THREE REPRESENTATIONS OF A HYPERBOLIC QUADRIC OF $PG(3,q)$ IN $AG(2,q)$

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Summary. We construct three different representations of a hyperbolic quadric of a projective Galois space PG(3,q) in the affine Galois plane $AG(2,q)$. To do this, we use the representation R, or $R(U_1, U_2, \pi, 3)$ of the projective space $P(r,k)$, over the field k, in the affine plane $A(2,k)$, over the same field k, called also "Crashing", cited in the bibliography [1]. Further applications of this representation are the construction of maximal partial line spreads in PG, q even, a geometric proof of the equivalence between the Desargues and the Veblen theorems and a geometric proof of the equivalence between the Pappus-Pascal theorem and the "Three stars theorem". Those results will soon appear.

1. First representation

Theorem 1. Theorem of the hyperbola and the hyperbolic quadric. *Let $PG(3,q)$ be the projective space of dimension three over the Galois field $GF(q)$, let $AG(2,q)$ be the affine plane over the same field and let R be an $R(U_1, U_2, \pi, 3)$-representation of $PG(3,q)$, as in* [1]. *Let $\mathcal{I}$ be a hyperbola of $AG(2,q)$ and let t_1 and t_2 be the asymptots of $\mathcal{I}$. Let T_1 and T_2 be the points of π represented through R by the lines t_1 and t_2 of $AG(2,q)$, respectively. For any point X of $\mathcal{I}$, let X_1 be the point common to t_1 and to the line through X, parallel to t_2, let X_2 be the point common to the line t_2 and to the line through X parallel to t_1, let ℓ_X be the line of $PG(3,q)$ represented through R, by the ordered pair (X_1, X_2) and, finally, let $\overline{\ell}_X$ be the line of $PG(3,q)$ represented through R by the ordered pair (X_2, X_1). Then, the following sets of $PG(3,q)$:*

$$\mathcal{R} = \{\ell_X\}_{X\in\mathcal{I}} \cup \{U_1T_2, U_2T_1\},$$
$$\overline{\mathcal{R}} = \{\overline{\ell}_X\}_{X\in\mathcal{I}} \cup \{U_1T_1, U_2T_2\},$$

where U_iT_j, $i,j = 1,2$, denotes the line of $PG(3,q)$ through the points U_i and T_j, are the two reguli of a hyperbolic quadric of $PG(3,q)$ meeting π at a non-degenerate conic, admitting v as a secant line.

Proof. Let $PG(3,q)$ be the three dimensional projective space over $GF(q)$, let $AG(2,q)$ be the affine plane over $GF(q)$ and let R be an $R(U_1, U_2, \pi, 3)$-representation of $PG(3,q)$. Let $\mathcal{I}$ be a hyperbola of $AG(2,q)$, let t_1 and t_2 be the asymptots of $\mathcal{I}$ and let O be the common point of t_1 and t_2. For any $i = 1, 2$, the line t_i represents, through R, a point T_i of $\pi - \{Y\}$ (see Fig. 1), where Y is the point $U_1U_2 \cap \pi$. The line of π joining T_1 and T_2 does not contain Y, since in $AG(2, q)$ the lines t_1 and t_2 meet at O. It follows that the lines U_1T_1 and U_2T_2 of $PG(3, q)$ are skew, like the lines U_1T_2 and U_2T_1. Obviously, the four lines U_iT_j, $i, j = 1, 2$, form a skew quadrangle, denoted by Q.

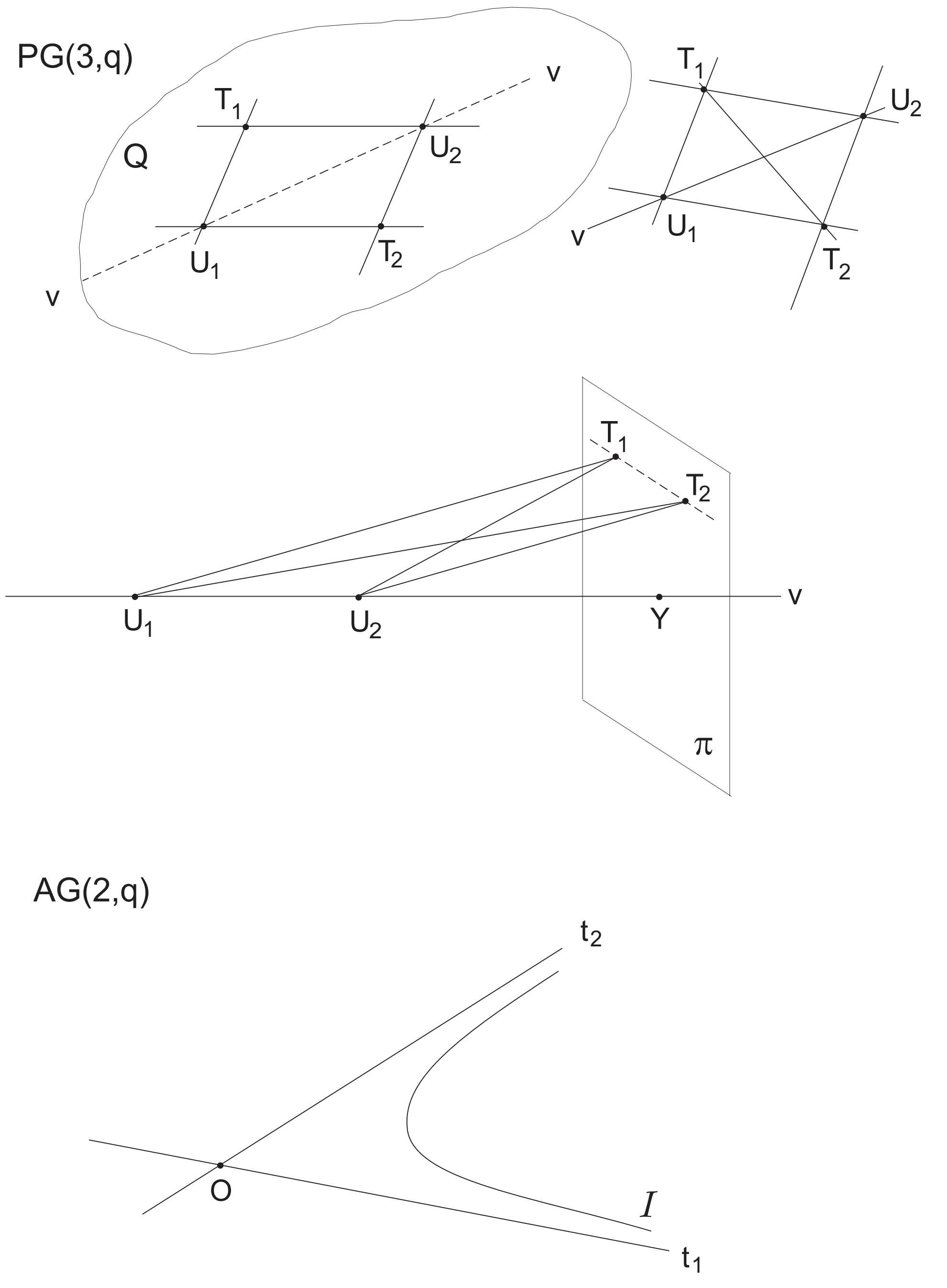

Figure 1.

The lines U_iT_j, $i, j = 1, 2$, of $PG(3, q)$ are represented by R in $AG(2, q)$ in the following way:

$$\begin{aligned} U_1T_1 &: \{(t_1\} \cup \{(t_1, t)\}_{t\in\mathcal{T}_1}, \\ U_1T_2 &: \{(t_2\} \cup \{(t_2, t)\}_{t\in\mathcal{T}_2}, \\ U_2T_1 &: \{(t_1\} \cup \{(t, t_1)\}_{t\in\mathcal{T}_1}, \\ U_2T_2 &: \{(t_2\} \cup \{(t, t_2)\}_{t\in\mathcal{T}_2}, \end{aligned}$$

where $\mathcal{T}_i, i = 1, 2$, is the set of the lines of $AG(2, q)$ parallel to t_i and distinct from t_i.

Now, let A be a point of $\mathcal{I}$ (see Fig. 2).

Let t_1' be the line of $AG(2, q)$ through A and parallel to t, and t_2' the line through A parallel to t_2. Moreover, let $A_1 = t_1 \cap t_2'$, $A_2 = t_2 \cap t_1'$. It is $A_1 \neq A_2$, since $A_i \in t_i - \{O\}$, for any $i = 1, 2$. The ordered pair of distinct points (A_1, A_2) of $AG(2, q)$ represents, by R, a line ℓ of $PG(3, q)$ not meeting v and not in π (see Fig. 2). By the representations of ℓ and U_iT_j, $i, j = 1, 2$, in $AG(2, q)$, we get:

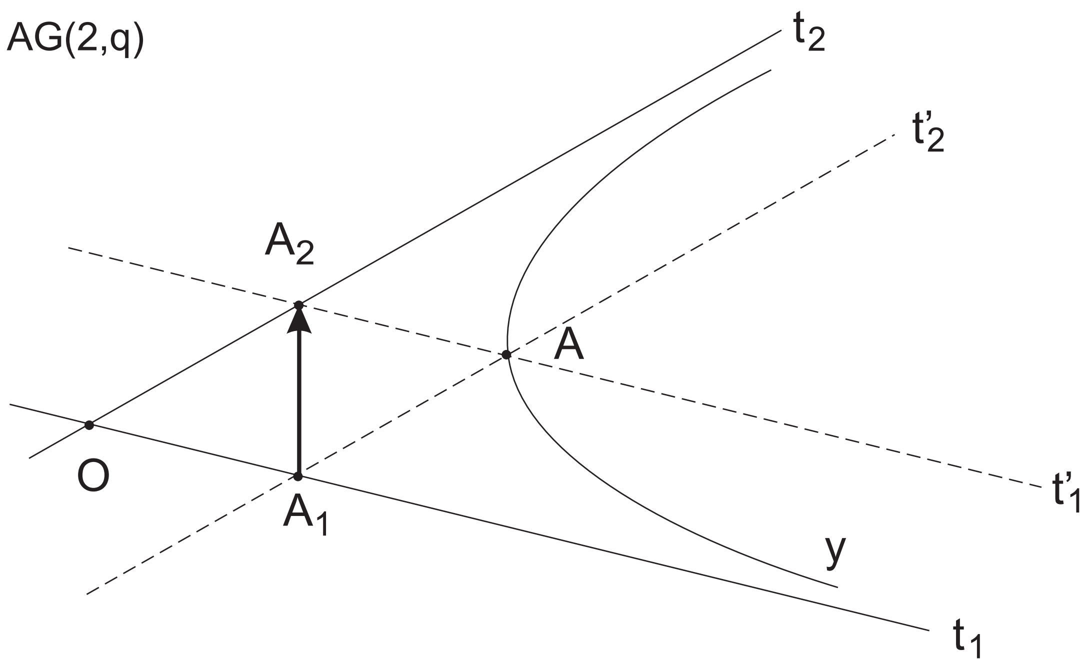

Figure 2.

1) The line ℓ meets the line U_1T_1 at L' represented by the ordered pair (t_1, t'_1). Such a point L' is distinct from U_1 and T_1, because the ordered pairs of distinct lines of $AG(2, q)$ represent the points of $PG(3, q)$ not in π (see Fig. 3).

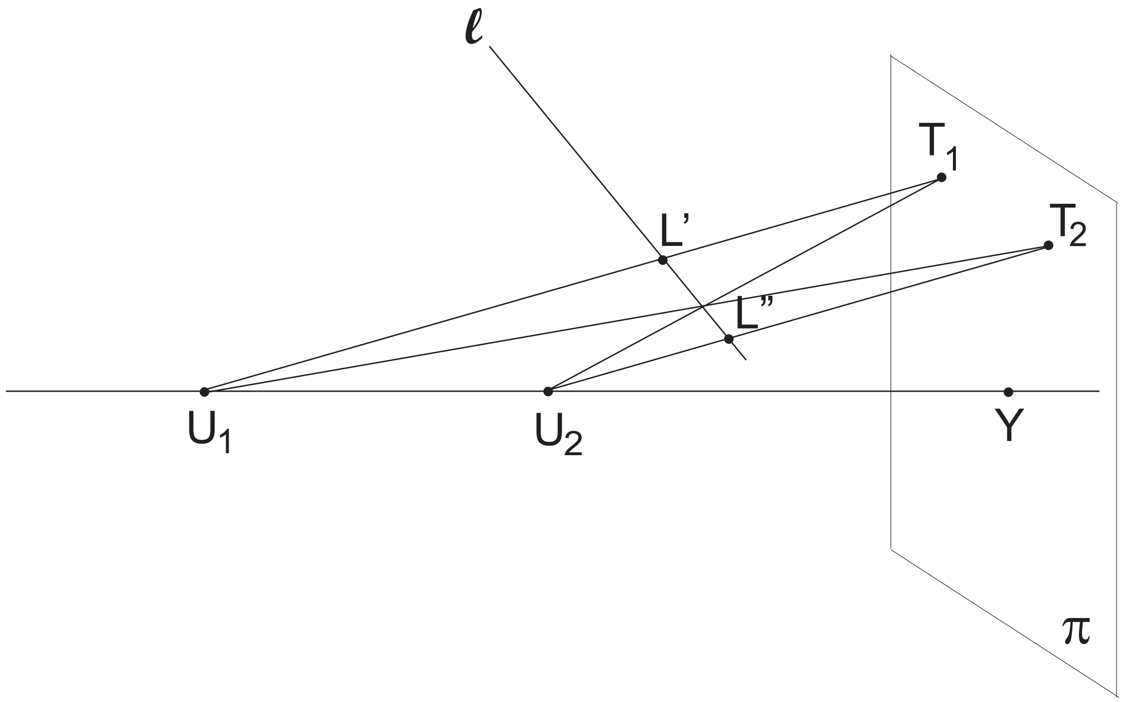

Figure 3.

2) The line ℓ meets the line U_2T_2 at the point L'' represented by the ordered pair (t'_2, t_2) and such a point is distinct from U_2 and T_2.

3) The line ℓ does not meet either U_1T_2, or U_2T_1.

By 3) and since the lines U_2T_1 and U_1T_2 of $PG(3, q)$ are mutually skew, it follows that the lines ℓ, U_1T_2 and U_2T_1 are two by two skew. Let us denote by $\mathcal{H}$ the hyperbolic quadric of $PG(3, q)$ containing ℓ, U_1T_2 and U_2T_1. Then, call $\mathcal{R}$ the regulus of $\mathcal{H}$ containing ℓ, U_1T_2 and U_2T_1. By 1) and 2), it follows that U_1T_1 and U_2T_2 belong to the regulus $\overline{\mathcal{R}}$ of $\mathcal{H}$ opposite to $\mathcal{R}$. The ordered pair (A_2, A_1) represents a line $\overline{\ell}$ of $PG(3, q)$ not meeting v and not in π.

By the representations of $\overline{\ell}$ and U_iT_j, $i, j = 1, 2$, in $AG(2, q)$, we get:

4) The line $\overline{\ell}$ meets U_2T_1, at the point $\overline{L}'$, represented by the ordered pair (t'_1, t_1); such a point $\overline{L}'$ is distinct from U_2 and T_1.

5) The line $\overline{\ell}$ meets U_1T_2, at the point $\overline{L}''$, represented by the ordered pair (t_2, t'_2); such a point is distinct from U_1 and T_2.

6) The line $\overline{\ell}$ does not meet either U_1T_1, or U_2T_2.

7) The line $\overline{\ell}$ meets ℓ at the point P of π represented by the line A_1A_2 of $AG(2,q)$.

By 4), 5), 6) and 7) it follows that the line $\overline{\ell}$ is a line of $\overline{\mathcal{R}}$ distinct from U_1T_1 and U_2T_2 (see Fig. 4).

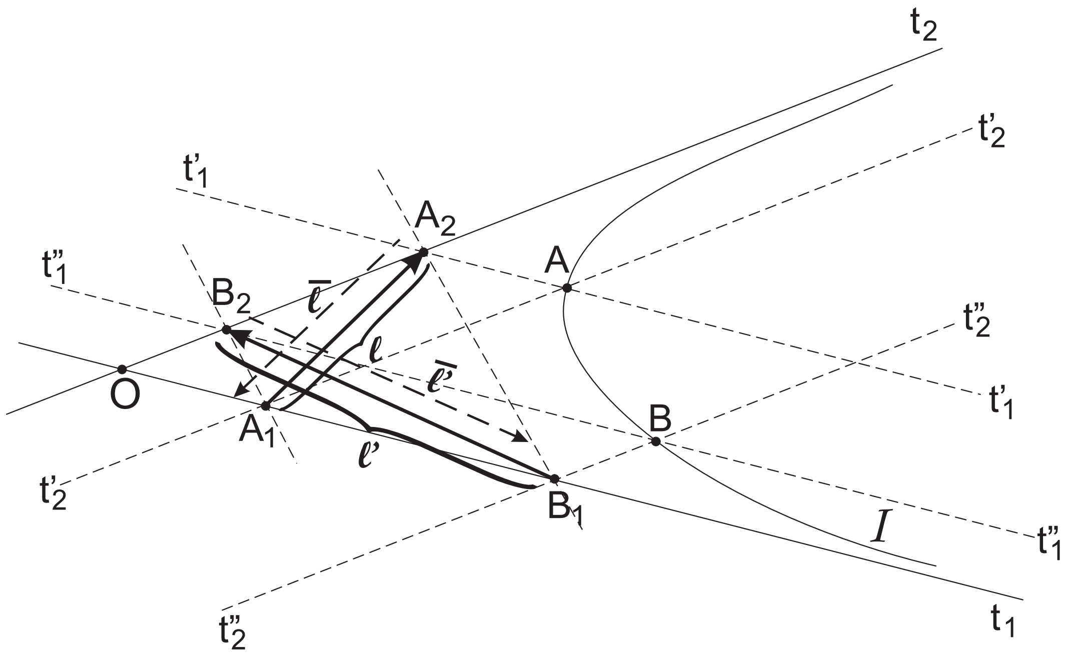

Figure 4.

Now, let B be a point of $\mathcal{I} - \{A\}$. Let t_1'' be the line of $AG(2,q)$ through B and parallel to t_1 and let t_2'' be the line of $AG(2,q)$ through B parallel to t_2. Let $B_1 = t_1 \cap t_2''$ and $B_2 = t_2 \cap t_1''$. The ordered pair of distinct points (B_1, B_2) represents a line ℓ' of $PG(3,q)$ not meeting v and not in π. Such a line meets U_1T_1 at the point of $PG(3,q)$ represented by the ordered pair (t_1, t_1''). The line ℓ' meets U_2T_1 at the point $PG(3,q)$ represented by the ordered pair (t_2'', t_2). Let us prove that ℓ' meets $\overline{\ell}$. To do this, choose a coordinate system in $AG(2,q)$ such that $T = T(0,0)$, $A_1 = A_1(1,0)$, $A_2 = A_2(0,1)$. In such a system, the coordinates of the point A are $(1,1)$ and the hyperbola $\mathcal{I}$ has the equation $xy = 1$. It follows that

$$\begin{aligned} B &= B\left(x_0, \frac{1}{x_0}\right), \\ B_1 &= B_1(x_0, 0), \\ B_2 &= B_2\left(0, \frac{1}{x_0}\right), \end{aligned}$$

with $x_0 \neq 0$. The slopes of the lines A_1B_2 and A_2B_1 are both equal to $a - \dfrac{\ell}{x_0}$, therefore such two lines are parallel. It follows that ℓ' meets $\overline{\ell}$. Since ℓ' meets U_1T_1, U_2T_2 and $\overline{\ell}$ which belong to $\overline{\mathcal{R}}$, it follows that $\ell' \in \mathcal{R}$. Similarly, we prove that the line ℓ' of $PG(3,q)$ represented by the ordered pair (B_2, B_1) belongs to $\overline{\mathcal{R}}$.

For any $X \in \mathcal{I}$, let X_1 be the point common to t_1 and to the line through X parallel to t_2 and let X_2 be the point common to t_2 and to the line through X parallel to t_1 (see Fig. 5).

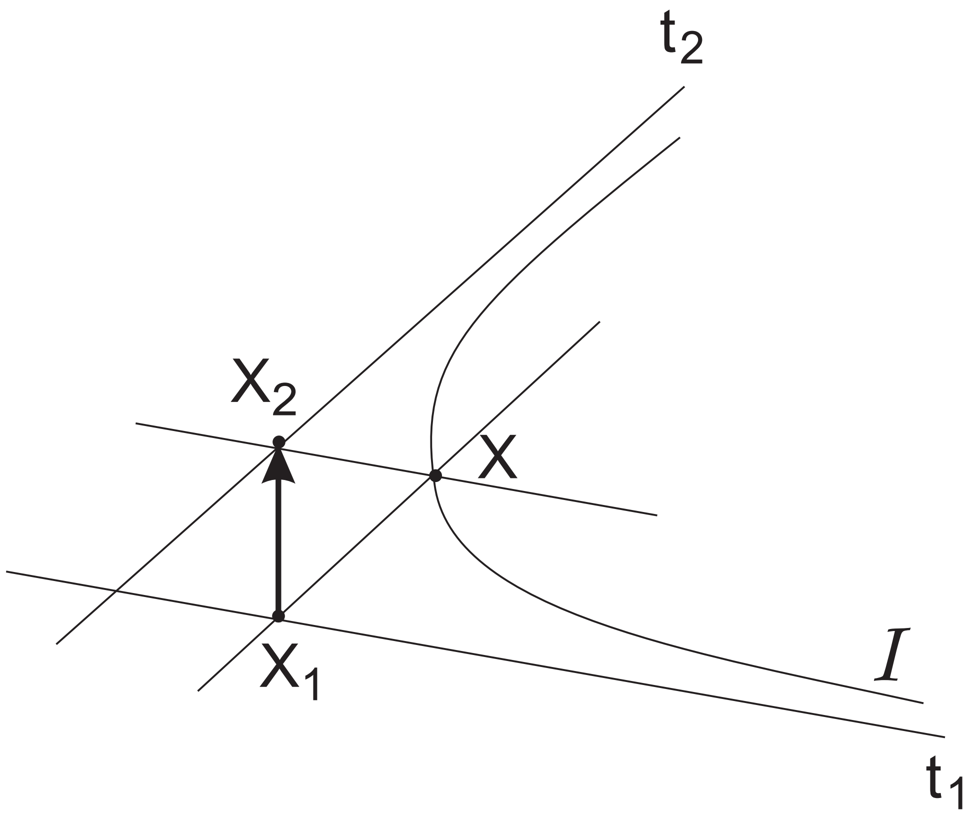

Figure 5.

Let ℓ_X be the line of $PG(3,q)$ represented by the ordered pair (X_1, X_2) and let $\overline{\ell}_X$ the line represented by the ordered pair (X_2, X_1). By the previous results, we get:

$$\begin{aligned} \mathcal{F}_1 &= \{\ell_X\}_{x\in\mathcal{I}} \subset \mathcal{R}, \\ \overline{\mathcal{F}}_1 &= \{\overline{\ell}_X\}_{x\in\mathcal{I}} \subset \overline{\mathcal{R}}. \end{aligned}$$

The above inclusions are proper, since there are lines of $\mathcal{R}$ not in $\mathcal{F}_1$ (U_1U_2 and U_2T_1) and lines of $\overline{\mathcal{R}}$ not in $\overline{\mathcal{F}}_1$ (U_1T_1 and U_2T_2).

We remark that there is no line of $\mathcal{H}$ contained in π. For, let b a line of π and in $\mathcal{H}$. then, either $b \in \mathcal{R}$, or $b \in \overline{\mathcal{R}}$.

Let $b \in \mathcal{R}$. The line b meets $\overline{\ell}$ (because b and $\overline{\ell}$ belong to opposite reguli of $\mathcal{H}$). Since that and since $b \subset \pi$, it follows that b meets $\overline{\ell}$ at the point P common to $\overline{\ell}$

and π, represented by R in $AG(2,q)$ by the line A_1A_2. But such a point P belongs also to the line b. Therefore, ℓ and b have P in common. Since ℓ and b are lines of the same regulus $\mathcal{R}$ of $\mathcal{H}$, it follows $b = \ell$: a contradiction, since ℓ is not a line of π, while $b \subset \pi$. The contradiction proves that $b \notin \mathcal{R}$. Similarly, we prove that $b \notin \overline{\mathcal{R}}$. So, we get a contradiction, because from $b \subset \mathcal{H}$, it follows that $b \in \mathcal{R} \cup \overline{\mathcal{R}}$. The contradiction proves that there is no line of $\mathcal{H}$ contained in π. So, the remark is proved.

From this remark it follows that $\mathcal{H}$ meets π at a non-degenerate conic. Obviously, every line of $\mathcal{H}$ is a line of $\mathcal{R}$ not meeting v, while every line of $\overline{\mathcal{F}}$ is a line of $\overline{\mathcal{R}}$ not meeting v.

Now, let us prove that every line of $\mathcal{R}$ not meeting v is a line of $\mathcal{F}$.

Let $\widetilde{\ell}$ be a line of $\mathcal{R}$ not meeting v. Since $\widetilde{\ell}$ does not meet v and, since $\widetilde{\ell}$ is not a line of π (we already proved that no line of $\mathcal{H}$ is contained in π), it follows that $\widetilde{\ell}$ is represented by an ordered pair (L_1, L_2) of distinct points of $AG(2,q)$. The line $\widetilde{\ell}$, which belongs to $\mathcal{R}$, meets U_1T_1, U_2T_2 and $\overline{\ell}$, which belong to $\overline{\mathcal{R}}$. By the representations of $\widetilde{\ell}, U_1T_1$ and U_2T_2 and since $\widetilde{\ell}$ meets U_1T_1 and U_2T_2, we get

$$L_1 \in t_1, \ L_2 \in t_2.$$

We remark that $L_1 \neq O$. In fact, if $L_1 = O$, the distinct points L_1 and L_2 belong both to t_2 and $\widetilde{\ell}$ contains T_2. It follows that $\widetilde{\ell} = U_1T_2$, since $\widetilde{\ell} \in \mathcal{R}$, $U_1T_2 \in \mathcal{R}$, $T_2 \in \widetilde{\ell}$, $T_2 \in U_1T_2$: a contradiction, since $\widetilde{\ell}$ does not meet v, while U_1T_2 meets v and U_1. The contradiction proves the remark. Similarly, we prove that $L_2 \neq O$.

By the above remark and since $L_1 \in t_1$, $L_2 \in t_2$, it follows

$$L_1 \in t_1 - \{O\}, \ L_2 \in t_2 - \{O\}.$$

By the previous results, it follows immediately that $L_1 \neq A_2$, $L_2 \neq A_1$.

As $\widetilde{\ell}$ meets $\overline{\ell}$, it follows that in $AG(2,q)$ the line L_1A_2 is parallel to L_2A_1 (maybe coinciding with it). Let L be the point of $AG(2,q)$ common to the line through L_2 parallel to t_1 and to the line through L_1 parallel to t_2. Let us prove that $L \in \mathcal{I}$. In the coordinate system that we chose before, let m be the slope of the lines parallel to A_1L_2 and A_2L_1. Such a slope does exist, since $L_1 \in t_1 - \{O\}$ and it is different from zero because $L_2 \in t_2 - \{O\}$. The points L_1 and L_2 have coordinates $L_1\left(-\dfrac{1}{m}, 0\right)$, $L_2(0, -m)$. It follows that L has coordinates $\left(-\dfrac{1}{m}, -m\right)$ and then $L \in \mathcal{I}$ (remember that in our coordinate system the hyperbola $\mathcal{I}$ has the equation $xy = 1$). By the above results and by the definition of $\mathcal{F}$, it follows that $\widetilde{\ell} \in \mathcal{F}$. So, every line of $\mathcal{R}$ not meeting v is a line of $\mathcal{F}$. So, the result is proved. Similarly, we prove that every line of $\overline{\mathcal{R}}$ not meeting v is a line of $\overline{\mathcal{F}}$. It follows that all the lines of $\mathcal{F}$ coincide with the lines of $\mathcal{R}$ not meeting v, while the lines of $\overline{\mathcal{F}}$ coincide with the lines of $\overline{\mathcal{R}}$ not meeting v. We remark that the lines U_2T_2 and U_2T_1 coincide with the lines of $\mathcal{R}$ meeting v. For, U_1T_2 and U_2T_1 are lines of $\mathcal{R}$ meeting v. Conversely, every line of $\mathcal{R}$ meeting v coincides either with U_1T_2, or with U_2T_1. For, let $\ell_{\mathcal{R}}$ be a line of $\mathcal{R}$ meeting v distinct from U_1T_2 and U_2T_1. Then, the point $L = \ell_{\mathcal{R}} \cap v$ is distinct from either U_1, or U_2. Then the line v, having three distinct points in common with $\mathcal{H}$, is a line of $\mathcal{H}$. It follows $v \in \overline{\mathcal{R}}$, since v meets $\ell_{\mathcal{R}}$, U_1T_2 and U_2T_1, belonging to $\mathcal{R}$. The line v meets also $U_2T_2 \in \overline{\mathcal{R}}$. It follows that $v = U_2T_2$, a contradiction, since $T_2 \neq Y$. The contradiction proves that no line of $\mathcal{R}$ meeting v and distinct from U_1T_2 and U_2T_1 exists, whence every line of $\mathcal{R}$ meeting v coincides either with U_1T_2, or with U_2T_1. So, the remark is proved. Similarly, we prove that U_1T_1 and U_2T_2 coincide with the lines of $\overline{\mathcal{R}}$ meeting v. By the above arguments, it follows that

$$\begin{aligned} \mathcal{R} &= \mathcal{F} \cup \{U_1T_2, U_2T_1\}, \\ \overline{\mathcal{R}} &= \overline{\mathcal{F}} \cup \{U_1T_1, U_2T_2\}. \end{aligned}$$

As all the hyperbolic quadrics of $PG(3,q)$ are equivalent, it follows that for every hyperbolic quadric $\mathcal{H}$ of $PG(3,q)$ there is a representation $R(U_1, U_2, \pi, 3)$ of $PG(3,q)$ which represents $\mathcal{H}$ by a hyperbola of $AG(2,q)$.

2. Second representation

Let $AG(2,q)$ be the affine plane over the Galois field $GF(q)$. In $AG(2,q)$, let t_1 and t_2 be two distinct lines meeting at a point O. Let A be a point of $t_1 - \{O\}$ (see Fig. 6), let B be a point of $t_2 - \{O\}$ and, finally, let t_3 be the line through A and B.

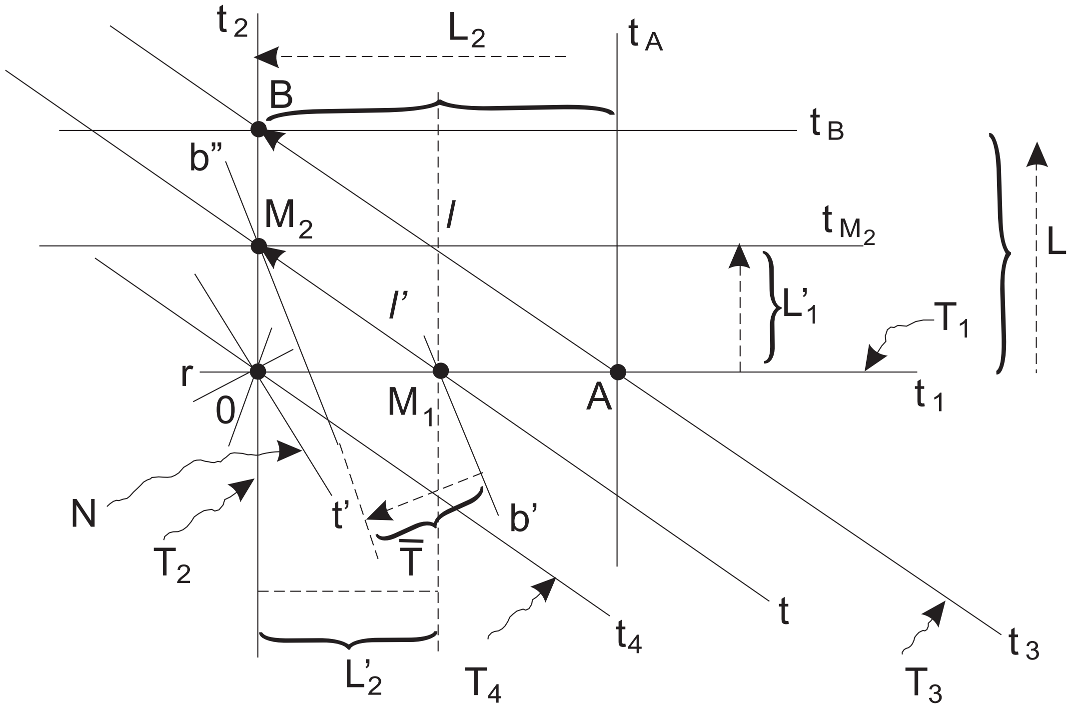

Figure 6.

From now on, the symbol d_{MN} denotes the direction of the line of $AG(2,q)$ through the distinct points M and N. Let t_4 be the line through O with direction d_{AB}. Let R be an $R(U_1, U_2, \pi, 3)$-representation of $PG(3,q)$ (see [1]). Let ℓ be the line of $PG(3,q)$ represented by the ordered pair of distinct points (A, B). Let r be the line of π (not through Y) represented by the proper pencil of lines with centre O. The lines v, r and ℓ are two by two skew. Let $\mathcal{H}$ be the hyperbolic quadric of $PG(3,q)$ containing v, r and ℓ and let $\mathcal{R}$ be the regulus of $\mathcal{H}$ determined by v, r, ℓ and $\overline{\mathcal{R}}$ the opposite regulus. Let T_1 be the point of $\pi - \{Y\}$ represented by the line t_1 and let T_2 be the point of $\pi - \{Y\}$ represented by the line t_2. Let F be the improper pencil (that is the pencil of parallel lines) consisting of the lines of $AG(2,q)$ with direction d_{AB} and let z be the line of π (through Y) represented by F. The line z meets v at Y, meets r at T_4 represented by the line t_4 and meets *ell* at the point T_3, represented by the line t_3. It follows that z is a line of $\overline{\mathcal{R}}$. The line U_1T_1 of $PG(3,q)$ meets v at the point U_1, meets r at T_1 and ℓ at L_2 represented by the ordered pair (t_1, t_B), where t_B is the line of $AG(2,q)$ through B and parallel to t_1. It follows that $U_1T_1 \in \overline{\mathcal{R}}$ (see Fig. 7).

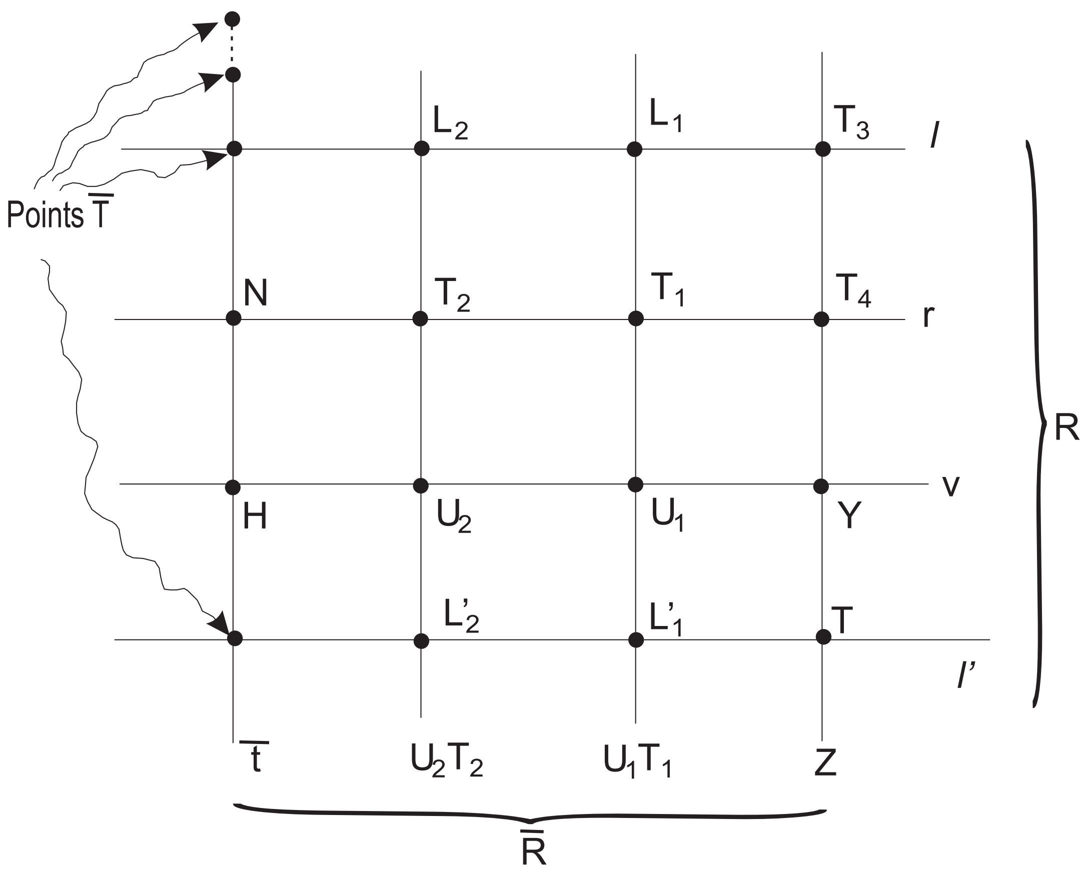

Figure 7.

Now, let M_1 and M_2 be two points of $AG(2,q)$ mutually distinct and such that $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$. Let t be the line of $AG(2,q)$ through M_1 and M_2. Let ℓ' be the line of $PG(3,q)$ represented by the ordered pair (M_1, M_2). The line ℓ' meets z at T, represented by the line t, meets U_1T_1 at L'_1, represented by the ordered pair (t_1, t_{M_2}), where t_{M_2} is the line of $AG(2,q)$ through M_2 and parallel to t_1, meeting U_2T_2 at the point L'_2, represented by the ordered pair (t_{M_1}, t_2), where t_{M_1} is the line of $AG(2,q)$ through M_1 and parallel to t_2. It follows that $\ell' \in \mathcal{R}$. By the above arguments it follows that every ordered pair of distinct points (M_1, M_2), with $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$, represents a line of $\mathcal{R}$, distinct from v and r. Conversely, let $m \in \mathcal{R} - \{v, r\}$. The line m is not a line of π, since m and r are skew and r is a line of π. It follows that m is represented by an ordered pair of distinct points (M_1, M_2). The line m, as a line of $\mathcal{R}$, meets U_1T_1, U_2T_2 and z, which are lines of $\overline{\mathcal{R}}$. Since m meets z (at point distinct from Y), it follows that $d_{M_1M_2} = d_{AB}$. Since m meets U_1T_1, it follows that $M_1 \in T_1$, while, since m meets U_2T_2, it follows that $M_2 \in t_2$. Moreover, by $M_1 \neq M_2$, $d_{M_1M_2} = d_{AB}$ and since neither t_1, nor t_2 have the direction d_{AB}, it follows that $M_1 \in t_1 - \{O\}$ and $M_2 \in t_2 - \{O\}$. By the previous arguments, it follows that every line of $\mathcal{R} - \{v, r\}$ is represented by an ordered pair of distinct points (M_1, M_2), with $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$. So, we prove that the ordered pairs of distinct points (M_1, M_2), with $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$, represent exactly all the lines of $\mathcal{R} - \{v, r\}$. If we denote by $\ell_{M_1M_2}$ the line of $PG(3,q)$ represented in $AG(2,q)$ by the ordered pair of distinct points (M_1, M_2), we get

$$\begin{aligned}\mathcal{R} = \{v, r\} \cup \{\ell_{M_1M_2} : M_1 \in t_1 - \{O\},\\ M_2 \in t_2 - \{O\},\ d_{M_1M_2} = d_{AB}\}.\end{aligned}$$

Now, let α be a plane of $PG(3,q)$ containing v but not through U_1T_1, U_2T_2 and z. Such a plane α is tangent to $\mathcal{H}$ at a point $H \in v - \{Y, U_1, U_2\}$ and contains therefore the line $\bar{t}$ of $\overline{\mathcal{R}}$ through H. Moreover, α is spanned in $AG(2,q)$ by a direction d' distinct from d_{AB} and distinct either from the direction of t_1, or that of t_2. Consider the points of $\bar{t} - \{H\}$. They are the intersections of the lines of $\mathcal{R}$ distinct from v with the plane α. Such points are therefore represented as follows:

1) The point $N = \bar{t} \cap r = \alpha \cap r$ is represented by the line t' of $AG(2,q)$ through O and of direction d'.

2) Each point $\overline{T}$ of $\bar{t} - \{H, N\}$ is represented by an ordered pair (b', b''), where b' and b'' are the lines of $AG(2,q)$ with direction d' and through the points M_1, M_2 respectively, with

$$\begin{aligned}M_1 \in t_1 - \{O\},\ M_2 \in t_2 - \{O\},\\ d_{M_1M_2} = d_{AB}.\end{aligned}$$

By varying of the direction d' in $\mathcal{D} - \{d_{AB}, d_1, d_2\}$, where $\mathcal{D}$ is the set of the directions of $AG(2,q)$ and d_1, d_2 are the directions of t_1 and t_2 respectively, we get the representations of all the lines of $\overline{\mathcal{R}} - \{z, U_1T_1, U_2T_2\}$, each of them being deprived of their point in common with v (see Fig. 8).

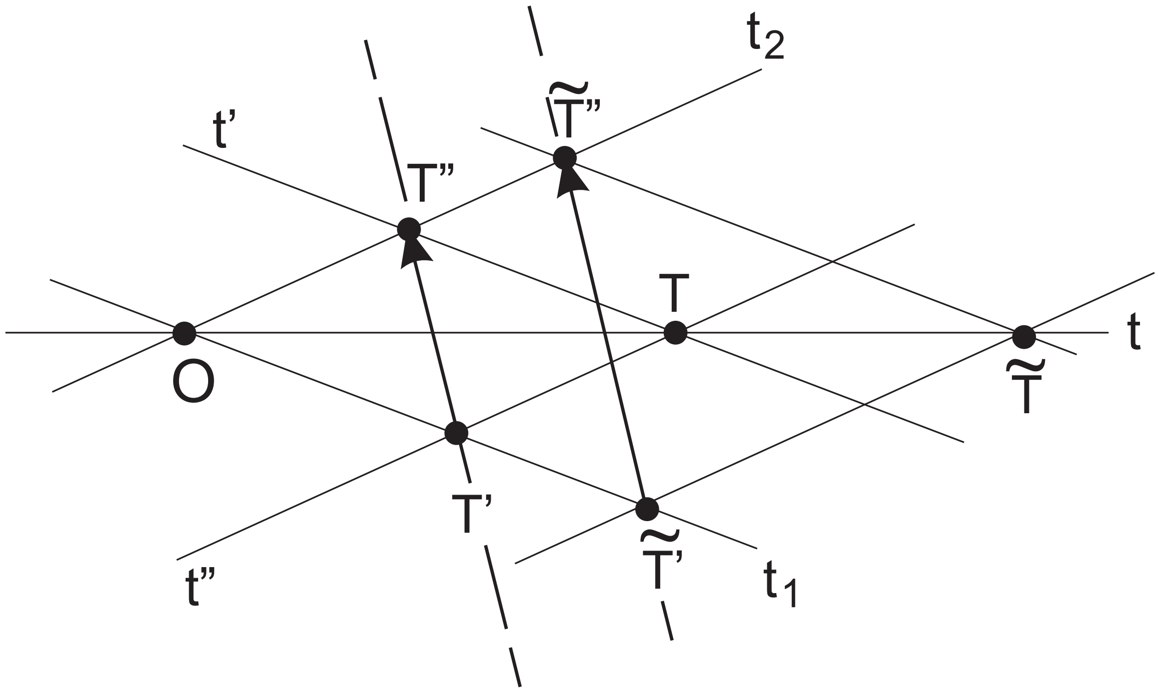

Figure 8.

Now, let t_1, t_2 and t be three distinct lines of $AG(2,q)$ through the same point O. Let T be a point of t distinct from O. Let t' be the line through R and parallel to t_1 and let $T'' = t_2 \cap t'$. Let t' be the lines through T parallel to t_2 and let $T' = t_1 \cap t''$. By the Desargues theorem it follows that the direction of $T'T''$ does not depend on T.

By that and by the previous arguments, it follows:

Theorem 2. *Let $PG(3,q)$ be the three-dimensional projective space over the field q and let $AG(2,q)$ be the affine plane over q. Let R be an $R(U_1, U_2, \pi, 3)$-representation of $PG(3,q)$. Let t_1, t_2 and t be three distinct lines of $AG(2,q)$ through the same point O and let d_1 and d_2 be the directions of t_1 and t_2, respectively. Let T_1 be the point of π represented by the line t_1 and let T_2 be the point of π represented by t_2. For any $T \in t - \{O\}$, let T'_T be the point common to the line t_1 and to the line through T parallel to t_2 and let T''_T be the point common to the line t_2 and to the line through T parallel to t_1. Then the direction d of the line joining T'_T and T''_T does not depend on the choice of T in $t - \{O\}$. Let $\ell(T'_T, T''_T)$*

be the line of $PG(3,q)$ represented by the ordered pair of distinct points (T'_T, T''_T) and let r be the line of π represented by the proper pencil (that is the pencil of lines through the same point O) of lines with centre O. Then, the following set of lines

$$\mathcal{R} = \{v, r\} \cup \{\ell(T'_T, T''_T)\}_{T \in t - \{O\}}$$

is the regulus of a hyperbolic quadric $\mathcal{H}$ in $PG(3,q)$.

Let z be the line of π represented by the improper pencil (that is the pencil of parallel lines) of lines of $AG(2,q)$ with direction d. Finally, let d' be a direction of $AG(2,q)$ distinct from d_1, d_2 and d and let n be the line of $AG(2,q)$ through O and having direction d'. For any $T \in t - \{O\}$, let $b'(T)$ and $b''(T)$ be the lines of $AG(2,q)$ having direction d' and through T'_T and T''_T, respectively. The following set

$$\{b'(T), b''(T)\}_{T \in t - \{O\}} \cup \{n\}$$

represents a line of the regulus $\overline{\mathcal{R}}$ of $\mathcal{H}$ opposite to $\mathcal{R}$, deprived of its point in common with v. Such a line will be denoted by $\ell(d')$. Then the regulus $\overline{\mathcal{R}}$ is

$$\overline{\mathcal{R}} = \{z, U_1T_1, U_2T_2\} \cup \{\ell(d')\}_{d' \in \mathcal{D} - \{d_1, d_2, d\}},$$

where $\mathcal{D}$ is the set of the directions of $AG(2,q)$.

Since the hyperbolic quadrics of $PG(3,q)$ are all equivalent, it follows that for any hyperbolic quadric $\mathcal{H}$ of $PG(3,q)$ there is an $R(U_1, U_2, \pi, 3)$-representation of $PG(3,q)$ which represents $\mathcal{H}$ by means of an ordered triple (t_1, t_2, t) of distinct lines of $AG(2,q)$ of the same pencil.

3. Third representation

Let $PG(3,q)$ be the three dimensional projective space over the field k and let $AG(2,q)$ be the affine plane over k. Let $\mathcal{S}$ be the set of points of $AG(2,q)$ and R an $R(U_1, U_2, \pi, 3)$-representation of $PG(3,q)$. In $AG(2,q)$ let t_1 and t_2 be two distinct lines meeting at O (see Fig. 9) and let r_0 be the line of π represented (see [1]), through R, by the proper pencil of lines of $AG(2,q)$ through O.

Let T_1 and T_2 be the points of π represented, through R, by the lines t_1 and t_2 of $AG(2,q)$. Let X be a point of $AG(2,q) - t_1 \cup t_2$ and let x be the line of π represented by the proper pencil of lines with centre X. Let I be the so defined set

$$I = \{(X_1, X_2) \in \mathcal{S} \times \mathcal{S} : X_1 \in t_1 - \{O\},\ X_2 \in t_2 - \{O\} \text{ and } X_1, X_2, X \text{ collinear}\}.$$

Denote by $\ell(X', X'')$ the line of $PG(3,q)$ represented by the ordered pair (X', X'') of distinct points of $AG(2,q)$, let

$$\mathcal{L} = \{\ell(X_1, X_2)\}_{(X_1, X_2) \in I}.$$

The lines U_1T_1 and U_2T_2 of $PG(3,q)$ are mutually skew, since the line of π through T_1 and T_2 does not contain $Y = \pi \cap v$ (t_1 and t_2 are not parallel). The line x and U_1T_1 are skew, like the line x and U_2T_2, since $x \in \pi$, $T_1 \notin x$, $T_2 \notin x$ ($X \notin t_1$, $X \notin t_2$). It follows that x, U_1T_1 and U_2T_2 are mutually skew.

Let $\mathcal{H}$ be the hyperbolic quadric of $PG(3,q)$ containing the lines x, U_1T_1 and U_2T_2 and let $\overline{\mathcal{R}}$ be the regulus of $\mathcal{H}$ to which such lines belong. The line r_0 meets x at the point of π represented by the line OX of $AG(2,q)$ and meets U_1T_1 and U_2T_2 at T_1 and T_2, respectively. It follows that r_0 belongs to the regulus $\mathcal{R}$ of $\mathcal{H}$ opposite to $\overline{\mathcal{R}}$.

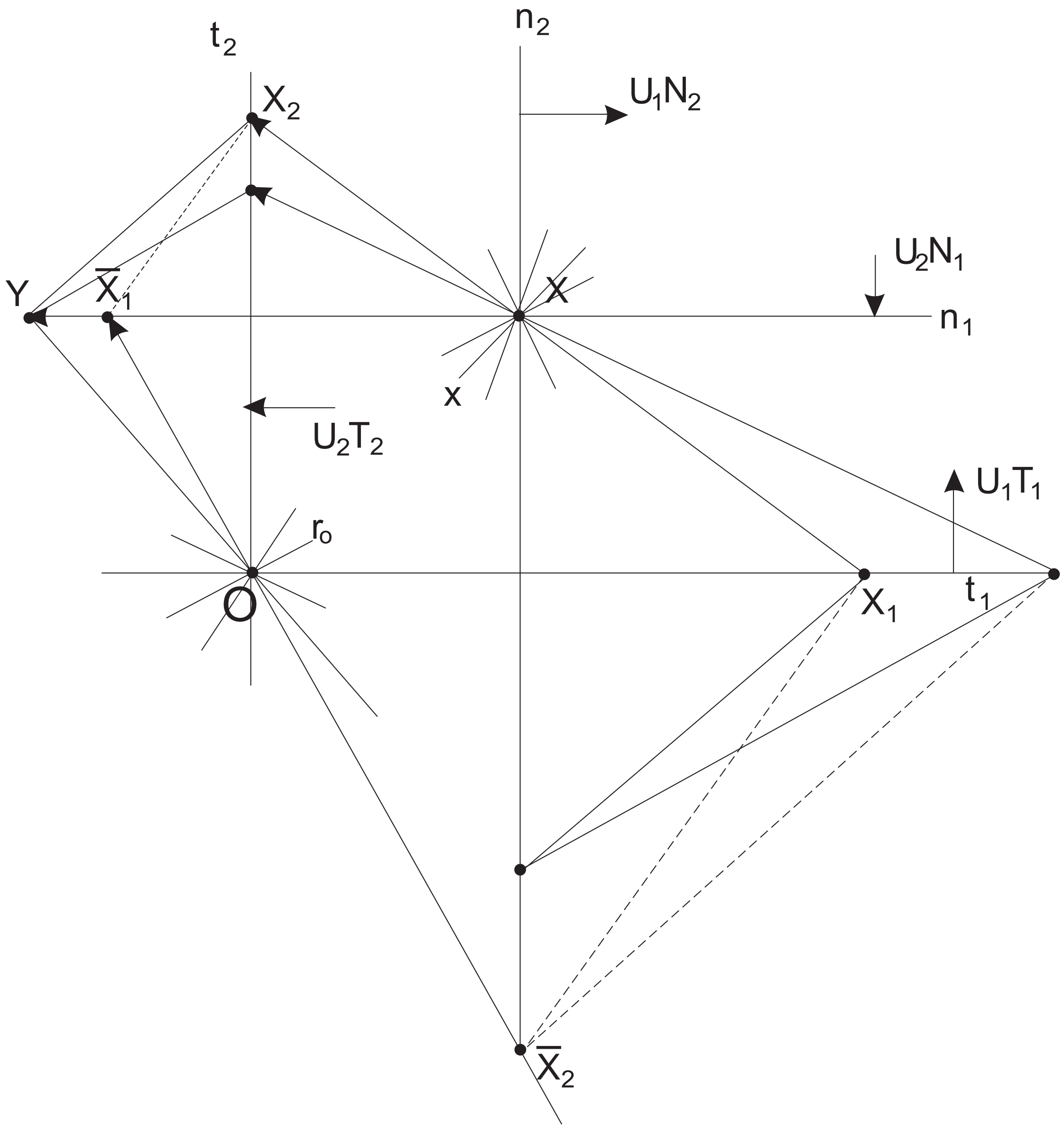

Figure 9.

Moreover, it is easy to check that every line of $\mathcal{L}$ meets x, U_1T_1 and U_2T_2. It follows that every line of $\mathcal{L}$ belongs to $\mathcal{R}$. Now, let n_1 and n_2 be the lines of $AG(2,q)$ through X and parallel to t_1 and t_2, respectively. Then, let N_1 and N_2 be the points of π represented by the lines n_1 and n_2, respectively. The line U_2N_1 of $PG(3,q)$ meets U_2T_2 at U_2, U_1T_1 at the point represented by the ordered pair (t_1, n_1) and x and N_1. It follows that $U_2N_1 \in \mathcal{R}$. The line U_1N_2 meets U_1T_1 at U_1, U_2T_2 at the point represented by the ordered pair (n_2, t_2) and x at N_2. Then $U_1N_2 \in \mathcal{R}$. By the above arguments, it follows

$$E = \{r_0, U_1N_2, U_2N_1\} \cup \mathcal{L} \subseteq \mathcal{R}.$$

Now, let us prove that *every line of $\mathcal{R}$ is a line of E.*

Proof. Assume that r' is a line of $\mathcal{R}$ not in E. Then r' is not in π, otherwise $r_0 \cap r' \neq \emptyset$, while r' and r_0 are skew, since r and r_0 are two distinct lines of $\mathcal{R}$ ($r_0 \in E$, $r' \in E$). The line r' does not contain U_1 and U_2, since r' is distinct from U_1N_2 and U_2N_1 ($U_1N_2 \in E$, $U_2N_1 \in E$, $r' \notin E$). Remark that r' does not meet v. For, $M = r' \cap v$. By $U_1 \notin r'$, $U_2 \notin r'$, it follows that $M \neq U_1$, $M \neq U_2$. Then, v contains the three distinct points M, U_1 and U_2 of $\mathcal{H}$ and then v is a line of $\mathcal{H}$. Since v meets U_1T_1 and U_2T_2, which belong to $\overline{\mathcal{R}}$, it follows that $v \in \mathcal{R}$ and then v meets $x \in \overline{\mathcal{R}}$: a contradiction, because x does not pass through $Y = \pi \cap v$ (x is represented in $AG(2,q)$ by the proper pencil of lines with centre X). The contradiction proves the remark. Therefore, r' is a line of $PG(3,q)$ not in π and not meeting v. It follows that r' is represented by an ordered pair of distinct points (X_1, X_2) of $AG(2,q)$. By $r' \in \mathcal{R}$ and $x \in \overline{\mathcal{R}}$, it follows that $r' \cap x \neq \emptyset$ and then the line of $AG(2,q)$ joining X_1 and X_2 contains X. By $r' \in \mathcal{R}$ and $U_1T_1 \in \overline{\mathcal{R}}$, it follows that $r' \cap U_1T_1 \neq \emptyset$ and then $X_1 \in t_1$. By $r' \in \mathcal{R}$ and $U_2T_2 \in \overline{\mathcal{R}}$ it follows $r' \cap U_2T_2 \neq \emptyset$ and then $X_2 \in r_2$. Moreover, we get $X_1 \neq O$, $x_2 \neq O$, that is $X_1 \in t_1 - \{O\}$, $X_2 \in t_2 - \{O\}$, since the points X_1, X_2 and X are collinear and $X_1 \neq X_2$. Then $X_1 \in t_1 - \{O\}$, $X_2 \in t_2 - \{O\}$ and X_1, X_2, X are collinear. By that, it follows $(X_1, X_2) \in I$ and then $r' = \ell(X_1, X_2) \in \mathcal{L} \subset E$, so $r' \in E$ is a contradiction, since $r' \notin E$. The contradiction proves that every line of $\mathcal{R}$ belongs to E and then $\mathcal{R} \subseteq E$. By that and by $E \subseteq \mathcal{R}$ it follows $\mathcal{R} = E$, that is

$$\mathcal{R} = \{r_0, U_1N_2, U_2N_1\} \cup \mathcal{L}.$$

Now, let $\overline{I}$ be the following set

$$\overline{I} = \{(\overline{X}_2, \overline{X}_1) \in \mathcal{S} \times \mathcal{S} : \overline{X}_1 \in n_1 - \{X\},\ X_2 \in n_2 - \{X\},\ \overline{X}_1, \overline{X}_2 \text{ and } O \text{ collinear}\}.$$

Let

$$\overline{\mathcal{L}} = \{\ell(\overline{X}_2, \overline{X}_1)\}_{(\overline{X}_2, \overline{X}_1) \in \overline{I}}.$$

In a similar way as before, we get

$$\overline{\mathcal{R}} = \{x, U_1T_1, U_2T_2\} \cup \overline{\mathcal{L}}.$$

So, the following theorem is proved.

Theorem 3. *Let $PG(3,q)$ be the projective three-dimensional space over the field q, let $AG(2,q)$ be the affine plane over q and let $\mathcal{S}$ be the set of points of $AG(2,q)$ and R an $R(U_1,U_2,\pi,3)$-representation of $PG(3,q)$ (see [1]). In $AG(2,q)$ let t_1 and t_2 be two distinct lines meeting at a point O and let r_0 be the line of π represented, through R, by the proper pencil of lines of $AG(2,q)$ with centre O. Let T_1 and T_2 be the points of π represented through R, by the lines t_1 and t_2 of $AG(2,q)$ respectively. Let X be a point of $AG(2,q)-(t_1\cup t_2)$ and let x be the line of π represented by the proper pencil of lines of $AG(2,q)$ with centre X.*

Let n_1 and n_2 be the lines of $AG(2,q)$ through X and parallel to t_1 and t_2, respectively. Then, let N_1 and N_2 be the points of π represented through R, by the lines n_1 and n_2, respectively. Let I and $\overline{I}$ be the following sets:

$$\begin{aligned} I &= \{(X_1,X_2)\in\mathcal{S}\times\mathcal{S} : X_1\in t_1-\{O\},\ X_2\in t_2-\{O\} \\ &\qquad\qquad \textit{and } X_1,X_2,X \textit{ collinear}\},\\ \overline{I} &= \{(\overline{X}_2,\overline{X}_1)\in\mathcal{S}\times\mathcal{S} : \overline{X}_1\in n_1-\{X\},\ \overline{X}_2\in n_2-\{X\} \\ &\qquad\qquad \textit{and } \overline{X}_1,\overline{X}_2,O \textit{ collinear}\}.\end{aligned}$$

Denote by $\ell(X',X'')$ the line of $PG(3,q)$ represented through R by the ordered pair of distinct points (X',X'') of $AG(2,q)$.

Let

$$\begin{aligned}\mathcal{L} &= \{\ell(X_1,X_2)\}_{(X_1,X_2)\in I},\\ \overline{\mathcal{L}} &= \{\ell(\overline{X}_2,\overline{X}_1)\}_{(\overline{X}_2,\overline{X}_1)\in \overline{I}}.\end{aligned}$$

Then the following sets of lines or $PG(3,q)$

$$\begin{aligned}\mathcal{R} &= \{r_0, U_1N_2, U_2N_1\}\cup\mathcal{L},\\ \overline{\mathcal{R}} &= \{x, U_1T_1, U_2T_2\}\cup\overline{\mathcal{L}},\end{aligned}$$

are the two reguli of a hyperbolic quadric of $PG(3,q)$ admitting the plane π as tangent plane, the contact point being the point of π represented by the line OX of $AG(2,q)$ and the line v being a secant line.

The above theorem allows us to represent in the plane $AG(2,q)$ the reguli and then the points of a hyperbolic quadric $\mathcal{H}$, deprived of two points which are not represented. For, the line r_0 is represented by the pencil of lines of $AG(2,q)$ with centre O, the line x is represented by the pencil of lines of $AG(2,q)$ with centre X, the lines U_1N_2, U_2N_1, U_1T_1 U_2T_2 are represented as follows:

$U_1N_2-\{U_1\}:\{n_2\}\cup\{n_2,n\}:n$ parallel to n_2 and distinct from n_2,

$U_2N_1-\{U_2\}:\{n_1\}\cup\{n,n_1\}:n$ parallel to n_1 and distinct from n_1,

$U_1T_1-\{U_1\}:\{t_1\}\cup\{t_1,t\}:t$ parallel to t_1 and distinct from t_1,

$U_2T_2-\{U_2\}:\{t_2\}\cup\{t,t_2\}:t$ parallel to t_2 and distinct from t_2,

Since the hyperbolic quadrics are all equivalent in $PG(3,q)$, it follows that for any hyperbolic quadric $\mathcal{H}$ of $PG(3,q)$ there is an $R(U_1,U_2,\pi,3)$-representation which allows us to represent $\mathcal{H}$ as in Theorem 3.

Those three different representations of a hyperbolic quadric of $PG(3, q)$ in $AG(2, q)$ show a further application of the representation called also ”crashing” cited in the bibliography.

Bibliography

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Accepted: 24.11.2012