n-FOLD POSITIVE IMPLICATIVE HYPER K-IDEALS

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Abstract. In this paper we are supposed to introduce the definitions of n-fold positive implicative hyper K-ideals. These definitions are the generalizations of the definitions of positive implicative hyper K-ideals, which have been defined in [13]. Then we obtain some related results. In particular we determine the relationships between those n-fold positive implicative hyper K-ideals which satisfy the simple condition.

Keywords: hyper K-algebra; weak hyper K-ideal; hyper K-ideal; n-fold positive implicative hyper K-ideals; simple condition.

Introduction

The theory of hyper compositional structure has been introduced by F. Marty in 1934 during the 8th congress of Scandinavian Mathematicians, where he presented his work [10]. Today the research in the hyper compositional structures field is very vivid. In particular Y.B. Jun, M.M. Zahedi, X.L. Xin and R.A. Borzooei introduced the notions of hyper BCK-algebra and hyper K-algebra in 2000 [4], [8]. The concepts of an n-fold positive implicative hyper K-ideals are the generalizations of the concepts of positive implicative hyper K-ideals, which are related to the concepts of positive implicative ideals of a BCK-algebra [15]. The relationships between positive implicative hyper K-ideals have been studied by M.M. Zahedi and T. Roodbari [12]. They defined 27 types of positive implicative hyper K-ideals, and proved some propositions and theorems in this field. Now in this manuscript we define 27 types of n-fold positive implicative hyper K-ideals, and we concentrate on their relationships. Then we study the relationships between those n-fold positive implicative hyper K-ideals which satisfy the simple condition.

1. Preliminaries

In this paper we use the definitions of hyper K-algebra and hyper K-ideal as the most important definitions.

Definition 1.1. [4] Let H be a nonempty set, and " $\circ$ " be a hyperoperation on H, that " $\circ$ " is a function from $H \times H$ to $P^*(H)$=$P(H)$-$\varnothing$. Then H is called a hyper K-algebra if it contains "0" and satisfies the following axioms:

$HK-1$ $\quad (x \circ z) \circ (y \circ z) < x \circ y$;

$HK-2$ $\quad (x \circ y) \circ z = (x \circ z) \circ y$;

$HK-3$ $\quad x < x$;

$HK-4$ $\quad x < y,\ y < x \Rightarrow x = y$;

$HK-5$ $\quad 0 < x$;

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean that the subset $\bigcup a \circ b$ of H for all $a \in A$ and $b \in B$.

Theorem 1.2.[2] *Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all non-empty subsets A, B and C of H the following relations hold:*

(1) $(x \circ y) < z \Leftrightarrow (x \circ z) < y$;

(2) $(x \circ z) \circ (x \circ y) < (y \circ z)$;

(3) $x \circ (x \circ y) < y$;

(4) $x \circ y < x$;

(5) $A \circ B < A$;

(6) $A \subseteq B \Rightarrow A < B$;

(7) $x \in x \circ 0$;

(8) $(\mathrm{A} \circ \mathrm{C}) \circ (\mathrm{A} \circ \mathrm{B}) < (\mathrm{B} \circ \mathrm{C})$;

(9) $(\mathrm{A} \circ \mathrm{C}) \circ (\mathrm{B} \circ \mathrm{C}) < (\mathrm{A} \circ \mathrm{B})$;

(10) $(\mathrm{A} \circ \mathrm{B}) < C \Leftrightarrow (\mathrm{A} \circ \mathrm{C}) < B$;

(11) $A \circ B < A$;

(12) $(\mathrm{A} \circ \mathrm{C}) \circ B = (\mathrm{A} \circ \mathrm{B}) \circ C$;

Theorem 1.3. [6] *Let x,y,z be some elements in hyper K-algebra H. Then the following hold:*

(1) $x < y$ *implies that* $z \circ y < z \circ x$,

(2) $x < y$ *implies that* $x \circ z < y \circ z$.

Definition 1.4. [2] Let I be a nonempty subset of a hyper K-algebra H and $0 \in I$. Then

(1) I is called a weak hyper K-ideal of H if $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$ for all $x, y \in H$.

(2) I is called a hyper K-ideal of H if $x \circ y < I$ and $y \in I$ imply that $y \in I$ for all $x, y \in H$.

Note that in any hyper K-algebra H, $\{0\} \subseteq H$ is a hyper K-ideal.

Theorem 1.5. [2] *Any hyper K-ideal of a hyper K-algebra H is a weak hyper K-ideal.*

Definition 1.6. [3] Let I be a nonempty subset of a hyper K-algebra H. Then we say that I is closed, whenever $x<y$, $y \in I$ imply that $x \in I$ for all $x,y \in H$.

Definition 1.7. [2] Let H be a hyper K-algebra. An element $a \in H$ is called a left (resp. right) scalar if $|a \circ x| = 1$ (resp. $|x \circ a| = 1$) for all $x \in H$.

Theorem 1.8. [12] *Let I be a hyper K-ideal of a hyper K-algebra H. Then the following statements are equivalent:*

(1) $(x \circ y) < I$,

(2) $(x \circ y) \cap I \neq \varnothing$.

Definition 1.9. [12] Let $H=\{0,1,2\}$ be a hyper K-algebra. We say that H satisfies the simple condition if the conditions $1 \not< 2$ and $2 \not< 1$ hold.

Definition 1.10. [12] A hyper K-algebra H is called simple if for all distinct elements $a, b \in H - 0$, $a \not< b$ and $b \not< a$.

Theorem 1.11. [12] *Let H satisfies the simple condition. Then,*

(i) $a \circ 0 = \{a\}$, *for all* $a \in H - \{0\}$,

(ii) $a \in a \circ b$, *for all distinct elements* $a, b \in H$,

(iii) $H - \{a\} \subseteq H \circ a$, *for all* $a \in H$,

(iv) $a \in b \circ c \Longleftrightarrow c \in b \circ a$, *for all distinct elements* $a, c \in H$, *and* $b \in H - \{0\}$,

(v) $x < x \circ a \Longleftrightarrow x \in x \circ a$, *for all* $a, x \in H$,

(vi) $A < A \circ b \Longleftrightarrow A \cap (A \circ b) \neq \varnothing$, *for all* $b \in H$ *and* $\varnothing \neq A \subseteq H$,

(vii) $(x \circ y) \circ z < x \circ (y \circ z)$, *for all* $x, y, z \in H$,

(viii) *If* $0 \in I \subseteq H$, *then* $A \circ B < I \Longleftrightarrow (A \circ B) \cap I \neq \varnothing$, *for all non-empty subsets A and B of H.*

In the rest of this paper, by H we denote a hyper K-algebra.

2. n-fold positive implicative hyper K-ideals

In this section we define the notions of n-fold positive implicative hyper K-ideals of types $1'$,$2'$,$3'$,and $4'$. Then we define 27 other types, and we give many examples to show that these notions are different from each other. Finally we prove some theorems and obtain some related result.

Definition 2.1. Let I be a nonempty subset of a hyper K-algebra H such that $o \in I$. If n is a natural number, then I is called an n-fold positive implicative hyper K-ideal of

(i) type $1'$, if for all $x,y \in H$, $x \circ y^{n+1} \subseteq I$ implies that $x \circ y^n \subseteq I$,

(ii) type $2'$, if for all $x,y \in H$, $x \circ y^{n+1} \subseteq I$ implies that $x \circ y^n < I$,

(iii) type $3'$, if for all $x,y \in H$, $x \circ y^{n+1} < I$ implies that $x \circ y^n \subseteq I$,

(v) type $4'$, if for all $x,y \in H$, $x \circ y^{n+1} < I$ implies that $x \circ y^n < I$.

Theorem 2.2. *Let A be a weak hyper K-ideal and I be hyper K-ideal of hyper K-algebra H such that $I \subseteq A$. If I is an n-fold positive implicative hyper K-ideal of type $1'$ or $3'$, so is A.*

Proof. Assume I is an n-fold positive implicative hyper K-ideal of type $1'$, and $x \circ y^{n+1} \subseteq A$. Then by Theorem 1.2 $x \circ y^{n+1} < A$. Since $0 \in (x \circ y^{n+1}) \circ (x \circ y^{n+1})$, and $0 \in I$, we obtain $0 \in ((x \circ y^{n+1}) \circ (x \circ y^{n+1})) \circ I$. Therefore we have

$$(x \circ (x \circ y^{n+1})) \circ y^{n+1} = (x \circ y^{n+1}) \circ (x \circ y^{n+1}) < I$$

On the other hand, I is an n-fold positive implicative hyper K-ideal of type $1'$. So, $(x \circ (x \circ y^{n+1})) \circ y^n \subseteq I$. Hence, $(x \circ (x \circ y^{n+1})) \circ y^n \subseteq A$, thus $(x \circ y^n) \circ (x \circ y^{n+1}) \subseteq A$. Moreover, A is a weak hyper K-ideal and $x \circ y^{n+1} \subseteq A$. So, $x \circ y^n \subseteq A$. It means A is an n-fold positive implicative hyper K-ideal of type $1'$.

Similarly, we can prove for type $3'$.

Theorem 2.3. *Let A and I be hyper K-ideals of hyper K-algebra H such that $I \subseteq A$. If I is an n-fold positive implicative hyper K-ideal of type $2'$ or $4'$, so is A.*

Proof. Assume I is an n-fold positive implicative hyper K-ideal of type $2'$, and $x \circ y^{n+1} \subseteq A$. Then by Theorem 1.2 $x \circ y^{n+1} < A$.

Since $(x \circ (x \circ y^{n+1})) \circ y^{n+1} = (x \circ y^{n+1}) \circ (x \circ y^{n+1}) < I$, so $(x \circ (x \circ y^{n+1})) \circ y^{n+1} < I$. By hypothesis I is an n-fold positive implicative hyper K-ideal of type $2'$, we have $(x \circ (x \circ y^{n+1})) \circ y^n < I$. Therefore, $(x \circ (x \circ y^{n+1})) \circ y^n < A$, thus, $(x \circ y^n) \circ (x \circ y^{n+1}) < A$. Moreover, A is a hyper K-ideal and $x \circ y^{n+1} \subseteq A$, therefore $x \circ y^n < A$. It means A is an n-fold positive implicative hyper K-ideal of type $2'$.

Similarly, we can prove for type $4'$.

Definition 2.4. Let I be a nonempty subset of a hyper K-algebra H, such that $0 \in I$. If n is a natural number, then I is called an n-fold positive implicative hyper K-ideal of:

(i) type 1, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) \subseteq I$,

(ii) type 2, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(iii) type 3, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) < I$,

(iv) type 4, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \subseteq I$,

(v) type 5, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(vi) type 6, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) < I$,

(vii) type 7, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) < I$,

(viii) type 8, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(ix) type 9, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) \subseteq I$,

(x) type 10, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xi) type 11, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) \subseteq I$,

(xii) type 12, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) < I$,

(xiii) type 13, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \subseteq I$,

(xiv) type 14, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xv) type 15, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) < I$,

(xvi) type 16, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) < I$,

(xvii) type 17, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xviii) type 18, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) \subseteq I$,

(xix) type 19, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) < I$,

(xx) type 20, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \subseteq I$,

(xxi) type 21, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xxii) type 22, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) \subseteq I$,

(xxiii) type 23, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) < I$,

(xxiv) type 24, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xxv) type 25, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) < I$,

(xxvi) type 26, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xxvii) type 27, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) \subseteq I$.

For simplicity of notation we use n-fold PIHKI instead of n-fold Positive Implicative Hyper K-ideal .

Remark. From this definition, we conclude that the notions of 1-fold PIHKI of type j and PIHKI of type j of H coincide, for any $j = 1, 2, ..., 27$.

Theorem 2.5. *Let I be a hyper K-ideal of hyper K-algebra H. If I is an n-fold PIHK of type* 2,3,5,6,7,8,10,12,15,16,19,21,23,24,25*, or* 26*. Then it is also, $n+1$-fold PIHKI of type* 2,3,5,6,7,8,10,12,15,16,19,21,23,24,25*, or* 26*, respectively.*

Proof. Let I be an n-fold PIHKI of type 2, $(x \circ y) \circ z^{n+1} \subseteq I$, and $y \circ z^{n+1} \subseteq I$. By Theorem 1.2 we have $x \circ z^{n+1} < x \circ z^n$. Since I is of type 2, we have $x \circ z^n < I$. On the other hand, I is a hyper K-ideal. So $x \circ z^{n+1} < I$. It means I is an n+1-fold PIHKI of type 2.

For other types the proof is similar.

Open problem. *If I is an n-fold PIHKI of type* 1,2,...*, or* 27*, then is it also n+1-fold PIHKI of type* 1,2,...*, or* 27*, respectively?*

Example 2.6. (1) The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

It is easy to check that $I = \{0, 2\}$ is a 2-fold PIHKI of types 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ,12 , 14, 15 ,16, 17, 19,21, 22, 23, 24, 25 and 26, while $I = \{0, 2\}$ is not a 2-fold PIHKI of type 11, because $((0 \circ 2) \circ 0^2) \cap I \neq \varnothing$, $2 \circ 0^2 \subseteq I$, but $0 \circ 0^2 \nsubseteq I$. Also $I = \{0, 2\}$ is not a 2-fold PIHKI of type 13, because $((0 \circ 2) \circ 0^2 \cap I \neq \varnothing$, $2 \circ 0^2 \subseteq I$, but $0 \circ 0^2 \subseteq I$. Similarly, by considering $x = 0$, $y = 2$, $z = 0$, we have $I = \{0, 2\}$ is not a 2-fold PIHKI of types 18, 20, 27.

(2) Consider the following hyper K-algebra

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

we have $I = \{0 , 2\}$ is a 2-fold PIHKI of types 11, 13.

(3) Consider the following hyper K-algebra

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,2}	{0,2}

It can be checked that $I = \{0\ ,\ 2\}$ is a 2-fold PIHKI of types 27.

(4) Let $(X\ ,\ *\ ;\ 0)$ be a BCK-algebra and define a hyper opration "$\circ$" on X by $x \circ y = \{x * y\}$ for all $x,\ y \in X$. If I is an n-fold positive implicative ideal of the BCK-algebra X, then it is easy to see that $(I\ ,\ *\ ;\ 0)$ is an n-fold PIHKI of types 1, 2, 3,..., or 27.

Theorem 2.7. *Let I be a non-empty subset of H. Then the following statements hold:*

(1) *If I is n-fold PIHKI of type* 4*, then I is n-fold PIHKI of types* 1, 6*,*

(2) *If I is n-fold PIHKI of type* 5*, then I is n-fold PIHKI of types* 2, 6*,*

(3) *If I is n-fold PIHKI of type* 6*, then I is n-fold PIHKI of type* 3*,*

(4) *If I is n-fold PIHKI of type* 8*, then I is n-fold PIHKI of type* 7*,*

(5) *If I is n-fold PIHKI of type* 9*, then I is n-fold PIHKI of types* 7, 8*,*

(6) *If I is n-fold PIHKI of type* 11*, then I is n-fold PIHKI of types* 10, 12*,*

(7) *If I is n-fold PIHKI of type* 10*, then I is n-fold PIHKI of type* 12*,*

(8) *If I is n-fold PIHKI of type* 13*, then I is n-fold PIHKI of types* 14, 15*,*

(9) *If I is n-fold PIHKI of type* 14*, then I is n-fold PIHKI of* 15*,*

(10) *If I is n-fold PIHKI of type* 18*, then I is n-fold PIHKI of* 16, 17*,*

(11) *If I is n-fold PIHKI of type* 17*, then I is n-fold PIHKI of type* 16*,*

(12) *If I is n-fold PIHKI of type* 20*, then I is n-fold PIHKI of type* 3*,*

(13) *If I is n-fold PIHKI of type* 21*, then I is n-fold PIHKI of type* 19*,*

(14) *If I is n-fold PIHKI of type* 24*, then I is n-fold PIHKI of type* 23*,*

(15) *If I is n-fold PIHKI of type* 22*, then I n-fold PIHKI of type* 24*,*

(16) *If I is n-fold PIHKI of type* 27*, then I is n-fold PIHKI of type* 26.

Proof. The proof is straightforward.

The following examples show that the converse of the statements of Theorem 2.7 are not true in general.

Example 2.8. The following tables show some hyper K-algebra structures on H $= \{0, 1, 2\}$.

(1):

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,1}	{0,1,2}

We can see that $I = \{0, 1\}$ is a 2-fold PIHKI of type 6, while I is not a 2-fold PIHKI of type 4, because $((2\circ 1)\circ 0^2)\subseteq I$ and $(1\circ 0^2)\cap I\neq\varnothing$, but $(2\circ 0^2)\nsubseteq I$.

(2):

∘	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

We can see that $I = \{0, 1\}$ is a 2-fold PIHKI of type 7, while I is not a 2-fold PIHKI of type 8, because $((2\circ 1)\circ 0^2)\subseteq I$ and $(1\circ 0^2)< I$, but $(2\circ 0^2)\cap I=\varnothing$. Also $I = \{0, 1\}$ is not a 2-fold PIHKI of type 9, since $((2\circ 1)\circ 0^2)\subseteq I$ and $(1\circ 0^2)< I$, but $(2\circ 0^2)\nsubseteq I$.

(3):

∘	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

$I = \{0, 2\}$ is a 2-fold PIHKI of type 10, while I is not a 2-fold PIHKI of type 11, because $((0\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)\subseteq I$, but $(0\circ 2^2)\subseteq I$.

(4):

∘	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

$I = \{0, 1\}$ is a 2-fold PIHKI of type 12, while I is not a 2-fold PIHKI of type 11, because $((2\circ 1)\circ 0^2)\cap I\neq\varnothing$ and $(1\circ 0^2)\subseteq I$, but $(2\circ 0^2)\nsubseteq I$.

(5):

∘	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,1}	{0,1,2}

$I = \{0, 1\}$ is a 2-fold PIHKI of type 15, while I is not a 2-fold PIHKI of type 13, because $((2\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)\cap I\neq\varnothing$, but $(2\circ 2^2)\nsubseteq I$.

(6):

∘	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

$I = \{0, 2\}$ is a 2-fold PIHKI of type 14, while I is not a 2-fold PIHKI of type 13, because $((0\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)\cap I\neq\varnothing$, but $(0\circ 2^2)\nsubseteq I$.

(7):

∘	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

$I = \{0, 2\}$ is a 2-fold PIHKI of type 15, while I is not a 2-fold PIHKI of type 14, because

$((1\circ 2)\circ 0^2)\cap I\neq\varnothing$ and $(2\circ 0^2)\cap I\neq\varnothing$, but $(1\circ 0^2)\cap I=\varnothing$.

(8):

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,2}	{0,2}

$I = \{0, 1\}$ is a 2-fold PIHKI of type 16, while I is not a 2-fold PIHKI of type 18, because $((2\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)\cap I\neq\varnothing$, but $(2\circ 2^2)\nsubseteq I$.

(9):

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

We see that $I = \{0, 2\}$ is a 2-fold PIHKI of type 17, while I is not a 2-fold PIHKI of type 18, because $((0\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)<I$, but $(0\circ 2^2)\nsubseteq I$.

(10):

$\circ$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

$I = \{0, 1\}$ is a 2-fold PIHKI of type 16, while I is not a 2-fold PIHKI of type 17, because $((2\circ 1)\circ 0^2)\cap I\neq\varnothing$ and $(1\circ 0^2)<I$, but $(2\circ 0^2)\cap I=\varnothing$.

Also we see that $I = \{0, 1\}$ is a 2-fold PIHKI of type 19, while I is not a 2-fold PIHKI of type 20, becaus $((2\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)\cap I=\varnothing$, but $(2\circ 2^2)\nsubseteq I$.

(11):

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

$I = \{0, 2\}$ is a 2-fold PIHKI of type 24, while I is not a 2-fold PIHKI of type 22, because $((1\circ 2)\circ 0^2)<I$ and $(2\circ 0^2)\subseteq I$, but $(1\circ 0^2)\nsubseteq I$.

(12):

$\circ$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

We see that $I = \{0, 1\}$ is a 2-fold PIHKI of type 25, but I is not a 2-fold PIHKI of type 26, because $((2\circ 1)\circ 0^2)<I$ and $(1\circ 0^2)<I$, $(2\circ 0^2)\cap I=\varnothing$.

(13):

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

$I = \{0, 2\}$ is a 2-fold PIHKI of type 26, while I is not a 2-fold PIHKI of type 27, because $((0\circ 1)\circ 2^2) < I$ and $(1\circ 2^2) < I$, but $(0\circ 2^2)\not\subseteq I$.

Also we see that $I = \{0, 1\}$ is a 2-fold PIHKI of types 2 and 3, while I is not a 2-fold PIHKI of type 1, because $((2\circ 1)\circ 0^2)\subseteq I$ and $(1\circ 0^2)\subseteq I$, but $(2\circ 0^2)\not\subseteq I$.

Theorem 2.9. *Let I be a hyper K-algebra of H. Then the following statement are equivalent:*

(1) $x\circ y^n < I$,

(2) $(x\circ y^n)\cap I\neq\varnothing$.

Proof. (1)⇒(2) Assume $x\circ y^n < I$, then there exist $a\in I$, and $t\in x\circ y^n$ such that $t < a$. Thus $0\in t\circ a$. Now, since $0\in I$ and $0\in 0\circ a$, then $t\circ a < I$. So $t\in I$. Hence, $(x\circ y^n)\cap I\neq\varnothing$.

(2)⇒(1) It is obvious.

Theorem 2.10. *Let I be a hyper K-ideal of a hyper K-algebra H. Then the following statements are equivalent:*

(1) *I is an n-fold PIHKI of type* 14,

(2) *I is an n-fold PIHKI of type* 15,

(3) *I is an n-fold PIHKI of type* 16,

(4) *I is an n-fold PIHKI of type* 17,

(5) *I is an n-fold PIHKI of type* 19,

(6) *I is an n-fold PIHKI of type* 21,

(7) *I is an n-fold PIHKI of type* 25,

(8) *I is an n-fold PIHKI of type* 26.

Proof. (1)⇒(2) Let I be an n-fold PIHKI of type 14. So for all $x,y,z\in H$, if $((x\circ y)\circ z^n)\cap I\neq\emptyset$, and $(y\circ z^n)\cap I\neq\emptyset$, then $(x\circ z^n)\cap I\neq\emptyset$. On the other hand, by Theorem 2.9 we have $(x\circ z^n) < I$. Thus I is of type 15.

(8)⇒(1) Let I be an n-fold PIHKI of type 26. So for all $x,y,z\in H$, if $((x\circ y)\circ z^n) < I$, and $y\circ z^n < I$, then $(x\circ z^n)\cap I\neq\emptyset$. Now, by Theorem 2.9 we have $((x\circ y)\circ z^n)\cap I\neq\emptyset$, and so, $(y\circ z^n)\cap I\neq\emptyset$ implies that $(x\circ z^n)\cap I\neq\emptyset$. Thus I is of type 14.

The proof of other statements can be obtained by the same way.

Theorem 2.11. *Let I be a hyper K-ideal of a hyper K-algebra H. Then the following statements are equivalent:*

(1) *I is an n-fold PIHKI of type* 13,

(2) *I is an n-fold PIHKI of type* 18,

(3) *I is an n-fold PIHKI of type* 20,

(4) *I is an n-fold PIHKI of type* 27.

Proof. By considering Theorem 2.9 the proof is easy.

Theorem 2.12. *Let I be a hyper K-ideal of a hyper K-algebra H. Then the following statements are equivalent:*

(1) *I is an n-fold PIHKI of type* 10,

(2) *I is an n-fold PIHKI of type* 23,

(3) *I is an n-fold PIHKI of type* 12,

(4) *I is an n-fold PIHKI of type* 24.

Proof. By considering Theorem 2.9 the proof is easy.

Theorem 2.13. *Let I be a hyper K-ideal of a hyper K-algebra H. Then the following statements hold:*

(1) *If I is of type* $3'$ *then it is of type* 3,7,8,9,13,14,15,,16,17,18,20,26, *and* 27.

(2) *If I is of type* $4'$ *then it is of type* 3,7,10,12,14,15,16,17,19,21,23,24,25, *and* 26.

Proof. Assume that $(x \circ y) \circ z^n \subseteq I$ and $y \circ z^n < I$. Since:

$$((x \circ z^n) \circ z^n) \circ ((x \circ y) \circ z^n) < (x \circ z^n) \circ (x \circ y) < (y \circ z^n) < I$$

then

$$((x \circ z^n) \circ z^n) \circ ((x \circ y) \circ z^n) < I.$$

Since I is a hyper K-ideal, $((x \circ z^n) \circ z^n) < I$. Then by our hypothesis $x \circ z^n \subseteq I$, i.e. I is of type 9. Thus, by Theorem 2.7 it is of type 7,8.

By the same way, it can be proved it is of type 27. Thus, by Theorem 2.7 it is of type 26. Other types can be obtained by Theorems 2.11, and 2.7, similarly.

By the same way, and by considering Theorems 2.7, 2.10, and 2.12, it can be proved 2 is true.

Theorem 2.14. *Let* $0 \in H$ *be a right scalar element of a hyper K-algebra H and I be an n-fold PIHKI of type* 11,13,14,21,22 *or* 24. *Then I is a hyper K-ideal.*

Proof. Let $x,y \in H$, I be an n-fold PIHKI of type 11, $(x \circ y) \cap I \neq \varnothing$, and $y \in I$. Since $0 \in H$ is a right scalar element, we have $((x \circ y) \circ 0^n) \cap I \neq \varnothing$ and $\{y\} = y \circ 0 = y \circ 0^n \subseteq I$. Thus $\{x\} = x \circ 0 = x \circ 0^n \subseteq I$, then $x \in I$. Therefore I is a hyper K-ideal. The proof of each of the n-fold PIHKI of types 13,14,20,21,22 or 24 is the same.

Example 2.15. (1) The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	{0,1}	{0,1,2}	{0,1,2}
1	{1}	{0,1}	{1,2}
2	{1,2}	{0,1,2}	{0,1,2}

Then $I = \{0\ ,\ 1\}$ is an n-fold PIHKI of type 21, for any $n{\in}N$, while I is not a hyper K-ideal, because $(2{\circ}1){\cap}I{\neq}\varnothing$, and $1{\in}I$, but $2 \notin I$. Also we see that $0{\in}H$ is not a right scalar element.

(2) Consider the following hyper K-algebra

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

We see that $0{\in}H$ is not a right scalar element and $I = \{0\ ,\ 2\}$ is an n-fold PIHKI of types 11, 14, 22, and 24, for any $n{\in}N$, while I is not a hyper K-ideal, because $(1{\circ}2){\cap}I{\neq}\varnothing$, and $2{\in}I$, but $1 \notin I$. $I = \{0\ ,\ 1\}$ is an n-fold PIHKI of type 13, for any $n{\in}N$, while I is not a hyper K-ideal, because $(2{\circ}1){\cap}I{\neq}\varnothing$, and $1{\in}I$, but $2 \notin I$.

Note that Example 2.15 shows the condition $0{\in}H$ is a right scalar is necessary in Theorem 2.14.

Theorem 2.16. *Let $0{\in}H$ be a right scalar element of a hyper K-algebra H and I be closed. If I is an n-fold PIHKI of type* 12, 15, 16, 19 *or* 23*, then I is a weak hyper K-ideal.*

Proof. Let I be an n-fold PIHKI of type 12, $x{,}y{\in}H$, $(x{\circ}y){\subseteq}I$ and $y{\in}I$. Since $0{\in}H$ is a right scalar element, $((x{\circ}y){\circ}0^n){\cap}I{\neq}\varnothing$ and $(y{\circ}0){\subseteq}I$ imply that $(x{\circ}0^n) < I$. So there exists $i{\in}I$ such that $x{\circ}0^n < i$. Therefore $(x{\circ}0^{n-1}){\circ}i{<}0$. Thus, there exists $k{\in}(x{\circ}0^{n-1}){\circ}i$ such that $k{<}0$. Hence, by $0{<}k$ we have $k{=}0$, i.e. $0{\in}(x{\circ}0^{n-1}){\circ}i$. It means $x{\circ}0^{n-1}{<}i$. Repeatedly using this way it follows $x{<}i$. Now since I is closed, we obtain that $x{\in}I$. Therefore I is a weak hyper K-ideal. The proof of each of the n-fold PIHKI of types 15, 16, 19, or 23 is the same.

Theorem 2.17. *Let $0{\in}H$ be a right scalar element of a hyper K-algebra H and I be an n-fold PIHKI of type* 18, 20, 26 *or* 27*. Then I is a weak hyper K-ideal.*

Proof. The proof is similar to the proof of Theorem 2.16.

Example 2.18. Consider the following hyper K-algebra

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

In this example $I = \{0\ ,\ 2\}$ is an n-fold PIHKI of types 12, 15, 16, 19, and 23, for any $n \in N$, while I is not a hyper K-ideal, because we see that I is not closed and $0 \in H$ is not a right scalar.

Definition 2.19. Let H be a hyper K-algebra and $I \subseteq H$ and $a \in I$. We define $I_{a^n} = \{\ x \in H \mid (x \circ a^n) \cap I \neq \varnothing\ \}$.

Theorem 2.20. *Let H be a hyper K-algebra. Then I is an n-fold PIHKI of type* 14 *if and only if for all $a \in H$, I_{a^n} is a hyper K-ideal.*

Proof. Let for all $x,y,a \in H$, $((x \circ y) \cap I_{a^n}) \neq \varnothing$, $y \in I_{a^n}$. Then $((x \circ y) \circ a^n) \cap I \neq \varnothing$, $(y \circ a^n) \cap I \neq \varnothing$. Since I is an n-fold PIHKI of type 14, $(x \circ a^n) \cap I \neq \varnothing$. Therefore $x \in I_{a^n}$, i.e. I is a hyper K-ideal.

Conversely, let for all $x,y,a \in H$, $((x \circ y) \circ a^n) \cap I \neq \varnothing$ and $(y \circ a^n) \cap I \neq \varnothing$. Then, $x \circ y \subseteq I_{a^n}$. So,by Theorem 2.9 $x \circ y < I_{a^n}$. Now, since $y \in I_{a^n}$ and I_{a^n} is a hyper K-ideal, we obtain $x \in I_{a^n}$. Thus, $(x \circ a^n) \cap I \neq \varnothing$, i.e. I is an n-fold PIHKI of type 14.

3. n-fold positive implicative hyper K-ideals in simple hyper K-algebras

In this part $(H, \circ, 0)$ is a simple hyper K-algebra, unless otherwise is stated.

Theorem 3.1. *Let $0 \in I \subseteq H$. Then*

(i) *I is an n-fold PIHKI of type* 2 *if and only if I is an n-fold PIHKI of type* 3*,*

(ii) *I is an n-fold PIHKI of type* 4 *if and only if I is an n-fold PIHKI of type* 9*,*

(iii) *I is an n-fold PIHKI of type* 5 *if and only if I is an n-fold PIHKI of type* 6(7,8)*,*

(iv) *I is an n-fold PIHKI of type* 11 *if and only if I is an n-fold PIHKI of type* 22*,*

(v) *I is an n-fold PIHKI of type* 10 *if and only if I is an n-fold PIHKI of type* 12(23,24)*,*

(vi) *I is an n-fold PIHKI of type* 13 *if and only if I is an n-fold PIHKI of type* 18(20,27)*,*

(vii) *I is an n-fold PIHKI of type* 14 *if and only if I is an n-fold PIHKI of type* 15(16,17,19,21,25,26)*.*

Proof. The proof follows from Definition 2.4 and Theorem 1.11.

Theorem 3.2. *Let $a \in H - \{0\}$ and $I = H - \{a\}$ be a hyper K-ideal. Then I is an n-fold PIHKI of type* 25(14,15,16,17,19,21,26) *if and only if* $|a \circ b^n| = 1$, *for all* $b \in I$.

Proof. Let I be an n-fold PIHKI of type 25. Then we prove that $|a \circ b^n| = 1$, for all $b \in I$. On the contrary, let $|a \circ b^n| > 1$, for some $b \in I$. By Theorem 1.11(ii) we have $a \in a \circ b^n$. So there exists $c \in H - \{a\}$ such that $c \in a \circ b^n$. Thus $(a \circ 0) \circ b^n = (a \circ b^n) \circ 0 < I$ and $0 \circ b^n < I$ imply that $a \circ b^n < I$. It means $(a \circ b) \circ b^{n-1} < I$. So $(a \circ b) \circ b^{n-1} < I$ and $b \circ b^{n-1} < I$ imply that $a \circ b^{n-} < I$. Repeatedly using this way it follows $a \circ b < I$. Since I is a hyper K-ideal and $b \in I$ we have $a \in I$, which is a contradiction. Therefore $|a \circ b^n| = 1$, for all $b \in I$.

Conversely, let $|a \circ b^n| = 1$, for all $b \in I$. We show that I is an n-fold PIHKI of type 25. On the contrary, let $(x \circ y) \circ z^n < I$ and $y \circ z^n < I$, but $x \circ z^n \nless I$, for some $x, y, z \in H$. $x \circ z^n \nless I$ implies that $x \neq z$. By Theorem 1.11(ii) $x \in x \circ z$. Thus by hypothesis we obtain $x = a$. If $x = y$, then $y \circ z^n = a \circ z^n = \{a\} \nless I$, which is a contradiction. If $x \neq y$, then $(x \circ y) \circ z^n = a \circ z^n = \{a\} \nless I$, which is a contradiction. Therefore I is an n-fold PIHKI of type 25.

Theorem 3.3. *Let Let $a \in H - \{0\}$ and $I = H - \{a\}$. If I is an n-fold PIHKI of type* 27(13,18,20), *then*

(i) $|a \circ b^n| = 1$, *for all* $b \in I$,

(ii) $b \circ c^n \neq H$, *for all* $b, c \in H$.

Proof. (i) On the contrary, let $|a \circ b^n| > 1$, for some $b \in I$. Then there exists $t \in H - \{a\}$ such that $t \in a \circ b^n$. So $(a \circ t) \circ b^n < I$. Thus $(a \circ t) \circ b^n < I$ and $t \circ b^n < I$ imply that $a \circ b^n \subseteq I$, which is a contradiction. Because By Theorem 1.11(ii) $a \in a \circ b^n \subseteq I$ and so $a \in I$. Therefore $|a \circ b^n| = 1$, for all $b \in I$.

(ii) If there exist $b, c \in H$ such that $b \circ c^n \neq H$, then $(b \circ 0) \circ c^n < I$ and $o \circ c^n < I$ imply that $H = b \circ c^n \subseteq I$, which is impossible. Therefore $b \circ c^n \neq H$, for all $b, c \in H$.

The following example shows that the converse of the above theorem is not true in general.

Example 3.4. The following table shows a simple hyper K-algebra structure on $H = \{0, 1, 2, 3\}$.

$\circ$	0	1	2	3
0	{0}	{0}	{0,2}	{0}
1	{1}	{0}	{1,2}	{1}
2	{2}	{2}	{0}	{2}
3	{3}	{3}	{2,3}	{0}

We can see that $2 \circ b^2 = \{2\}$, for all $b \in H - \{2\}$, and $b \circ c^2 \neq H$, for all $b, c \in H$, but $I = H - \{2\}$ is not a 2-fold PIHKI of type 27. Because $(2 \circ 0) \circ 2^2 < I$ and $0 \circ 2^2 < I$, while $2 \circ 2^2 = \{0, 2\} \not\subseteq I$.

Theorem 3.5. *Let $a \in H - \{0\}$ and $I = H - \{a\}$. Then I is an n-fold PIHKI of type* 10(12, 23, 24) *if and only if $|a \circ b^n| = 1$, for all $b \in I$.*

Proof. The proof is similar to the proof of Theorem 3.2, by imposing some modifications.

Theorem 3.6. *Let $a \in H - \{0\}$ and $I = H - \{a\}$. If $|a \circ b^n| = 1$, for all $b \in I$, then I is an n-fold PIHKI of type* 6(5, 7, 8).

Proof. Let $(x \circ y) \circ z^n \subseteq I$, and $(y \circ z^n) < I$. We show that $x \circ z^n < I$. If $x = z$, it is clear that $x \circ z^n < I$. Now let $x \neq z$. Consider two cases: case(1): $x \neq a$, and case(2): $x = a$. Case(1): By Theorem 1.11(ii) we obtain $x \in x \circ z^n$ and so $x \circ z^n < I$. case(2): We consider the following two sub-cases and show that $(x \circ y) \circ z^n \not\subseteq I$ or $y \circ z \not< I$.

case(i'): $y = x$ implies that $\{a\} = y \circ z^n = x \circ z^n = a \circ z^n \not< I$.

case(ii'): $y \neq x$ implies that $\{a\} = (x \circ y) \circ z^n \not\subseteq I$. Therefore I is an n-fold PIHKI of type 7.

The following example shows that the converse of the above theorem is not true in general.

Example 3.7. The following table shows a simple hyper K-algebra structure on $H = \{0, 1, 2, 3\}$.

$\circ$	0	1	2	3
0	{0}	{0}	{0,2}	{0}
1	{1}	{0}	{1,2}	{1}
2	{2}	{2}	{0}	{2}
3	{3}	{3}	{2,3}	{0}

We can see that $I = H - \{1\}$ is an n-fold PIHKI of type 5,6,7 and 8, but $|1 \circ 2^n| \neq 1$.

Theorem 3.8. *Let $a \in H - \{0\}$ and $I = H - \{a\}$. Then I is an n-fold PIHKI of type* 10(12,14,15,16,17,19,21,23,24,25,26) *if and only if I is a hyper K-ideal of H.*

Proof. The proof follows from Theorems 3.2 and 3.5.

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