RARELY b-CONTINUOUS FUNCTIONS

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Abstract. In this paper we introduce a new class of functions called rarely b-continuous. Some characterizations and several properties concerning rare b-continuity are obtained.

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1. Introduction and preliminaries

In 1979, Popa [15] introduced the notion of rarely continuous functions as a generalization of weak continuity. The function has been further investigated by Long and Herrington in [12] and by various authors [5], [6], [7], [8], [9], [16]. The first author of this article introduced and investigated weak b-continuity [17] as a generalization of weak continuity. The purpose of the present paper is to introduce concept of rare b-continuity in topological spaces as a generalization of rare continuity and weak b-continuity. We investigate several properties of rarely b-continuous functions. Rare b-continuity implied by rare precontinuity and rare quasi continuity and implies rare β-continuity. The notion of $I.b$-continuity is also introduced which is weaker than b-continuity and stronger than b-continuity. It is shown that if Y is a regular space, then the function $f : X \to Y$ is $I.b$-continuous on X if and only if f is rarely b-continuous on X.

Throughout this paper, X and Y are topological spaces. Recall that a rare set R is a set R such that $int(R) = \varnothing$. A subset S of a space (X, τ) is called regular open [18] (resp. regular closed [18]) if $S = int(cl(S))$ (resp. $S = cl(int(S))$.

A subset S of a space (X, τ) is called semi-open [11] (resp.preopen [13], α-open [14], semi-preopen [2] or β-open [1], b-open [3] or γ-open [4]) if $S \subset cl(int(S))$ (resp. $S \subset int(cl(S))$, $S \subset int(cl(int(S)))$, $S \subset cl(int(cl(S)))$, $S \subset cl(int(S)) \cup int(cl(S))$) The complement of a semi-open (resp.preopen, α-open, β-open, b-open) set is said to be semi-closed (resp., preclosed, α-closed, β-closed, b-closed).

The family of all open (resp., regular open, semi-open,preopen, α-open, β-open, b-open) sets of X denoted by $O(X)$ (resp., $RO(X)$, $SO(X)$, $PO(X)$, $\alpha O(X)$, $\beta O(X)$, $BO(X)$).

The family of all b-closed sets of X is denoted by $BC(X)$ and the family of all b-open (resp. open, regular open) sets of X containing a point $x \in X$ is denoted by $BO(X, x)$ (resp., $O(X, x)$, $RO(X, x)$)

If S is a subset of a space X, then the b-closure of S, denoted by $bcl(S)$, is the smallest b-closed set containing S. The b-interior of S, denoted by $bint(S)$ is the largest b-open set contained in S. Our next definition contains some types of functions used throughout this paper.

Definition 1 A function $f : X \to Y$ is called:

(a) Weakly continuous [10] (resp. weakly b-continuous [17]) if for each $x \in X$ and each open set G containing $f(x)$, there exists $U \in O(X, x)$ (resp., $U \in BO(X, x)$) such that $f(U) \subset cl(G)$.

(b) b-continuous [4] if $f^{-1}(V)$ is b-open in X for every open set V of Y;

(c) Rarely continuous [15] (resp., rarely precontinuous [8], rarely quasicontinuous [16], rarely β-continuous [7]) at $x \in X$ if for each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap cl(R_G) = \varnothing$ and $U \in O(X, x)$ (resp., $U \in PO(X, x)$, $U \in SO(X, x)$, $U \in \beta O(X, x)$) such that $f(U) \subset G \cup R_G$.

2. Rarely b-continuous functions

Definition 2 A function $f : X \to Y$ is called rarely b-continuous at $x \in X$ if for each open set $G \subset Y$ containing $f(x)$, there exists a rare set R_G with $G \cap cl(R_G) = \varnothing$ and $U \in BO(X, x)$ such that $f(U) \subset G \cup R_G$.

Theorem 3 *The following statements are equivalent for a function $f : X \to Y$:*

(a) *The function is rarely b-continuous at $x \in X$.*

(b) *For each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap cl(R_G) = \varnothing$ such that*

$$x \in bint(f^{-1}(G \cup R_G)).$$

(c) *For each* $G \in O(Y, f(x))$, *there exists a rare set* R_G *with* $cl(G) \cap R_G = \varnothing$ *such that*

$$x \in bint(f^{-1}(cl(G) \cup R_G)).$$

(d) *For each* $G \in RO(Y, f(x))$, *there exists a rare set* R_G *with* $G \cap cl(R_G) = \varnothing$ *such that*

$$x \in bint(f^{-1}(G \cup R_G)).$$

(e) *For each* $G \in O(Y, f(x))$, *there exists* $U \in BO(X, x)$ *such that*

$$int(f(U) \cap (Y - G)) = \varnothing.$$

(f) *For each* $G \in O(Y, f(x))$, *there exists* $U \in BO(X, x)$ *such that*

$$int(f(U)) \subset cl(G).$$

Proof. (a) $\Rightarrow$ (b): Let $x \in X$ and $G \in O(Y, f(x))$. Then, there exists a rare set set R_G with $G \cap cl(R_G) = \varnothing$ and $U \in BO(X, x)$ such that $f(U).\subset G \cup R_G$. It follows that $x \in U \subset f^{-1}(G \cup R_G)$, then we have $x \in bint(f^{-1}(G \cup R_G))$.

(b) $\Rightarrow$ (c): Suppose that $G \in O(Y, f(x))$. Then there exists a rare set R_G with $G \cap cl(R_G) = \varnothing$ such that $x \in bint(f^{-1}(G \cup R_G))$. Since $G \cap cl(R_G) = \varnothing$, $R_G \subset Y - G$ where $Y - G = (Y - cl(G)) \cup (cl(G) - G)$. Now, we have $R_G \subset (R_G \cap (Y - cl(G))) \cup (cl(G) - G)$. Set $R^* = R_G \cap (Y - cl(G))$. It follows that R^* is a rare set with $cl(G) \cap R^* = \varnothing$. Therefore, $x \in bint(f^{-1}(G \cup R_G)) \subset bint(f^{-1}(cl(G) \cup R^*))$.

(c) $\Rightarrow$ (d): Assume that $x \in X$ and $G \in RO(Y, f(x))$. Then, there exists a rare set R_G with $cl(G) \cap R_G = \varnothing$ such that $x \in bint(f^{-1}(cl(G) \cup R_G))$. Set $R^* = R_G \cup (cl(G) - G)$. It follows that R^* is a rare set and $G \cap cl(R^*) = \varnothing$. Hence $x \in bint(f^{-1}(cl(G) \cup R_G)) = bint\,[f^{-1}(G \cup (cl(G) - G) \cup R_G)] = bint\,[f^{-1}(G \cup R^*)]$.

(d) $\Rightarrow$ (e): Let $G \in O(Y, f(x))$. Then, using $f(x) \in G \subset int(cl(G))$ and the fact that $int(cl(G)) \in RO(Y, f(x))$, there exists a rare set R_G with $int(cl(G)) \cap cl(R_G) = \varnothing$ such that $x \in bint\,[f^{-1}(int(cl(G)) \cup R_G)]$. Suppose $U = bint\,[f^{-1}(int(cl(G)) \cup R_G)]$. Then, $U \in BO(X, x)$ and, therefore, $f(U) \subset int(cl(G)) \cup R_G$. We have $int[f(U) \cap (Y - G)] = int(f(U)) \cap (int(Y - G)) \subset int[cl(G) \cup R_G] \cap (Y - cl(G)) \subset ((cl(G) \cup int(R_G))) \cap (Y - cl(G)) = \varnothing$.

(e) $\Rightarrow$ (f): Since $int(f(U) \cap (Y - G)) = int(f(U)) \cap int(Y - G)) = int[f(U)] \cap (Y - cl(G)) = \varnothing$, we have $int[f(U)] \subset cl(G)$.

(f) $\Rightarrow$ (a): $G \in O(Y, f(x))$. Then, by (f), there exists $U \in BO(X, x)$ such that $int[f(U)] \subset cl(G)$. Then, $f(U) = [f(U) - int(f(U))] \cup int(f(U)) \subset [f(U) - int(f(U))] \cup cl(G) = [f(U) - int(f(U))] \cup G \cup (cl(G) - G)$. Set $R^* = [f(U) - int(f(U))] \cap (Y - G)$ and $R^{**} = (cl(G) - G)$. Then, R^* and R^{**} are rare sets. Moreover, $R_G = R^* \cup R^{**}$ is a rare set and $cl(R_G) \cap G = \varnothing$ and $f(U) \subset G \cup R_G$. ■

Theorem 4 *A function* $f : X \to Y$ *is rarely* b*-continuous if and only if* $f^{-1}(G) \subset bint(f^{-1}(G \cup R_G))$ *for every open set* G *in* Y, *where* R_G *is a rare set with* $G \cap cl(R_G) = \varnothing$.

Proof. Clear from the Theorem 3. ■

Remark 5 Rare b-continuity is implied by rare quasi-continuity and rare precontinuity, and implies rare β-continuity, but the converse implications are not true in general as the following examples shows.

Example 6 Let τ be the usual topology for $\mathbb{R}$ and for $A = [0,1] \cup ((1,2) \cap \mathbb{Q})$ define $\sigma = \{\varnothing, \mathbb{R}, A, \mathbb{R} - A\}$. Then, the identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ is rarely b-continuous but it is neither rarely quasi-continuous nor rarely precontinuous.

Example 7 Let τ be the usual topology for $\mathbb{R}$ and $\sigma = \{\varnothing, \mathbb{R}, [1,2) \cap \mathbb{Q}\}$. Then, the identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ is rarely β-continuous but it is not rarely b-continuous.

Definition 8 A function $f : X \to Y$ is called $I.b$-continuous at $x \in X$ if for each open set $G \subset Y$ containing $f(x)$, there exists a b-open set U containing x such that $int[f(U)] \subset G$.

If f has this property at each point $x \in X$, then we say that f is $I.b$-continuous on X.

Remark 9 It is clear that $I.b$-continuity is weaker than b-continuity and stronger than rare b-continuity.

Theorem 10 *Let Y be a regular space. Then the function $f : X \to Y$ is $I.b$-continuous on X if and only if f is rarely b-contiuous on X.*

Proof. *Necessity* is clear.

Sufficiency. Let f be rarely b-continuous on X. Suppose that $f(x) \in G$, where G is an open set in Y and $x \in X$. By the regularity of Y, there exists an open set G_1 in Y such that $f(x) \in G_1$ and $cl(G_1) \subset G$. Since f is rarely b-continuous, then there exists $U \in BO(X,x)$ such that $int[f(U)] \subset cl(G_1)$. This implies $int[f(U)] \subset G$ which means that $I.b$-continuous on X. ■

Definition 11 A function $f : X \to Y$ is called strongly b-open if for every $U \in BO(X)$, $f(U)$ is open.

Theorem 12 *If a function $f : X \to Y$ is strongly b-open and rarely b-continuous then f is weakly b-continuous.*

Proof. Suppose that $x \in X$ and G is any open set of Y containing $f(x)$. Since f is rarely b-continuous, there exists a rare set R_G with $G \cap cl(R_G) = \varnothing$ and $U \in BO(X,x)$ such that $f(U) \subset G \cup R_G$.Then $f(U) \cap (Y - cl(G)) \subset R_G$. Since f is strongly b-open $f(U) \cap (Y - cl(G))$ is open. But the rare set R_G has no interior point. Then $f(U) \cap (Y - cl(G)) = \varnothing$. This implies that $f(U) \subset cl(G)$. Hence f is weakly b-continuous. ■

Lemma 13 (Andrijevic [3]) *The intersection of an open set and a b-open set is a b-open set.*

Theorem 14 *If a function $f : X \to Y$ is rarely b-continuous at x and for each open set G containing $f(x)$, $f^{-1}(cl(R_G))$ is closed in X, then f is b-continuous at x where R_G is a rare set with $G \cap cl(R_G) = \varnothing$.*

Proof. Let $G \in O(Y, f(x))$. Since f is rarely b-continuous at x, there exist a rare set R_G with $G \cap cl(R_G) = \varnothing$ and $U \in BO(X, x)$ such that $f(U) \subset G \cup R_G$. Since $G \cap cl(R_G) = \varnothing$, we have

$$f(x) \notin cl(R_G) \text{ and } x \in X - f^{-1}(cl(R_G)).$$

Set $V = U \cap (X - f^{-1}(cl(R_G)))$ then, by Lemma 13,

$$V \in BO(X, x) \text{ and } f(V) \subset f(U) \cap (Y - cl(R_G)) \subset G.$$

Therefore, f is b-continuous at x. ■

Theorem 15 *If a function $f : X \to Y$ is rarely b-continuous then the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for every $x \in X$ is rarely b-continuous.*

Proof. Suppose that $x \in X$ and W is any open set containing $g(x)$. Then there exist open sets U and V in X and Y respectively such that $(x, f(x)) \in U \times V \subset W$. Since f is rarely b-continuous, there exists $G \in BO(X, x)$ such that $int[f(G)] \subset cl(V)$. Let $O = U \cap G$. By Lemma 13, $O \in BO(X, x)$ and we have

$$int[g(O)] \subset int[U \times f(G)] \subset U \times cl(V) \subset cl(W).$$

Therefore, g is rarely b-continuous. ■

Definition 16 *A topological space (X, τ) is said to be b-compact [4] if every b-open cover of X has a finite subcover.*

Definition 17 Let $\mathcal{A} = \{G_i\}$ be a class of subsets of X.By rarely union sets [5] of $\mathcal{A}$ we mean $\{G_i \cup R_{G_i}\}$, where each R_{G_i} is a rare set such that each of $\{G_i \cap cl(R_{G_i})\}$ is empty.

Definition 18 A topological space (X, τ) is called rarely almost compact [5] if each open cover of X has a finite subfamily whose rarely union sets cover the space.

Definition 19 A subset K of a space X is said to be:

(a) b-compact relative to X [4] if for every cover $\{V_\alpha : \alpha \in I\}$ of K by b-open sets of X, there exists a finite subset I_0 of I such that $K \subset \cup\{V_\alpha : \alpha \in I_0\}$,

(b) rarely almost compact relative to X [5] if for every cover of K by open sets of X, there exists a finite subfamily whose rarely union sets cover K.

Theorem 20 *Let $f : X \to Y$ be rarely b-continuous and K be a b-compact set in X. Then $f(K)$ is a rarely almost compact subset of Y.*

Proof. Suppose that $\mathcal{G}$ is an open cover of $f(K)$. Set $\mathcal{G}^* = \{V \in \mathcal{G} : V \cap f(K) \neq \varnothing\}$. Then $\mathcal{G}^*$ is an open cover of $f(K)$. Hence for each $x \in K$, there is some $V_x \in \mathcal{G}^*$ such that $f(x) \in V_x$. Since f is rarely b-continuous there exist a rare set R_{V_x} with $V_x \cap cl(R_{V_x}) = \varnothing$ and a b-open set U_x containing x such that $f(U_x) \subset V_x \cup R_{V_x}$. Hence there is a subfamily $\{U_{x_i}\}_{x_i \in K_0}$ which covers K, where K_0 is a finite subset of K. The subfamily $\{V_{x_i} \cup R_{V_{x_i}}\}_{x_i \in K_0}$ also covers $f(K)$. ■

Lemma 21 *If $g : Y \to Z$ is continuous and one-to-one, then g preserves rare sets [12].*

Theorem 22 *If $f : X \to Y$ is a rarely b-continuous surjection and $g : Y \to Z$ is continuous and one-to-one, then $gof : X \to Z$ is rarely b-continuous.*

Proof. Suppose that $x \in X$ and $g(f(x)) \in V$, where V is open set Z. By hypothesis, g is continuous, therefore there exists an open set $G \subset Y$ containing $f(x)$ such that $g(G) \subset V$. Since f is rarely b-continuous, there exists rare set R_G with $G \cap cl(R_G) = \varnothing$ and a b-open set U containing x such that $f(U) \subset G \cup R_G$. It follows from Lemma 21 that $g(R_G)$ is a rare set in Z. Since R_G is a subset of $Y - G$ and g is injective, we have $cl(g(R_G)) \cap V = \varnothing$. This implies that $g(f(U)) \subset V \cup g(R_G)$. Hence the result follows. ■

Definition 23 A function $f : X \to Y$ is called pre-b-open if for every $U \in BO(X)$, $f(U) \in BO(Y)$.

Theorem 24 *If $f : X \to Y$ is a pre b-open surjection and $g : Y \to Z$ a function such that $gof : X \to Z$ is rarely b-continuous. Then g is rarely b-continuous.*

Proof. Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let G be an open set containing $g(f(x))$. Then there exists a rare set R_G with $G \cap cl(R_G) = \varnothing$ and a b-open set U containing x such that $g(f(U)) \subset G \cup R_G$. But $f(U)$ is a b-open set containing $f(x) = y$ such that $g(f(U)) = (gof)(U) \subset G \cup R_G$. This shows that g is rarely b-continuous at y. ■

Definition 25 A function $f : X \to Y$ satisfies interiority rare b condition if $bint(f^{-1}(G \cup R_G)) \subset f^{-1}(G)$ for each open set G in Y, where R_G is a rare set with $G \cap cl(R_G) = \varnothing$.

Theorem 26 *If $f : X \to Y$ is rarely b-continuous and satisfies interiority rare b condition then f is b-continuous.*

Proof. Since f is rarely b-continuous by Theorem 4, we have

$$f^{-1}(G) \subset bint(f^{-1}(G \cup R_G)),$$

where G is an open set in Y and R_G is a rare set with $G \cap cl(R_G) = \varnothing$. On the other hand by the interiority rare b condition we have $bint(f^{-1}(G \cup R_G)) \subset f^{-1}(G)$. Therefore $f^{-1}(G)$ is b-open in X and consequently f is b-continuous. ■

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