 A subset $K$ of a space $X$ is said to be:

(a) b-compact relative to $X$ [4] if for every cover $\{V_\alpha : \alpha \in I\}$ of $K$ by b-open sets of $X$, there exists a finite subset $I_0$ of $I$ such that $K \subset \bigcup\{V_\alpha : \alpha \in I_0\}$,
(b) rarely almost compact relative to \( X \) [5] if for every cover of \( K \) by open sets of \( X \), there exists a finite subfamily whose rarely union sets cover \( K \).

**Theorem 20** Let \( f : X \to Y \) be rarely \( b \)-continuous and \( K \) be a \( b \)-compact set in \( X \). Then \( f(K) \) is a rarely almost compact subset of \( Y \).

**Proof.** Suppose that \( G \) is an open cover of \( f(K) \). Set \( G^* = \{ V \in G : V \cap f(K) \neq \emptyset \} \). Then \( G^* \) is an open cover of \( f(K) \). Hence for each \( x \in K \), there is some \( V_x \in G^* \) such that \( f(x) \in V_x \). Since \( f \) is rarely \( b \)-continuous there exist a rare set \( R_{V_x} \) with \( V_x \cap \text{cl}(R_{V_x}) = \emptyset \) and a \( b \)-open set \( U_x \) containing \( x \) such that \( f(U_x) \subset V_x \cup R_{V_x} \). Hence there is a subfamily \( \{ U_{x_i} \}_{x_i \in K_0} \) which covers \( K \), where \( K_0 \) is a finite subset of \( K \). The subfamily \( \{ V_{x_i} \cup R_{V_{x_i}} \}_{x_i \in K_0} \) also covers \( f(K) \). ■

**Lemma 21** If \( g : Y \to Z \) is continuous and one-to-one, then \( g \) preserves rare sets [12].

**Theorem 22** If \( f : X \to Y \) is a rarely \( b \)-continuous surjection and \( g : Y \to Z \) is continuous and one-to-one, then \( g \circ f : X \to Z \) is rarely \( b \)-continuous.

**Proof.** Suppose that \( x \in X \) and \( g(f(x)) \in V \), where \( V \) is open set \( Z \). By hypothesis, \( g \) is continuous, therefore there exists an open set \( G \subset Y \) containing \( f(x) \) such that \( g(G) \subset V \). Since \( f \) is rarely \( b \)-continuous, there exists rare set \( R_G \) with \( G \cap \text{cl}(R_G) = \emptyset \) and a \( b \)-open set \( U \) containing \( x \) such that \( f(U) \subset G \cup R_G \). It follows from Lemma 21 that \( g(R_G) \) is a rare set in \( Z \). Since \( R_G \) is a subset of \( Y - G \) and \( g \) is injective, we have \( \text{cl}(g(R_G)) \cap V = \emptyset \). This implies that \( g(f(U)) \subset V \cup g(R_G) \). Hence the result follows. ■

**Definition 23** A function \( f : X \to Y \) is called pre-\( b \)-open if for every \( U \in BO(X) \), \( f(U) \in BO(Y) \).

**Theorem 24** If \( f : X \to Y \) is a pre \( b \)-open surjection and \( g : Y \to Z \) a function such that \( g \circ f : X \to Z \) is rarely \( b \)-continuous. Then \( g \) is rarely \( b \)-continuous.

**Proof.** Let \( y \in Y \) and \( x \in X \) such that \( f(x) = y \). Let \( G \) be an open set containing \( g(f(x)) \). Then there exists a rare set \( R_G \) with \( G \cap \text{cl}(R_G) = \emptyset \) and a \( b \)-open set \( U \) containing \( x \) such that \( g(f(U)) \subset G \cup R_G \). But \( f(U) \) is a \( b \)-open set containing \( f(x) = y \) such that \( g(f(U)) = (g \circ f)(U) \subset G \cup R_G \). This shows that \( g \) is rarely \( b \)-continuous at \( y \). ■

**Definition 25** A function \( f : X \to Y \) satisfies interiority rare \( b \) condition if \( \text{bint}(f^{-1}(G \cup R_G)) \subset f^{-1}(G) \) for each open set \( G \) in \( Y \), where \( R_G \) is a rare set with \( G \cap \text{cl}(R_G) = \emptyset \).

**Theorem 26** If \( f : X \to Y \) is rarely \( b \)-continuous and satisfies interiority rare \( b \) condition then \( f \) is \( b \)-continuous.
Proof. Since \( f \) is rarely \( b \)-continuous by Theorem 4, we have

\[ f^{-1}(G) \subset \text{bint}(f^{-1}(G \cup R_G)), \]

where \( G \) is an open set in \( Y \) and \( R_G \) is a rare set with \( G \cap \text{cl}(R_G) = \emptyset \). On the other hand by the interiority rare \( b \) condition we have \( \text{bint}(f^{-1}(G \cup R_G)) \subset f^{-1}(G) \). Therefore \( f^{-1}(G) \) is \( b \)-open in \( X \) and consequently \( f \) is \( b \)-continuous.

References


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