LIE IDEAL AND GENERALIZED JORDAN LEFT DERIVATION ON SEMIPRIME RINGS

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Abstract. Let R be a 2-torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let g be a generalized Jordan left (right) derivation associated with Jordan left (right) derivation d on R. Then g is a generalized left (right) derivation on R. It is proved that if $Q_r(S)$ is the Martindale quotient ring of S then there exists $q \in Q_r(S)$ such that $g(x) = qx + d(x)$ for all $x \in R$. (In right derivation case).

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1. In this paper, R will denote a 2-torsion free semiprime ring. A ring R is said to be semiprime if $xRx = 0$ with $x \in R$ implies $x = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subseteq U$, where $[x, y]$ denotes the Lie product $xy - yx$ of x and y. A mapping $A : R \times R \to R$ is said to be biadditive if it is additive in both the variables. Let C be the extended centroid of R and $S = RC$ be the central closure of R. The notion of extended centroid and central closure of semiprime ring R are given by Amitsur in [1]. It can also be found in [5]. Let $Q_r(S)$ be the Martindale quotient ring corresponding to S. An additive mapping $t : R \to R$ is said to be a left centralizer (resp. Jordan left centralizer) if $t(xy) = t(x)y$ (resp. $t(x^2) = t(x)x$) for all $x, y \in R$. Clearly, every left centralizer is a Jordan left centralizer but the converse may not be true in general. In [19], it is proved that if the ring is 2-torsion free semiprime then the converse is also true. An additive mapping $d : R \to R$ is said to be a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) for all $x, y \in R$. Clearly, every derivation is a Jordan derivation but converse need not be true. A counter

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example is a reverse derivation given by Herstein in [12]. An additive mapping $*: R \to R$ is said to be a reverse derivation if $(ab)^* = b^*a + ba^*$. However, if ring is 2-torsion free prime or semiprime then Jordan derivation is also a derivation [12], [8]. An additive mapping $d : R \to R$ is said to be a left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(x^2) = 2xd(x)$) for all $x, y \in R$. Every left derivation is a Jordan left derivation but the converse is not true. If the ring is 2-torsion free, prime then every Jordan left derivation is a left derivation [2]. An additive mapping $g : R \to R$ is said to be a generalized derivation (resp. generalized Jordan derivation) associated with the derivation d if $g(xy) = g(x)y + xd(y)$ (resp. $g(x^2) = g(x)x + xd(x)$) for all $x, y \in R$. An additive mapping $g : R \to R$ is said to be generalized left derivation (resp. generalized Jordan left derivation) associated with the Jordan left derivation $d : R \to R$ if $g(xy) = xg(y) + yd(x)$ ($g(x^2) = xg(x) + xd(x)$) for all $x, y \in R$. Every generalized left derivation is a generalized Jordan left derivation but converse is not true. An example is given by Ashraf and Ali in [2]. In [2] a question asked: "Is every generalized Jordan left derivation on semiprime ring generalized left derivation?" We show here that if a ring is 2-torsion free semiprime in which $x^2 = 0$ implies $x = 0$ the above question has affirmative answer.

2. To prove our theorems we need some results.

Lemma 1 ([3])**.** *Let R be a $2-$torsion free ring and U be a Lie ideal of R such that $x^2 \in U$ for all $x \in U$. If $d : R \to R$ is an additive mapping satisfying $d(x^2) = 2xdx$ for all $x \in U$, then*

(i) $d(xy + yx) = 2xdy + 2ydx$ *for all* $x, y \in U$.

(ii) $.d(xyx) = x^2dy + 3xydx - yxdx$ *for all* $x, y \in U$.

(iii) $d(xyz + zyx) = (xz + zx)dy + 3xydz + 3zydx - yxdz - yzdx$ *for all* $x, y, z \in U$.

(iv) $[x, y]xdx = x[x, y]dx$ *for all* $x, y \in U$.

(v) $[x, y]A(x, y) = 0$ *for all* $x, y \in U$.

Here $A(x, y) = d(xy) - yd(x) - xdy$.

From Lemma 1, one can easily see that

$$A(x, y) + A(y, x) = 0 \text{ and } A(x, y) - A(y, x) = d[x, y].$$

Lemma 2. *Let R be a $2-$torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let U be a Lie ideal of R such that $x^2 \in U$ for all $x \in U$. Let $A, B : U \times U \to U$ are biadditive mappings such that $A(x, y)B(x, y) = 0$ for all $x, y \in U$ then $A(x, y)B(u, v) = 0$ for all $x, y, u, v \in U$.*

Proof. Replace x by $x + u$ in $A(x, y)B(x, y) = 0$ to get

$$A(x, y)B(u, y) = -A(u, y)B(x, y).$$

Now, by the above relation, we have

$$\begin{aligned}\{A(x,y)B(u,y)\}^2 &= A(x,y)B(u,y)A(x,y)B(u,y)\\ &= -A(x,y)B(x,y)A(u,y)B(u,y).\end{aligned}$$

Thus, by given hypothesis, we get $\{A(x,y)B(u,y)\}^2 = 0$, for all $x, y, u \in U$. We get $A(x,y)B(u,y) = 0$. Again, replacing y by $y+v$ in $A(x,y)B(u,y) = 0$ and using the same argument, we get the required result. ∎

Lemma 3. *Let R be a semiprime ring and $x, y \in R$ such that $[u,v]A(x,y) = 0$ for all $u, v \in R$, then $A(x,y) \in Z(R)$.*

Proof. Let $z, w \in R$ be arbitrary elements

$$\begin{aligned}[z,A(x,y)]w[z,A(x,y)] &= [z,A(x,y)]wzA(x,y) - [z,A(x,y)]wA(x,y)z\\ &= [z,A(x,y)wz]A(x,y) - A(x,y)[z,wz]A(x,y)\\ &\quad - [z,A(x,y)w]A(x,y)z + A(x,y)[z,w]A(x,y)z.\end{aligned}$$

$$\Rightarrow \quad [z,A(x,y)]w[z,A(x,y)] = 0 \text{ for all } w, z \in R.$$

Since R is semiprime, we get $[z, A(x,y)] = 0$ for all $z \in R$ and so $A(x,y) \in Z(R)$. ∎

Theorem 1. *Let R be a $2-$torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let U be a Lie ideal of R such that $x^2 \in U$ for all $x \in U$. If $d: R \to R$ is an additive mapping satisfying $d(x^2) = 2xdx$ for all $x \in U$, then $d(xy) = xdy + ydx$ for all $x, y \in U$.*

Proof. By Lemma 1 (v), $[x,y]A(x,y) = 0$ for all $x, y \in U$. Since both $[x,y]$ and $A(x,y)$ are biadditive mappings, so by Lemma 2, we get that $[u,v]A(x,y) = 0$ for all $x, y, u, v \in U$. Now for fixed x, y in U, $A(x,y) \in Z(U)$ by Lemma 3. And this is true for all $x, y \in U$.

Now, we have

$$\begin{aligned}2(A(x,y))^2 &= A(x,y)(A(x,y) + A(x,y))\\ &= A(x,y)(A(x,y) - A(y,x)), \text{ since } A(x,y) = -A(y,x)\\ &= A(x,y)d[x,y], \text{ since } A(x,y) - A(y,x) = d[x,y],\end{aligned}$$

that is

$$2(A(x,y))^2 = A(x,y)d[x,y].$$

Since $[x,y]A(x,y) = 0$ and $A(x,y) \in Z(U)$, we have

$$A(x,y)[x,y] + [x,y]A(x,y) = 0$$

and, by Lemma 1(i), we get

$$2A(x,y)d[x,y] + 2[x,y]dA(x,y) = 0.$$

Since R is $2-$torsion free, $A(x,y)d[x,y] = -[x,y]dA(x,y)$. Thus, we get

$$2(A(x,y))^2 + [x,y]dA(x,y) = 0.$$

Multiplying by $A(x,y)$ on left and using the fact that $A(x,y) \in Z(U)$, we get $2(A(x,y))^3 = 0$, since $[x,y]A(x,y) = 0$. Again, using the fact that R is $2-$torsion free, we get $(A(x,y))^3 = 0$. Now, since we are assuming $x^2 = 0$ implies $x = 0$, so we have $A(x,y) = 0$, and thus, $d(xy) = xdy + ydx$, for all $x, y \in U$. ∎

Corollary 1. *Let R be a $2-$torsion free semiprime ring with unit element. Then, every Jordan left derivation is a left derivation.*

Proof. Since R is semiprime ring with unity, the assumption $x^2 = 0$ implies $x = 0$ is clearly satisfied. ∎

Corollary 2. *Let R be a $2-$torsion free semisimple ring in which $x^2 = 0$ implies $x = 0$ (or R has unity). Then, every Jordan left derivation is a left derivation.*

Proof. Since every semisimple ring is semiprime, the result follow from Theorem 1. ∎

Lemma 4 ([19] Proposition 1.4)**.** *Let R be a $2-$torsion free semiprime ring and $T : R \to R$ an additive mapping which satisfies $T(x^2) = T(x)x$ for all $x \in R$. Then T is a left centralizer.*

Lemma 5. *An additive mapping $g : R \to R$ is a generalized left (right) derivation if and only if g is of the form $g = d + t$, where d is a left (right) derivation and t is a right (left) centralizer.*

Proof. Let g be a generalized left derivation on R i.e. $g(xy) = xg(y) + yd(x)$ where d is a left derivation on R. Suppose $t = g - d$, then

$$\begin{aligned} t(xy) &= (g-d)(xy) = xg(y) + yd(x) - xd(y) - yd(x) \\ &= x(g-d)(y) = xt(y). \end{aligned}$$

This shows that t is a right centralizer. Hence $g = d + t$.

Conversely, suppose $g = d + t$. Then,

$$\begin{aligned} g(xy) &= (d+t)(xy) = xd(y) + yd(x) + xt(y) \\ &= x(d+t)(y) + yd(x) = xg(y) + yd(x). \end{aligned}$$

Thus, g is generalized left derivation. ∎

Theorem 2. *Let R be a 2-torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let $g : R \to R$ be a generalized Jordan left derivation associated with Jordan left derivation $d : R \to R$ then g is a generalized left derivation on R.*

Proof. Since g is a generalized Jordan left derivation, we have

$$g(x^2) = xg(x) + xd(x), \quad \forall x \in R$$

Since d is given to be a Jordan left derivation, by Theorem 1, it is a left derivation on R. Let us denote $g - d$ by t. Then, we have

$$t(x^2) = (g-d)(x^2) = g(x^2) - d(x^2) = xg(x) + xd(x) - 2xd(x) = x(g-d)(x) = xt(x),$$

which shows that t is a Jordan right centralizer. By Lemma 4, t is a right centralizer. Hence, g is of the form $g = d + t$, where d is a left derivation and t is a right centralizer. By Lemma 5, g is a generalized left derivation on R. This proves the theorem. ∎

The third theorem is inspired by T.K. Lee [16].

We prove here that every generalized Jordan right derivation on semiprime ring has the form $g(x) = qx + d(x)$ for all $x \in R$ for some $q \in Q_r(S)$.

Remark 1. The concept of left derivation (left centralizer) and right derivation (right centralizer) are analogues.

Corollary 3. *Let R be a 2-torsion free semiprime ring with unity then every generalized Jordan left derivation is a generalized left derivation.*

Corollary 4. *Let R be a semisimple ring in which $x^2 = 0$ implies $x = 0$ (or R has unity) then every generalized Jordan left derivation is a generalized left derivation.*

Lemma 6 ([15] Lemma 2)**.** *Let $f : R \to S$ be an additive map satisfying $f(xy) = f(x)y$ for all $x, y \in R$. Then there exists $q \in Q_r(S)$ such that $f(x) = qx$ for all $x \in R$.*

Theorem 3. *Let R be a 2-torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let $g : R \to R$ be a generalized Jordan right derivation associated with Jordan right derivation $d : R \to R$. Then there exist $q \in Q_r(S)$, the Martindale quotient ring of S, such that $g(x) = qx + d(x)$ for all $x \in R$.*

Proof. Since g is a generalized Jordan right derivation associated with the Jordan right derivation d on R, by Theorem 2, g is a generalized right derivation on R. Hence g is of the form $g = d + t$, where d is a right derivation and t is left centralizer. That is, $(g - d)(xy) = t(xy) = t(x)y$. Now, by Lemma 6, there exists $q \in Q_r(S)$ such that $(g - d)(x) = t(x) = qx$ for all $x \in R$. Thus $g(x) = qx + d(x)$ for all $x \in R$. Hence the theorem. ∎

Remark 2. Corollaries 3 and 4 are also hold true for Theorem 3.

3. Example. Semiprimeness of ring R is essential in Theorem 1. For, let S be a commutative ring. Suppose

$$R = \left\{ X = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\}.$$

R is not a semiprime ring, since

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $Y = \begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix}$, but $\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Define $d : R \longrightarrow R$ as

$$d \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, clearly, d is an additive mapping.

Now,

$$\begin{aligned} d(X^2) &= \begin{pmatrix} 0 & 0 & 0 & ab-ba \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= 2 \begin{pmatrix} 0 & 0 & 0 & ab-ba \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2XdX, \end{aligned}$$

for all $X \in R$, since S is commutative. This means that d is a Jordan left derivation.

But

$$d(XY) = \begin{pmatrix} 0 & 0 & 0 & ay-bx \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq XdY + YdX = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

since S is commutative. That is d is not a left derivation.

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