RECOGNITION OF A_{10} AND $L_4(4)$ BY TWO SPECIAL CONJUGACY CLASS SIZES[1]

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Abstract. It is well-known that A_{10} is the smallest (by order) nonabelian simple group with connected prime graph and $L_4(4)$ is the smallest nonabelian simple group of Lie type with connected prime graph. In 2009, A.V. Vasil'ev first dealt with the groups with connected prime graph and proved that Thompson's conjecture holds for A_{10} and $L_4(4)$ (see [1]). In this work, the authors characterize finite simple groups A_{10} and $L_4(4)$ by their orders and largest and smallest conjugacy class sizes greater than 1, and partially generalize A.V. Vasil'ev's work.

Keywords: finite simple groups, conjugacy class size, prime graph.

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1. Introduction

Throughout this paper, groups under consideration are finite. For any group G, $\pi(G)$ denotes the set of prime divisors of $|G|$. We associate to $\pi(G)$ a simple graph called prime graph of G, denoted by $\Gamma(G)$. Prime graph $\Gamma(G)$ is defined as

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follows: the vertex of $\Gamma(G)$ is the set of all prime divisors of the order of G, two distinct vertexes p and q are adjacent by edge if and only if there is an element of order pq in G(see [10]). Denote the connected components of the prime graph by $T(G) = \{\pi_i(G)|1 \leqslant i \leqslant t(G)\}$, where $t(G)$ is the number of the prime graph components of G. If the order of G is even, we always assume that $2 \in \pi_1(G)$. In addition, for $x \in G$, $cl_G(x)$ denotes the conjugacy class in G containing x and $C_G(x)$ denotes the centralizer of x in G. Let $cs(G) = \{n \in \mathbb{N}|G$ has a conjugacy class C such that $|C| = n\}$. For $p \in \pi(G)$, we denote G_p and $Syl_p(G)$ a Sylow $p-$subgroup of G and the set of all of its Sylow $p-$subgroups, respectively. We also denote $Soc(G)$ the socle of G which is the subgroup generated by the set of a minimal normal subgroups of G. The other notation and terminologies in this paper are standard and the reader is referred to [8] if necessary. The second author G.Y. Chen once worked on J.G. Thompson's conjecture posed by J.G. Thompson in 1980s, which is about characterizing finite simple groups by the set of lengths of its conjugacy classes as following (ref. to [[9], Problem 12.38]):

Thompson's conjecture. *Let G be a finite group with $Z(G) = 1$ and L is a finite non-abelian simple group satisfying that $cs(G) = cs(L)$, then $G \simeq L$.*

In 1994, G.Y. Chen proved in his Ph.D. dissertation [3] that if G is a group with $Z(G) = 1$, and L a non-abelian simple group with non-connected prime graph such that $cs(G) = cs(L)$, then $G \simeq L$ (also ref. to [4], [5], [6]). In 2009, A.V. Vasil'ev first dealt with the groups with connected prime graph and proved that Thompson's conjecture holds for A_{10} and $L_4(4)$ (see [1]). In 2011, N. Ahanjideh in [2] proved that Thompson's conjecture is true for $L_n(q)$. Recently, G.Y. Chen and J.B. Li contributed their interests on special class sizes of finite simple groups, and characterize successfully sporadic simple groups (see J.B. Li's Ph.D. dissertation [15])and simple K_3-groups (to prepared) by their orders and few special class sizes greater than 1. In their papers, they provided two new ways to characterize finite simple group by group order and largest class size, or smallest class size greater than 1. More importantly, one of two methods doesn't consider about connection of prime graph of group. Thus it is may be effective to deal with simple groups which have connected prime graph. In this paper, we focus our attention on simple groups A_{10} and $L_4(4)$ which have connected prime graphs, and characterize A_{10} and $L_4(4)$ by their orders, and largest and smallest conjugacy class sizes greater than 1, respectively. In addition, we partially generalize A.V. Vasil'ev's work (see [1]) and prove that Thompson's conjecture holds for A_{10} and $L_4(4)$ at the same time. That is the following theorem. For convenience, $lcs(G)$ and $scs(G)$ denote largest and smallest conjugacy class size greater than 1 of group G, respectively.

Main Theorem. *Let G be a group and L one of A_{10} and $L_4(4)$. Then $G \simeq L$ if and only if $|G| = |L|$ and $lcs(G) = lcs(L)$ and $scs(G) = scs(L)$.*

If Main Theorem is proved, then the following corollary holds, which proves Thompson's conjecture for A_{10} and $L_4(4)$.

Corollary. *Thompson's conjecture holds for finite simple group A_{10} and $L_4(4)$.*

Proof. Let G be a group and L one of A_{10} and $L_4(4)$. Under the hypothesis of Thompson's conjecture, it is proved in [1] that $|G| = |L|$. Hence the corollary follows from Main Theorem.

2. Preliminaries

First, we generalize a simple fact which is used many times in G.Y. Chen and J.B. Li's works. It is important to prove our Main Theorem.

Lemma 2.1. *Let G be a group, $\overline{G} = G/Z(G)$. $\overline{N}$ is a minimal normal subgroup of $\overline{G}$, and N is the pre-image of $\overline{N}$ in G. If $p \in \pi(\overline{N})$ for some $p \in \pi(G)$ and $N_p \in Syl_p(N)$ satisfying $|N_p| < scs(G)$, then $\overline{N}$ is not solvable.*

Proof. Assume that $\overline{N}$ is solvable. Then $\overline{N}$ is an elementary abelian $p-$group with $|\overline{N}| = p^t$, $t \geq 1$, and N is a nilpotent normal subgroup of G by the hypothesis. Hence N_p is a normal subgroup of G, and N_p is not a subgroup of $Z(G)$. So there exists an element x of $N_p - Z(G)$ satisfying that

$$1 < |cl_G(x)| = |G : C_G(x)| \leq |N_p| < scs(G),$$

violating the hypothesis.

By Lemma 2.1, the fact can easily be obtained as a corollary following.

Corollary 2.2. *Let G be a group, $\overline{G} = G/Z(G)$. If $|G_p| < scs(G)$ for any $p \in \pi(G)$, then $Soc(\overline{G}) \trianglelefteq \overline{G} \lesssim Aut(Soc(\overline{G}))$.*

Proof. Suppose that $\overline{N}$ is any minimal normal subgroup of $\overline{G}$, and N is the pre-image of $\overline{N}$ in G. By the hypothesis, N satisfies that $|N_p| \leq |G_p| < scs(G)$, so every minimal normal subgroup of $\overline{G}$ is not solvable by Lemma 2.1. Let $S_1, S_2, \ldots, S_k (k \geq 1)$ be all minimal normal subgroup of $\overline{G}$. Let $M = Soc(\overline{G})$, hence $M = Soc(\overline{G}) = S_1 \times S_2 \times \cdots \times S_k$ and S_i is a direct product of some isomorphic non-abelian simple groups for $i = 1, 2, \ldots, k$. Now, we assert that $C_{\overline{G}}(M) = 1$. If not, there exists a minimal normal subgroup $\overline{S}$ of $\overline{G}$ such that $\overline{S} \leq C_{\overline{G}}(M) \bigcap M$. Thus $\overline{S}$ is an abelian group, a contradiction. By N/C theorem, we have $M \trianglelefteq \overline{G} = \overline{G}/C_{\overline{G}}(M) \lesssim \mathrm{Aut}(M)$, as desired.

Lemma 2.3. *Let K be a normal subgroup of a group G, and $\overline{G} = G/K$. If $\overline{x}$ is the image of an element x of G in $\overline{G}$, and $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$. In particular, if $K = Z(G)$, then $C_{\overline{G}}(\overline{x}) = C_G(x)/Z(G)$.*

Proof. This is an immediate consequence of Theorem 1.6.2 in [14] or Lemma 5 in [1]. For $\pi(A_{10}), \pi(L_4(4)) \subseteq \{2,\ 3,\ 5,\ 7,\ 17\}$, we need to list all the non-abelian simple groups L satisfying with $\pi(L) \subseteq \{2, 3, 5, 7, 17\}$.

Lemma 2.4. *Let L be a non-abelian simple group. If $\pi(L) \subseteq \{2, 3, 5, 7, 17\}$, then L is isomorphic to one of simple groups of Table 1. Especially, $\{2, 3\} \subseteq \pi(L)$, and if $L \neq S_6(2), S_8(2)$, then $\pi(Out(L)) \subseteq \{2, 3\}$.*

Proof. This is Lemma 2.5 in [7].

Table 1. Non-abelian simple groups L with $\pi(L) \subseteq \{2,3,5,7,17\}$

L	Order of L	$\lvert\mathrm{Out}(L)\rvert$	L	Order of L	$\lvert\mathrm{Out}(L)\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	2	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	$\lvert D_8\rvert$
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	$\lvert S_3\rvert$
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2^2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\lvert D_{12}\rvert$	$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	He	$2^{10} \cdot 3^3 \cdot 7^3 \cdot 17$	2
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	$\lvert S_3\rvert$			

A group G is said to be an almost simple group related to L if and only if $L \trianglelefteq G \leq \mathrm{Aut}(L)$ for some non-abelian simple group L. Almost simple groups related to L with $\pi(L) \subseteq \{2,3,5,7,17\}$ are listed in the following lemma.

Lemma 2.5. *Let L be a non-abelian simple group such that $\pi(L) \subseteq \{2,3,5,7,17\}$. If $L \trianglelefteq G \leq Aut(L)$, then G is isomorphic to one of the groups listed in Table 2.*

Table 2. Almost simple groups $L \trianglelefteq G \leq Aut(L)$ with $\pi(L) \subseteq \{2,3,5,7,17\}$

L	G	L	G	L	G	L	G
A_5	L	$L_3(4)$	L	$S_6(2)$	L	$L_4(4)$	L
	$L \cdot 2$		$L \cdot 2_1$	A_{10}	L		$L \cdot 2_1$
$L_2(7)$	L		$L \cdot 3$		$L \cdot 2$		$L \cdot 2_2$
	$L \cdot 2$		$L \cdot 6$	$S_4(7)$	L		$L \cdot 2_3$
A_6	L		$L \cdot 2_2$		$L \cdot 2$		$L \cdot 2^2$
	$L \cdot 2_1$		$L \cdot 3 \cdot 2_2$	$O_8^+(2)$	L	$L_2(16)$	L
	$L \cdot 2_2$		$L \cdot 2_3$		$L \cdot 2$		$L \cdot 2$
	$L \cdot 2_3$		$L \cdot 3 \cdot 2_3$		$L \cdot 3$		$L \cdot 4$
	$L \cdot 2^2$		$L \cdot 2^2$		$L \cdot S_3$	$O_8^-(2)$	L
$L_2(8)$	L		$L \cdot D_{12}$	$U_4(3)$	L		$L \cdot 2$
	$L \cdot 3$	$L_2(49)$	L		$L \cdot 2_1$	$S_4(4)$	L
A_7	L		$L \cdot 2_1$		$L \cdot 4$		$L \cdot 2$
	$L \cdot 2$		$L \cdot 2_2$		$L \cdot 2_2$		$L \cdot 4$
$U_3(3)$	L		$L \cdot 2_3$		$L \cdot (2^2)_{122}$		
	$L \cdot 2$		$L \cdot 2^2$		$L \cdot (2^2)_{133}$		
A_8	L	$U_3(5)$	L		$L \cdot D_8$		
	$L \cdot 2$		$L \cdot 2$	$L_2(17)$	L		
$U_4(2)$	L		$L \cdot 3$		$L \cdot 2$		
	$L \cdot 2$		$L \cdot S_3$	He	L		
A_9	L	J_2	L		$L \cdot 2$		
	$L \cdot 2$		$L \cdot 2$	$S_8(2)$	L		

Proof. All almost simple groups not related to $L_2(17)$, $L_2(16)$, $S_4(4)$, $S_8(2)$, $O_8^-(2)$, $L_4(4)$, and He listed in Table 2 were given in Proposition 1 in [11]. Those related to one of $L_2(17)$, $L_2(16)$, $S_4(4)$, $S_8(2)$, $O_8^-(2)$, $L_4(4)$, and He are easily obtained by an algorithm from [12].

Lemma 2.6. *Let $R = R_1 \times \cdots \times R_k$, where R_i is a direct product of n_i isomorphic copies of a non-abelian simple group H_i, where H_i and H_j are not isomorphic if $i \neq j$. Then* $\mathrm{Aut}(R) \simeq \mathrm{Aut}(R_1) \times \cdots \times \mathrm{Aut}(R_k)$ *and* $\mathrm{Aut}(R_i) \simeq (\mathrm{Aut}(H_i) \wr S_{n_i}$, *where in this wreath product* $\mathrm{Aut}(H_i)$ *appears in its right regular representation and the symmetric group S_{n_i} in its natural permutation representation. Moreover, these isomorphisms induce outer automorphisms* $\mathrm{Out}(R) \simeq \mathrm{Out}(R_1) \times \cdots \times \mathrm{Out}(R_k)$ *and* $\mathrm{Out}(R_i) \simeq (\mathrm{Out}(H_i) \wr S_{n_i}$.

Proof. This is Theorem 3.3.20 in [13].

3. Proof of the Main Theorem

We divide the sufficient proof of Main Theorem into two lemmas.

Lemma 3.1. *Let G be a group with $|G| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$. If $lcs(G) = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7$ and $scs(G) = 2^4 \cdot 3 \cdot 5$. Then $G \simeq A_{10}$.*

Proof. It is clear that one has that $Z(G) \leqslant C_G(x)$ for any $x \in G$. Set x, and $y \in G$ such that $lcs(G) = |cl_G(x)| = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7$ and $scs(G) = |cl_G(y)| = 2^4 \cdot 3 \cdot 5$. Since $|G| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ and $lcs(G) = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7$, $Z(G)$ is a proper subgroup of G with $|Z(G)| \mid 2^3$ by the hypothesis. Thus 3, 5, and $7 \notin \pi(Z(G))$. Considering $\overline{G} = G/Z(G)$. For any prime $p \in \pi(G)$, the order of Sylow $p-$subgroup of G is less than $scs(G)$. By Corollary 2.2, we know that every minimal normal subgroup of $\overline{G} = G/Z(G)$ is non-solvable and $Soc(\overline{G}) \trianglelefteq \overline{G} \leq \mathrm{Aut}(Soc(\overline{G}))$. Let $M = Soc(\overline{G})$ and $S_1, S_2, \cdots, S_k (k \geq 1)$ be all minimal normal subgroups of $\overline{G}$, hence $M = Soc(\overline{G}) = S_1 \times S_2 \times \cdots \times S_k$ and S_i is a direct product of some isomorphic non-abelian simple groups for $i = 1, 2, \ldots, k$.

We assert that $3 \in \pi(M)$. Otherwise, M is a simple K_3-group with $\pi(M) = \{2, 5, 7\}$. This is impossible by Table 1.

We assert that $5 \in \pi(M)$. Otherwise, M is a simple K_3-group with $\pi(M) = \{2, 3, 7\}$ and $5 \in \pi(\mathrm{Out}(G))$. By Table 1, we find that M is isomorphic to one of the following simple groups: $L_2(7), L_2(8)$, and $U_3(3)$. By Lemma 2.4, we see that $|\mathrm{Out}(L_2(7))| = |\mathrm{Out}(U_3(3))| = 2$, and $|\mathrm{Out}(L_2(8))| = 3$, a contradiction to $5 \in \pi(\mathrm{Out}(G))$.

We also assert that $7 \in \pi(M)$. Otherwise, $\pi(M) = \{2, 3, 5\}$ and $7 \in \pi(\mathrm{Out}(G))$. By Table 1, M may be isomorphic to one of the following groups:

$$A_5, A_6, U_4(2), A_5 \times A_5, A_5 \times A_6, \text{ and } A_6 \times A_6.$$

By Lemma 2.4 and Lemma 2.6, we see that outer automorphism groups of these groups are $2-$groups, contradicting to $7 \in \pi(\mathrm{Out}(G))$. Hence $\{3, 5, 7\} \subseteq \pi(M)$. By Table 1 again, M may be isomorphic to one of the following groups:

$$A_7, A_8, L_3(4), A_9, J_2, A_{10}, A_5 \times L_2(7), A_5 \times L_2(8), A_5 \times U_3(3), A_5 \times A_7,$$
$$L_2(7) \times A_6, \text{ and } L_2(8) \times A_6.$$

Now, let us recall that $M \trianglelefteq \overline{G} \leq \mathrm{Aut}(M)$. If M is isomorphic to one of $A_7, A_8, L_3(4), A_9, A_5 \times L_2(7), A_5 \times L_2(8), A_5 \times U_3(3), L_2(7) \times A_6$, and $L_2(8) \times A_6$, then we have that $5 \parallel |\overline{G}|$ by Table 1 and Lemma 2.6. Hence $5 \in \pi(Z(G))$, a contradiction. If M is isomorphic to one of J_2, and $A_5 \times A_7$, then by the same reasoning $3 \in \pi(Z(G))$, a contradiction.

Hence $M \simeq A_{10}$ and G must be isomorphic to A_{10} by comparing the orders of M and G, as claimed.

Lemma 3.2. *Let G be a group with $|G| = 2^{12}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}17$. If $lcs(G) = 2^{10}{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}17$ and $scs(G) = 3^2 \cdot 5 \cdot 7 \cdot 17$. Then $G \simeq L_4(4)$.*

Proof. First, for any $x \in G$, $Z(G)$ is contained in $C_G(x)$. By the hypothesis, there exist y, and $z \in G$ such that $scs(G) = |cl_G(y)| = 3^2 \cdot 5 \cdot 7 \cdot 17$ and $lcs(G) = |cl_G(z)| = 2^{10}{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}17$. Since $|G| = 2^{12}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}17$, and $lcs(G) = 2^{10}{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}17$, we have that $Z(G)$ is a proper subgroup of G of order dividing 36. Similar to Lemma 3.1, Considering $\overline{G} = G/Z(G)$. For any prime $p \in \pi(G)$, the order of Sylow $p-$subgroup of G is less than $scs(G)$. Hence by Corollary 2.2, we know that every minimal normal subgroup of $\overline{G}$ is non-solvable and $Soc(\overline{G}) \trianglelefteq \overline{G} \leq \mathrm{Aut}(Soc(\overline{G}))$. Let $M = Soc(\overline{G})$ and $S_1,\ S_2,\ \ldots,\ S_k (k \geq 1)$ be all minimal normal subgroups of $\overline{G}$. Hence $M = Soc(\overline{G}) = S_1 \times S_2 \times \cdots \times S_k$ and S_i is a direct product of some isomorphic non-abelian simple groups for $i = 1, 2, \ldots, k$. Similar to the proof Lemma 3.1, we prove $5,\ 7$, and $17 \in \pi(M)$.

If $5 \notin \pi(M)$, then $5 \in \pi(\mathrm{Out}(M))$. Applying Table 1 and possible order of M, M may be isomorphic to one of the following groups:

$$L_2(7),\ L_2(8),\ L_2(17),\ U_3(3),\ L_2(7) \times L_2(17),\ \text{and } L_2(8) \times L_2(17).$$

By Lemmas 2.4 and 2.6, we see that outer automorphism groups of groups above are $\{2,\ 3\}-$groups, a contradiction to $5 \in \pi(\mathrm{Out}(M))$. Therefore, $5 \in \pi(M)$.

If $7 \notin \pi(M)$, then $7 \in \pi(\mathrm{Out}(M))$. By Table 1 and $5 \in \pi(M)$, M may be isomorphic to one of the following groups:

$$A_5,\ A_6,\ L_2(16),\ U_4(2),\ S_4(4),\ A_5 \times A_5,\ A_5 \times A_6,\ A_6 \times A_6,\ A_5 \times L_2(17),$$
$$A_5 \times L_2(16), A_6 \times L_2(17),\ \text{and } A_6 \times L_2(16).$$

By Table 1 and Lemma 2.6, we see that outer automorphism groups of these groups are $2-$ groups, a contradiction. Hence $7 \in \pi(M)$.

If $17 \notin \pi(M)$, then $17 \in \pi(\mathrm{Out}(G))$. By Table 1 and $\{5, 7\} \subseteq \pi(M)$, M may be isomorphic to one of the following groups:

$$A_7,\ A_8,\ L_3(4),\ A_9,\ J_2,\ S_6(2),\ A_{10},\ A_5 \times L_2(7),\ A_5 \times L_2(8),\ A_5 \times A_7,\ A_5 \times A_8,$$
$$A_5 \times L_3(4),\ A_6 \times L_2(7),\ A_6 \times L_2(8),\ A_6 \times A_7,\ A_6 \times A_8,\ A_6 \times L_3(4),$$
$$\text{and } A_5 \times L_2(7) \times A_6.$$

By Table 1 and Lemma 2.6, we know that 17 is not a prime divisor of outer automorphism of those groups above, a contradiction. Hence $17 \in \pi(M)$. For convenience, we assume that $7 \in \pi(S_i)$, and $17 \in \pi(S_j)$ for $i, j \in \{1, 2, \cdots, k\}$.

If $i \neq j$, then S_i and S_j are two non-isomorphic simple groups. By Table 1 and possible order of M again, we see that M may be isomorphic to one of the following groups:

$$\begin{array}{l} L_2(7) \times L_2(16),\ L_2(7) \times L_2(17),\ L_2(7) \times S_4(4),\ L_2(8) \times L_2(16), \\ L_2(8) \times L_2(17),\ L_2(8) \times S_4(4), A_7 \times L_2(16),\ A_7 \times L_2(17),\ U_3(3) \times L_2(16), \\ A_8 \times L_2(16), A_8 \times L_2(17),\ L_3(4) \times L_2(16),\ L_3(4) \times L_2(17), \\ A_5 \times L_2(7) \times L_2(16),\ A_5 \times L_2(7) \times L_2(17),\ A_5 \times L_2(8) \times L_2(16), \\ \text{and } A_6 \times L_2(7) \times L_2(16). \end{array}$$

If M is isomorphic to one of the following groups:

$$\begin{array}{l} L_2(7) \times L_2(16),\ L_2(7) \times L_2(17), Ł_2(8) \times L_2(16),\ U_3(3) \times L_2(16),\ A_8 \times L_2(17), \\ L_3(4) \times L_2(17),\ L_2(8) \times L_2(17),\ A_7 \times L_2(17),\ \text{and } A_5 \times L_2(7) \times L_2(17), \end{array}$$

then, by Table 1 and Lemma 2.6, we come to $5 \in \pi(Z(G))$ by comparing the orders of M, $\overline{G}$, and $\mathrm{Aut}(M)$, a contradiction.

If $M \simeq L_2(7) \times S_4(4)$, then

$$\mathrm{Aut}(M) = \mathrm{Aut}(L_2(7)) \times \mathrm{Aut} S_4(4)) = L_2(7) \cdot 2 \times S_4(4) \cdot 4$$

by Lemma 2.6 and Table 2.

Recall that $M \trianglelefteq \overline{G} \leq \mathrm{Aut}(M)$, then $|Z(G)| = 3$ or 6. So there exists an element w of order 7 in G such that $C_G(w)/Z(G) = C_{\overline{G}}(\overline{w}) \geq \langle \overline{w} \rangle \times S_4(4)$ by Lemma 2.3, where $\overline{w}$ is the image of w in $\overline{G}$. Hence $|C_G(w)| \geq 2^8 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17$ such that $1 < |cl_G(w)| < scs(G)$, a contradiction to minimality of $scs(G)$.

If $M \simeq L_2(8) \times S_4(4)$, then by same way above we come to $\mathrm{Aut}(M) = L_2(8) \cdot 3 \times S_4(4) \cdot 4$ such that $|Z(G)| \leq 2$. So there exists an element w of order 7 in G such that such that $1 < |cl_G(w)| < scs(G)$, a contradiction.

In a similar way used above, we can deal with the remaining cases of M, and can always find an element of G such that its conjugacy class length is less than $scs(G)$, leading to a contradiction.

Hence $i = j$. Without loss of generality, assume that 7, and $17 \in \pi(S_1)$. Then S_1 is a non-abelian simple group and isomorphic to $O_8^-(2)$ or $L_4(4)$ by Table 1. Therefore $k = 1$, and M may be isomorphic to one of following groups: $O_8^-(2)$, and $L_4(4)$.

If $M \simeq O_8^-(2)$, then, by Table 1 and Table 2, $\overline{G} \simeq O_8^-(2)$ or $O_8^-(2) \cdot 2$ and $\mathrm{Aut}(M) = O_8^-(2) \cdot 2$. Comparing the orders of M, $\overline{G}$, and $\mathrm{Aut}(M)$, we see that $|Z(G)| = 5$ and $\overline{G} \simeq O_8^-(2)$. If G is a split extension $O_8^-(2)$ by $Z(G)$, then $G = O_8^-(2) \times Z(G)$. Therefore, by [8], there exists a non-central element w of order 2 in G such that $1 < |cl_G(w)| < scs(G)$, leading to a contradiction. Hence G is not a split extension $O_8^-(2)$ by $Z(G)$, which implies that 5 divides $|\mathrm{Mult}(O_8^-(2))|$, a contradiction to $|\mathrm{Mult}(O_8^-(2))| = 2$ by [8].

Hence $M \simeq L_4(4)$ and so G must be isomorphic to $L_4(4)$ by $|G| = |M|$, as desired.

Proof of the Main Theorem. The necessity is obvious by [8] and the sufficiency follows from Lemmas 3.1 and 3.2.

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