THREE REPRESENTATIONS OF A HYPERBOLIC QUADRIC OF $PG(3, q)$ IN $AG(2, q)$

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Summary. We construct three different representations of a hyperbolic quadric of a projective Galois space $PG(3, q)$ in the affine Galois plane $AG(2, q)$. To do this, we use the representation $R$, or $R(U_1, U_2, \pi, 3)$ of the projective space $P(r, k)$, over the field $k$, in the affine plane $A(2, k)$, over the same field $k$, called also ”Crashing”, cited in the bibliography [1]. Further applications of this representation are the construction of maximal partial line spreads in $PG$, $q$ even, a geometric proof of the equivalence between the Desargues and the Veblen theorems and a geometric proof of the equivalence between the Pappus-Pascal theorem and the ”Three stars theorem”. Those results will soon appear.

1. First representation

Theorem 1. Theorem of the hyperbola and the hyperbolic quadric. Let $PG(3, q)$ be the projective space of dimension three over the Galois field $GF(q)$, let $AG(2, q)$ be the affine plane over the same field and let $R$ be an $R(U_1, U_2, \pi, 3)$-representation of $PG(3, q)$, as in [1]. Let $I$ be a hyperbola of $AG(2, q)$ and let $t_1$ and $t_2$ be the asymptotes of $I$. Let $T_1$ and $T_2$ be the points of $\pi$ represented through $R$ by the lines $t_1$ and $t_2$ of $AG(2, q)$, respectively. For any point $X$ of $I$, let $X_1$ be the point common to $t_1$ and to the line through $X$, parallel to $t_2$, let $X_2$ be the point common to the line $t_2$ and to the line through $X$ parallel to $t_1$, let $\ell_X$ be the line of $PG(3, q)$ represented through $R$, by the ordered pair $(X_1, X_2)$ and, finally, let $\overline{\ell}_X$ be the line of $PG(3, q)$ represented through $R$ by the ordered pair $(X_2, X_1)$. Then, the following sets of $PG(3, q)$:

$$R = \{\ell_X\}_{X \in I} \cup \{U_1T_2, U_2T_1\},$$
$$\overline{R} = \{\overline{\ell}_X\} \cup \{U_1T_1, U_2T_2\},$$

where $U_iT_j$, $i, j = 1, 2$, denotes the line of $PG(3, q)$ through the points $U_i$ and $T_j$, are the two reguli of a hyperbolic quadric of $PG(3, q)$ meeting $\pi$ at a non-degenerate conic, admitting $v$ as a secant line.

Proof. Let $PG(3, q)$ be the three dimensional projective space over $GF(q)$, let $AG(2, q)$ be the affine plane over $GF(q)$ and let $R$ be an $R(U_1, U_2, \pi, 3)$-representation of $PG(3, q)$. Let $I$ be a hyperbola of $AG(2, q)$, let $t_1$ and $t_2$ be
the asymptotic of $\mathcal{I}$ and let $O$ be the common point of $t_1$ and $t_2$. For any $i = 1, 2$, the line $t_i$ represents, through $R$, a point $T_i$ of $\pi - \{Y\}$ (see Fig. 1), where $Y$ is the point $U_1U_2 \cap \pi$. The line of $\pi$ joining $T_1$ and $T_2$ does not contain $Y$, since in $AG(2, q)$ the lines $t_1$ and $t_2$ meet at $O$. It follows that the lines $U_1T_1$ and $U_2T_2$ of $PG(3, q)$ are skew, like the lines $U_1T_2$ and $U_2T_1$. Obviously, the four lines $U_iT_j$, $i, j = 1, 2$, form a skew quadrangle, denoted by $Q$.

Figure 1.
The lines $U_iT_j$, $i, j = 1, 2$, of $PG(3, q)$ are represented by $R$ in $AG(2, q)$ in the following way:

\[
\begin{align*}
U_1T_1 & : \{(t_1) \cup \{(t_1, t)\}_{t \in T_1}, \\
U_1T_2 & : \{(t_2) \cup \{(t_2, t)\}_{t \in T_2}, \\
U_2T_1 & : \{(t_1) \cup \{(t, t_1)\}_{t \in T_1}, \\
U_2T_2 & : \{(t_2) \cup \{(t, t_2)\}_{t \in T_2},
\end{align*}
\]

where $T_i$, $i = 1, 2$, is the set of the lines of $AG(2, q)$ parallel to $t_i$ and distinct from $t_i$.

Now, let $A$ be a point of $I$ (see Fig. 2).

Let $t'_1$ be the line of $AG(2, q)$ through $A$ and parallel to $t$, and $t'_2$ the line through $A$ parallel to $t_2$. Moreover, let $A_1 = t_1 \cap t'_2$, $A_2 = t_2 \cap t'_1$. It is $A_1 \neq A_2$, since $A_i \in t_i - \{O\}$, for any $i = 1, 2$. The ordered pair of distinct points $(A_1, A_2)$ of $AG(2, q)$ represents, by $R$, a line $\ell$ of $PG(3, q)$ not meeting $v$ and not in $\pi$ (see Fig. 2). By the representations of $\ell$ and $U_iT_j$, $i, j = 1, 2$, in $AG(2, q)$, we get:

\[\text{Figure 2.}\]
1) The line $\ell$ meets the line $U_1T_1$ at $L'$ represented by the ordered pair $(t_1, t'_1)$. Such a point $L'$ is distinct from $U_1$ and $T_1$, because the ordered pairs of distinct lines of $AG(2, q)$ represent the points of $PG(3, q)$ not in $\pi$ (see Fig. 3).

![Figure 3.](image)

2) The line $\ell$ meets the line $U_2T_2$ at the point $L''$ represented by the ordered pair $(t'_2, t_2)$ and such a point is distinct from $U_2$ and $T_2$.

3) The line $\ell$ does not meet either $U_1T_2$, or $U_2T_1$.

By 3) and since the lines $U_2T_1$ and $U_1T_2$ of $PG(3, q)$ are mutually skew, it follows that the lines $\ell$, $U_1T_2$ and $U_2T_1$ are two by two skew. Let us denote by $\mathcal{H}$ the hyperbolic quadric of $PG(3, q)$ containing $\ell$, $U_1T_2$ and $U_2T_1$. Then, call $\mathcal{R}$ the regulus of $\mathcal{H}$ containing $\ell$, $U_1T_2$ and $U_2T_1$. By 1) and 2), it follows that $U_1T_1$ and $U_2T_2$ belong to the regulus $\overline{\mathcal{R}}$ of $\mathcal{H}$ opposite to $\mathcal{R}$. The ordered pair $(A_2, A_1)$ represents a line $\overline{\ell}$ of $PG(3, q)$ not meeting $\nu$ and not in $\pi$.

By the representations of $\overline{\ell}$ and $U_iT_j$, $i, j = 1, 2$, in $AG(2, q)$, we get:

4) The line $\overline{\ell}$ meets $U_2T_1$, at the point $\overline{L'}$, represented by the ordered pair $(t'_1, t_1)$; such a point $\overline{L'}$ is distinct from $U_2$ and $T_1$.

5) The line $\overline{\ell}$ meets $U_1T_2$, at the point $\overline{L''}$, represented by the ordered pair $(t_2, t'_2)$; such a point is distinct from $U_1$ and $T_2$. 

6) The line $\ell$ does not meet either $U_1T_1$, or $U_2T_2$.

7) The line $\ell$ meets $\ell$ at the point $P$ of $\pi$ represented by the line $A_1A_2$ of $AG(2, q)$.

By 4), 5), 6) and 7) it follows that the line $\ell$ is a line of $\mathcal{R}$ distinct from $U_1T_1$ and $U_2T_2$ (see Fig. 4).

Figure 4.

Now, let $B$ be a point of $\mathcal{I} - \{A\}$. Let $t''_1$ be the line of $AG(2, q)$ through $B$ and parallel to $t_1$ and let $t''_2$ be the line of $AG(2, q)$ through $B$ parallel to $t_2$. Let $B_1 = t_1 \cap t''_2$ and $B_2 = t_2 \cap t''_1$. The ordered pair of distinct points $(B_1, B_2)$ represents a line $\ell'$ of $PG(3, q)$ not meeting $\nu$ and not in $\pi$. Such a line meets $U_1T_1$ at the point of $PG(3, q)$ represented by the ordered pair $(t_1, t''_1)$. The line $\ell'$ meets $U_2T_1$ at the point $PG(3, q)$ represented by the ordered pair $(t''_2, t_2)$. Let us prove that $\ell'$ meets $\ell$. To do this, choose a coordinate system in $AG(2, q)$ such that $T = T(0, 0)$, $A_1 = A_1(1, 0)$, $A_2 = A_2(0, 1)$. In such a system, the coordinates of the point $A$ are $(1, 1)$ and the hyperbola $\mathcal{I}$ has the equation $xy = 1$. It follows that

\[
B = B\left(x_0, \frac{1}{x_0}\right),
\]

\[
B_1 = B_1(x_0, 0),
\]

\[
B_2 = B_2\left(0, \frac{1}{x_0}\right),
\]
with \( x_0 \neq 0 \). The slopes of the lines \( A_1B_2 \) and \( A_2B_1 \) are both equal to \( a - \frac{\ell}{x_0} \), therefore such two lines are parallel. It follows that \( \ell' \) meets \( \bar{\ell} \). Since \( \ell' \) meets \( U_1T_1, U_2T_2 \) and \( \bar{\ell} \) which belong to \( \overline{R} \), it follows that \( \ell' \in R \). Similarly, we prove that the line \( \ell' \) of \( PG(3,q) \) represented by the ordered pair \((B_2,B_1)\) belongs to \( \overline{R} \).

For any \( X \in I \), let \( X_1 \) be the point common to \( t_1 \) and to the line through \( X \) parallel to \( t_2 \) and let \( X_2 \) be the point common to \( t_2 \) and to the line through \( X \) parallel to \( t_1 \) (see Fig. 5).

Let \( \ell_X \) be the line of \( PG(3,q) \) represented by the ordered pair \((X_1,X_2)\) and let \( \bar{\ell}_X \) the line represented by the ordered pair \((X_2,X_1)\). By the previous results, we get:

\[
\mathcal{F}_1 = \{ \ell_X \}_{X \in I} \subset R,
\]
\[
\overline{\mathcal{F}_1} = \{ \bar{\ell}_X \}_{X \in I} \subset \overline{R}.
\]

The above inclusions are proper, since there are lines of \( R \) not in \( \mathcal{F}_1 \) \((U_1U_2 \) and \( U_2T_1)\) and lines of \( \overline{R} \) not in \( \overline{\mathcal{F}_1} \) \((U_1T_1 \) and \( U_2T_2)\).

We remark that there is no line of \( H \) contained in \( \pi \). For, let \( b \) a line of \( \pi \) and in \( H \), then, either \( b \in R \), or \( b \in \overline{R} \).

Let \( b \in R \). The line \( b \) meets \( \bar{\ell} \) (because \( b \) and \( \bar{\ell} \) belong to opposite reguli of \( H \)). Since that and since \( b \subset \pi \), it follows that \( b \) meets \( \bar{\ell} \) at the point \( P \) common to \( \bar{\ell} \)
and $\pi$, represented by $R$ in $AG(2, q)$ by the line $A_1A_2$. But such a point $P$ belongs also to the line $b$. Therefore, $\ell$ and $b$ have $P$ in common. Since $\ell$ and $b$ are lines of the same regulus $\mathcal{R}$ of $\mathcal{H}$, it follows $b = \ell$: a contradiction, since $\ell$ is not a line of $\pi$, while $b \subset \pi$. The contradiction proves that $b \notin \mathcal{R}$. Similarly, we prove that $b \notin \overline{\mathcal{R}}$. So, we get a contradiction, because from $b \subset \mathcal{H}$, it follows that $b \in \mathcal{R} \cup \overline{\mathcal{R}}$. The contradiction proves that there is no line of $\mathcal{H}$ contained in $\pi$. So, the remark is proved.

From this remark it follows that $\mathcal{H}$ meets $\pi$ at a non-degenerate conic. Obviously, every line of $\mathcal{H}$ is a line of $\mathcal{R}$ not meeting $v$, while every line of $\mathcal{F}$ is a line of $\overline{\mathcal{R}}$ not meeting $v$.

Now let us prove that every line of $\mathcal{R}$ not meeting $v$ is a line of $\mathcal{F}$. Let $\tilde{\ell}$ be a line of $\mathcal{R}$ not meeting $v$. Since $\tilde{\ell}$ does not meet $v$ and, since $\tilde{\ell}$ is not a line of $\pi$ (we already proved that no line of $\mathcal{H}$ is contained in $\pi$), it follows that $\tilde{\ell}$ is represented by an ordered pair $(L_1, L_2)$ of distinct points of $AG(2, q)$. The line $\tilde{\ell}$, which belongs to $\mathcal{R}$, meets $U_1T_1$, $U_2T_2$ and $\tilde{\ell}$, which belong to $\overline{\mathcal{R}}$. By the representations of $\ell, U_1T_1$ and $U_2T_2$ and since $\ell$ meets $U_1T_1$ and $U_2T_2$, we get

$$L_1 \in t_1, \ L_2 \in t_2.$$ 

We remark that $L_1 \neq O$. In fact, if $L_1 = O$, the distinct points $L_1$ and $L_2$ belong both to $t_2$ and $\tilde{\ell}$ contains $T_2$. It follows that $\tilde{\ell} = U_1T_2$, since $\tilde{\ell} \in \mathcal{R}, U_1T_2 \in \mathcal{R},$ $T_2 \in \tilde{\ell}$. $T_2 \in U_1T_2$: a contradiction, since $\tilde{\ell}$ does not meet $v$, while $U_1T_2$ meets $v$ and $U_1$. The contradiction proves the remark. Similarly, we prove that $L_2 \neq O$.

By the above remark and since $L_1 \in t_1, \ L_2 \in t_2$, it follows

$$L_1 \in t_1 - \{O\}, \ L_2 \in t_2 - \{O\}.$$ 

By the previous results, it follows immediately that $L_1 \neq A_2, \ L_2 \neq A_1$.

As $\ell$ meets $\tilde{\ell}$, it follows that in $AG(2, q)$ the line $L_1A_2$ is parallel to $L_2A_1$ (maybe coinciding with it). Let $L$ be the point of $AG(2, q)$ common to the line through $L_2$ parallel to $t_1$ and to the line through $L_1$ parallel to $t_2$. Let us prove that $L \in \mathcal{T}$. In the coordinate system that we chose before, let $m$ be the slope of the lines parallel to $A_1L_2$ and $A_2L_1$. Such a slope does exist, since $L_1 \in t_1 - \{O\}$ and it is different from zero because $L_2 \in t_2 - \{O\}$. The points $L_1$ and $L_2$ have coordinates $L_1 \left( -\frac{1}{m}, 0 \right), \ L_2(0, -m)$. It follows that $L$ has coordinates $\left( -\frac{1}{m}, -m \right)$ and then $L \in \mathcal{T}$ (remember that in our coordinate system the hyperbola $\mathcal{T}$ has the equation $xy = 1$). By the above results and by the definition of $\mathcal{F}$, it follows that $\tilde{\ell} \in \mathcal{F}$. So, every line of $\mathcal{R}$ not meeting $v$ is a line of $\mathcal{F}$. So, the result is proved.

Similarly, we prove that every line of $\overline{\mathcal{R}}$ not meeting $v$ is a line of $\overline{\mathcal{F}}$. It follows that all the lines of $\mathcal{F}$ coincide with the lines of $\mathcal{R}$ not meeting $v$, while the lines of $\overline{\mathcal{F}}$ coincide with the lines of $\overline{\mathcal{R}}$ not meeting $v$. We remark that the lines $U_3T_2$ and $U_3T_1$ coincide with the lines of $\mathcal{R}$ meeting $v$. For, $U_1T_2$ and $U_3T_1$ are lines of $\mathcal{R}$ meeting $v$. Conversely, every line of $\mathcal{R}$ meeting $v$ coincides either with $U_1T_2$, or with $U_2T_1$. For, let $\ell_\mathcal{R}$ be a line of $\mathcal{R}$ meeting $v$ distinct from $U_1T_2$ and $U_2T_1$. Then, the point $L = \ell_\mathcal{R} \cap v$ is distinct from either $U_1$, or $U_2$. Then the line $v$, 


having three distinct points in common with \( \mathcal{H} \), is a line of \( \mathcal{H} \). It follows \( v \in \overline{\mathcal{R}} \), since \( v \) meets \( \ell_R, U_1T_2 \) and \( U_2T_1 \), belonging to \( \mathcal{R} \). The line \( v \) meets also \( U_2T_2 \in \overline{\mathcal{R}} \). It follows that \( v = U_2T_2 \), a contradiction, since \( T_2 \neq Y \). The contradiction proves that no line of \( \mathcal{R} \) meeting \( v \) and distinct from \( U_1T_2 \) and \( U_2T_1 \) exists, whence every line of \( \mathcal{R} \) meeting \( v \) coincides either with \( U_1T_2 \), or with \( U_2T_1 \). So, the remark is proved. Similarly, we prove that \( U_1T_1 \) and \( U_2T_2 \) coincide with the lines of \( \overline{\mathcal{R}} \) meeting \( v \). By the above arguments, it follows that

\[
\mathcal{R} = \mathcal{F} \cup \{U_1T_2, U_2T_1\}, \\
\overline{\mathcal{R}} = \mathcal{F} \cup \{U_1T_1, U_2T_2\}.
\]

As all the hyperbolic quadrics of \( PG(3, q) \) are equivalent, it follows that for every hyperbolic quadric \( \mathcal{H} \) of \( PG(3, q) \) there is a representation \( R(U_1, U_2, \pi, 3) \) of \( PG(3, q) \) which represents \( \mathcal{H} \) by a hyperbola of \( AG(2, q) \).

2. Second representation

Let \( AG(2, q) \) be the affine plane over the Galois field \( GF(q) \). In \( AG(2, q) \), let \( t_1 \) and \( t_2 \) be two distinct lines meeting at a point \( O \). Let \( A \) be a point of \( t_1 - \{O\} \) (see Fig. 6), let \( B \) be a point of \( t_2 - \{O\} \) and, finally, let \( t_3 \) be the line through \( A \) and \( B \).

Figure 6.
From now on, the symbol \( d_{MN} \) denotes the direction of the line of \( AG(2, q) \) through the distinct points \( M \) and \( N \). Let \( t_4 \) be the line through \( O \) with direction \( d_{AB} \). Let \( R \) be an \( R(U_1, U_2, \pi, 3) \)-representation of \( PG(3, q) \) (see [1]). Let \( \ell \) be the line of \( PG(3, q) \) represented by the ordered pair of distinct points \( (A, B) \). Let \( r \) be the line of \( \pi \) (not through \( Y \)) represented by the proper pencil of lines with centre \( O \). The lines \( v, r \) and \( \ell \) are two by two skew. Let \( \mathcal{H} \) be the hyperbolic quadric of \( PG(3, q) \) containing \( v, r \) and \( \ell \) and let \( \mathcal{R} \) be the regulus of \( \mathcal{H} \) determined by \( v, r, \ell \) and \( \mathcal{R} \) the opposite regulus. Let \( T_1 \) be the point of \( \pi - \{Y\} \) represented by the line \( t_1 \) and let \( T_2 \) be the point of \( \pi - \{Y\} \) represented by the line \( t_2 \). Let \( F \) be the improper pencil (that is the pencil of parallel lines) consisting of the lines of \( AG(2, q) \) with direction \( d_{AB} \) and let \( z \) be the line of \( \pi \) (through \( Y \)) represented by \( F \). The line \( z \) meets \( v \) at \( Y \), meets \( r \) at \( T_4 \) represented by the line \( t_4 \) and meets \( \ell \) at the point \( T_3 \), represented by the line \( t_3 \). It follows that \( z \) is a line of \( \mathcal{R} \). The line \( U_1T_1 \) of \( PG(3, q) \) meets \( v \) at the point \( U_1 \), meets \( r \) at \( T_1 \) and \( \ell \) at \( L_2 \) represented by the ordered pair \( (t_1, t_B) \), where \( t_B \) is the line of \( AG(2, q) \) through \( B \) and parallel to \( t_1 \). It follows that \( U_1T_1 \in \mathcal{R} \) (see Fig. 7).

![Figure 7](image-url)
Now, let $M_1$ and $M_2$ be two points of $AG(2,q)$ mutually distinct and such that $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$. Let $t$ be the line of $AG(2,q)$ through $M_1$ and $M_2$. Let $\ell'$ be the line of $PG(3,q)$ represented by the ordered pair $(M_1, M_2)$. The line $\ell'$ meets $z$ at $T$, represented by the line $t$, meets $U_1T_1$ at $L_1'$, represented by the ordered pair $(t_1, t_{M_2})$, where $t_{M_2}$ is the line of $AG(2,q)$ through $M_2$ and parallel to $t_1$, meeting $U_2T_2$ at the point $L_2'$, represented by the ordered pair $(t_{M_1}, t_2)$, where $t_{M_1}$ is the line of $AG(2,q)$ through $M_1$ and parallel to $t_2$. It follows that $\ell' \in \mathcal{R}$. By the above arguments it follows that every ordered pair of distinct points $(M_1, M_2)$, with $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$, represents a line of $\mathcal{R}$, distinct from $v$ and $r$. Conversely, let $m \in \mathcal{R} - \{v, r\}$. The line $m$ is not a line of $\pi$, since $m$ and $r$ are skew and $r$ is a line of $\pi$. It follows that $m$ is represented by an ordered pair of distinct points $(M_1, M_2)$. The line $m$, as a line of $\mathcal{R}$, meets $U_1T_1$, $U_2T_2$ and $z$, which are lines of $\overline{\mathcal{R}}$. Since $m$ meets $z$ (at point distinct from $Y$), it follows that $d_{M_1M_2} = d_{AB}$. Since $m$ meets $U_1T_1$, it follows that $M_1 \in T_1$, while, since $m$ meets $U_2T_2$, it follows that $M_2 \in t_2$. Moreover, by $M_1 \neq M_2$, $d_{M_1M_2} = d_{AB}$ and since neither $t_1$, nor $t_2$ have the direction $d_{AB}$, it follows that $M_1 \in t_1 - \{O\}$ and $M_2 \in t_2 - \{O\}$. By the previous arguments, it follows that every line of $\mathcal{R} - \{v, r\}$ is represented by an ordered pair of distinct points $(M_1, M_2)$, with $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$. So, we prove that the ordered pairs of distinct points $(M_1, M_2)$, with $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$, represent exactly all the lines of $\mathcal{R} - \{v, r\}$. If we denote by $\ell_{M_1M_2}$ the line of $PG(3,q)$ represented in $AG(2,q)$ by the ordered pair of distinct points $(M_1, M_2)$, we get

$$\mathcal{R} = \{v, r\} \cup \\{\ell_{M_1M_2} : M_1 \in t_1 - \{O\}, M_2 \in t_2 - \{O\}, d_{M_1M_2} = d_{AB}\}.$$

Now, let $\alpha$ be a plane of $PG(3,q)$ containing $v$ but not through $U_1T_1$, $U_2T_2$ and $z$. Such a plane $\alpha$ is tangent to $\mathcal{H}$ at a point $H \in v - \{Y, U_1, U_2\}$ and contains therefore the line $\overline{t}$ of $\overline{\mathcal{R}}$ through $H$. Moreover, $\alpha$ is spanned in $AG(2,q)$ by a direction $d'$ distinct from $d_{AB}$ and distinct either from the direction of $t_1$, or that of $t_2$. Consider the points of $\overline{t} - \{H\}$. They are the intersections of the lines of $\mathcal{R}$ distinct from $v$ with the plane $\alpha$. Such points are therefore represented as follows:

1) The point $N = \overline{t} \cap r = \alpha \cap r$ is represented by the line $t'$ of $AG(2,q)$ through $O$ and of direction $d'$.

2) Each point $\overline{T}$ of $\overline{t} - \{H, N\}$ is represented by an ordered pair $(b', b'')$, where $b'$ and $b''$ are the lines of $AG(2,q)$ with direction $d'$ and through the points $M_1$, $M_2$ respectively, with $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$.
By varying of the direction \( d' \) in \( D - \{ d_{AB}, d_1, d_2 \} \), where \( D \) is the set of the directions of \( AG(2, q) \) and \( d_1, d_2 \) are the directions of \( t_1 \) and \( t_2 \) respectively, we get the representations of all the lines of \( \mathcal{R} - \{ z, U_1T_1, U_2T_2 \} \), each of them being deprived of their point in common with \( v \) (see Fig. 8).

Figure 8.

Now, let \( t_1, t_2 \) and \( t \) be three distinct lines of \( AG(2, q) \) through the same point \( O \). Let \( T \) be a point of \( t \) distinct from \( O \). Let \( t' \) be the line through \( R \) and parallel to \( t_1 \) and let \( T''' = t_2 \cap t' \). Let \( t' \) be the lines through \( T \) parallel to \( t_2 \) and let \( T' = t_1 \cap t'' \). By the Desargues theorem it follows that the direction of \( T''T''' \) does not depend on \( T \).

By that and by the previous arguments, it follows:

**Theorem 2.** Let \( PG(3, q) \) be the three-dimensional projective space over the field \( q \) and let \( AG(2, q) \) be the affine plane over \( q \). Let \( R \) be an \( R(U_1, U_2, \pi, 3) \)-representation of \( PG(3, q) \). Let \( t_1, t_2 \) and \( t \) be three distinct lines of \( AG(2, q) \) through the same point \( O \) and let \( d_1 \) and \( d_2 \) be the directions of \( t_1 \) and \( t_2 \), respectively. Let \( T_1 \) be the point of \( \pi \) represented by the line \( t_1 \) and let \( T_2 \) be the point of \( \pi \) represented by \( t_2 \). For any \( T \in t - \{ O \} \), let \( T_T' \) be the point common to the line \( t_1 \) and to the line through \( T \) parallel to \( t_2 \) and let \( T_T'' \) be the point common to the line \( t_2 \) and to the line through \( T \) parallel to \( t_1 \). Then the direction \( \ell \) of the line joining \( T_T' \) and \( T_T'' \) does not depend on the choice of \( T \) in \( t - \{ O \} \). Let \( \ell(T_T', T_T'') \)
be the line of $\text{PG}(3, q)$ represented by the ordered pair of distinct points $(T'_1, T''_1)$ and let $r$ be the line of $\pi$ represented by the proper pencil (that is the pencil of lines through the same point $O$) of lines with centre $O$. Then, the following set of lines

$$\mathcal{R} = \{v, r\} \cup \{\ell(T'_1, T''_1)\}_{T \in \{O\}}$$

is the regulus of a hyperbolic quadric $\mathcal{H}$ in $\text{PG}(3, q)$.

Let $z$ be the line of $\pi$ represented by the improper pencil (that is the pencil of parallel lines) of lines of $\text{AG}(2, q)$ with direction $d$. Finally, let $d'$ be a direction of $\text{AG}(2, q)$ distinct from $d_1, d_2$ and $d$ and let $n$ be the line of $\text{AG}(2, q)$ through $O$ and having direction $d'$. For any $T \in t - \{O\}$, let $b'(T)$ and $b''(T)$ be the lines of $\text{AG}(2, q)$ having direction $d'$ and through $T'_1$ and $T''_1$, respectively. The following set

$$\{b'(T), b''(T)\}_{T \in \{O\}} \cup \{n\}$$

represents a line of the regulus $\overline{\mathcal{R}}$ of $\mathcal{H}$ opposite to $\mathcal{R}$, deprived of its point in common with $v$. Such a line will be denoted by $\ell(d')$. Then the regulus $\overline{\mathcal{R}}$ is

$$\overline{\mathcal{R}} = \{z, U_1T_1, U_2T_2\} \cup \{\ell(d')\}_{d' \in \mathcal{D} - \{d_1, d_2, d\}},$$

where $\mathcal{D}$ is the set of the directions of $\text{AG}(2, q)$.

Since the hyperbolic quadrics of $\text{PG}(3, q)$ are all equivalent, it follows that for any hyperbolic quadric $\mathcal{H}$ of $\text{PG}(3, q)$ there is an $R(U_1, U_2, \pi, 3)$-representation of $\text{PG}(3, q)$ which represents $\mathcal{H}$ by means of an ordered triple $(t_1, t_2, t)$ of distinct lines of $\text{AG}(2, q)$ of the same pencil.

3. Third representation

Let $\text{PG}(3, q)$ be the three dimensional projective space over the field $k$ and let $\text{AG}(2, q)$ be the affine plane over $k$. Let $\mathcal{S}$ be the set of points of $\text{AG}(2, q)$ and $R$ an $R(U_1, U_2, \pi, 3)$-representation of $\text{PG}(3, q)$. In $\text{AG}(2, q)$ let $t_1$ and $t_2$ be two distinct lines meeting at $O$ (see Fig. 9) and let $r_0$ be the line of $\pi$ represented (see [1]), through $R$, by the proper pencil of lines of $\text{AG}(2, q)$ through $O$.

Let $T_1$ and $T_2$ be the points of $\pi$ represented, through $R$, by the lines $t_1$ and $t_2$ of $\text{AG}(2, q)$. Let $X$ be a point of $\text{AG}(2, q) - t_1 \cup t_2$ and let $x$ be the line of $\pi$ represented by the proper pencil of lines with centre $X$. Let $I$ be the so defined set

$I = \{(X_1, X_2) \in \mathcal{S} \times \mathcal{S} : X_1 \in t_1 - \{O\}, X_2 \in t_2 - \{O\} \text{ and } X_1, X_2, X \text{ collinear}\}$.

Denote by $\ell(X', X'')$ the line of $\text{PG}(3, q)$ represented by the ordered pair $(X', X'')$ of distinct points of $\text{AG}(2, q)$, let

$$\mathcal{L} = \{\ell(X_1, X_2)\}_{(X_1, X_2) \in I}.$$

The lines $U_1T_1$ and $U_2T_2$ of $\text{PG}(3, q)$ are mutually skew, since the line of $\pi$ through $T_1$ and $T_2$ does not contain $Y = \pi \cap v$ $(t_1$ and $t_2$ are not parallel). The
line $x$ and $U_1 T_1$ are skew, like the line $x$ and $U_2 T_2$, since $x \in \pi$, $T_1 \notin x$, $T_2 \notin x$ ($X \notin t_1$, $X \notin t_2$). It follows that $x$, $U_1 T_1$ and $U_2 T_2$ are mutually skew.

Let $\mathcal{H}$ be the hyperbolic quadric of $PG(3, q)$ containing the lines $x$, $U_1 T_1$ and $U_2 T_2$ and let $\overline{R}$ be the regulus of $\mathcal{H}$ to which such lines belong. The line $r_0$ meets $x$ at the point of $\pi$ represented by the line $OX$ of $AG(2, q)$ and meets $U_1 T_1$ and $U_2 T_2$ at $T_1$ and $T_2$, respectively. It follows that $r_0$ belongs to the regulus $R$ of $\mathcal{H}$ opposite to $\overline{R}$.

![Figure 9.](image-url)
Moreover, it is easy to check that every line of \( \mathcal{L} \) meets \( x, U_1T_1 \) and \( U_2T_2 \). It follows that every line of \( \mathcal{L} \) belongs to \( \mathcal{R} \). Now, let \( n_1 \) and \( n_2 \) be the lines of \( AG(2, q) \) through \( X \) and parallel to \( t_1 \) and \( t_2 \), respectively. Then, let \( N_1 \) and \( N_2 \) be the points of \( \pi \) represented by the lines \( n_1 \) and \( n_2 \), respectively. The line \( U_2N_1 \) of \( PG(3, q) \) meets \( U_2T_2 \) at \( U_2, U_1T_1 \) at the point represented by the ordered pair \( (t_1, n_1) \) and \( x \) and \( N_1 \). It follows that \( U_2N_1 \in \mathcal{R} \). The line \( U_1N_2 \) meets \( U_1T_1 \) at \( U_1, U_2T_2 \) at the point represented by the ordered pair \( (n_2, t_2) \) and \( x \) at \( N_2 \). Then \( U_1N_2 \in \mathcal{R} \). By the above arguments, it follows

\[
E = \{r_0, U_1N_2, U_2N_1\} \cup \mathcal{L} \subseteq \mathcal{R}.
\]

Now, let us prove that every line of \( \mathcal{R} \) is a line of \( E \).

**Proof.** Assume that \( r' \) is a line of \( \mathcal{R} \) not in \( E \). Then \( r' \) is not in \( \pi \), otherwise \( r_0 \cap r' \neq \emptyset \), while \( r' \) and \( r_0 \) are skew, since \( r \) and \( r_0 \) are two distinct lines of \( \mathcal{R} \) \( (r_0 \in E, r' \in E) \). The line \( r' \) does not contain \( U_1 \) and \( U_2 \), since \( r' \) is distinct from \( U_1N_2 \) and \( U_2N_1 (U_1N_2 \in E, U_2N_1 \in E, r' \notin E) \). Remark that \( r' \) does not meet \( v \). For, \( M = r' \cap v \). By \( U_1 \notin r', U_2 \notin r' \), it follows that \( M \neq U_1, M \neq U_2 \). Then, \( O \) contains the three distinct points \( U, U_1 \) and \( U_2 \) of \( \mathcal{H} \) and then \( v \) is a line of \( \mathcal{H} \). Since \( v \) meets \( U_1T_1 \) and \( U_2T_2 \), which belong to \( \overline{\mathcal{R}} \), it follows that \( v \in \mathcal{R} \) and then \( v \) meets \( x \in \overline{\mathcal{R}} \): a contradiction, because \( x \) does not pass through \( Y = \pi \cap v \) (\( x \) is represented in \( AG(2, q) \) by the proper pencil of lines with centre \( X \)). The contradiction proves the remark. Therefore, \( r' \) is a line of \( PG(3, q) \) not in \( \pi \) and not meeting \( v \). It follows that \( r' \) is represented by an ordered pair of distinct points \( (X_1, X_2) \) of \( AG(2, q) \). By \( r' \in \mathcal{R} \) and \( x \in \overline{\mathcal{R}} \), it follows that \( r' \cap x \neq \emptyset \) and then the line of \( AG(2, q) \) joining \( X_1 \) and \( X_2 \) contains \( x \). By \( r' \in \mathcal{R} \) and \( U_1T_1 \in \overline{\mathcal{R}} \), it follows that \( r' \cap U_1T_1 \neq \emptyset \) and then \( X_1 \in t_1 \). By \( r' \in \mathcal{R} \) and \( U_2T_2 \in \overline{\mathcal{R}} \) it follows \( r' \cap U_2T_2 \neq \emptyset \) and then \( X_1 \in t_2 \). Moreover, we get \( X_1 \neq O, X_2 \neq O \), that is \( X_1 \in t_1 - \{O\}, X_2 \in t_2 - \{O\} \), since the points \( X_1, X_2, X \) are collinear and \( X_1 \neq X_2 \). Then \( X_1 \in t_1 - \{O\}, X_2 \in t_2 - \{O\} \) and \( X_1, X_2, X \) are collinear. By that, it follows \( (X_1, X_2) \in I \) and then \( r' = \ell(X_1, X_2) \in \mathcal{L} \subseteq E \), so \( r' \in E \) is a contradiction, since \( r' \notin E \). The contradiction proves that every line of \( \mathcal{R} \) belongs to \( E \) and then \( \mathcal{R} \subseteq E \). By that and by \( E \subseteq \mathcal{R} \) it follows \( \mathcal{R} = E \), that is

\[
\mathcal{R} = \{r_0, U_1N_2, U_2N_1\} \cup \mathcal{L}.
\]

Now, let \( \overline{I} \) be the following set

\[
\overline{I} = \{(X_2, X_1) \in S \times S : X_1 \in n_1 - \{X\}, X_2 \in n_2 - \{X\}, X_1, X_2 \text{ and } O \text{ collinear}\}.
\]

Let

\[
\overline{I} = \{\ell(X_2, X_1) \mid (X_2, X_1) \in \overline{I}\}.
\]

In a similar way as before, we get

\[
\overline{\mathcal{R}} = \{x, U_1T_1, U_2T_2\} \cup \overline{I}.
\]

So, the following theorem is proved.
Theorem 3. Let $PG(3,q)$ be the projective three-dimensional space over the field $q$, let $AG(2, q)$ be the affine plane over $q$ and let $S$ be the set of points of $AG(2, q)$ and $R$ an $R(U_1, U_2, \pi, 3)$-representation of $PG(3, q)$ (see [1]). In $AG(2, q)$ let $t_1$ and $t_2$ be two distinct lines meeting at a point $O$ and let $r_0$ be the line of $\pi$ represented, through $R$, by the proper pencil of lines of $AG(2, q)$ with centre $O$. Let $T_1$ and $T_2$ be the points of $\pi$ represented through $R$, by the lines $t_1$ and $t_2$ of $AG(2, q)$ respectively. Let $X$ be a point of $AG(2, q) - (t_1 \cup t_2)$ and let $x$ be the line of $\pi$ represented by the proper pencil of lines of $AG(2, q)$ with centre $X$.

Let $n_1$ and $n_2$ be the lines of $AG(2, q)$ through $X$ and parallel to $t_1$ and $t_2$, respectively. Then, let $N_1$ and $N_2$ be the points of $\pi$ represented through $R$, by the lines $n_1$ and $n_2$, respectively. Let $I$ and $\overline{I}$ be the following sets:

$$I = \{(X_1, X_2) \in S \times S : X_1 \in t_1 \setminus \{O\}, \; X_2 \in t_2 \setminus \{O\}, \; \text{and} \; X_1, X_2, X \text{ collinear}\},$$

$$\overline{I} = \{(\overline{X}_2, \overline{X}_1) \in S \times S : \overline{X}_1 \in n_1 \setminus \{X\}, \; \overline{X}_2 \in n_2 \setminus \{X\}, \; \text{and} \; \overline{X}_1, \overline{X}_2, O \text{ collinear}\}.$$

Denote by $\ell(X', X'')$ the line of $PG(3, q)$ represented through $R$ by the ordered pair of distinct points $(X', X'')$ of $AG(2, q)$.

Let

$$\mathcal{L} = \{\ell(X_1, X_2) : (X_1, X_2) \in I\},$$

$$\overline{\mathcal{L}} = \{\ell(\overline{X}_2, \overline{X}_1) : (\overline{X}_2, \overline{X}_1) \in \overline{I}\}.$$

Then the following sets of lines or $PG(3, q)$

$$\mathcal{R} = \{r_0, U_1 N_2, U_2 N_1\} \cup \mathcal{L},$$

$$\overline{\mathcal{R}} = \{x, U_1 T_1, U_2 T_2\} \cup \overline{\mathcal{L}},$$

are the two reguli of a hyperbolic quadric of $PG(3, q)$ admitting the plane $\pi$ as tangent plane, the contact point being the point of $\pi$ represented by the line $OX$ of $AG(2, q)$ and the line $v$ being a secant line.

The above theorem allows us to represent in the plane $AG(2, q)$ the reguli and then the points of a hyperbolic quadric $\mathcal{H}$, deprived of two points which are not represented. For, the line $r_0$ is represented by the pencil of lines of $AG(2, q)$ with centre $O$, the line $x$ is represented by the pencil of lines of $AG(2, q)$ with centre $X$, the lines $U_1 N_2, U_2 N_1, U_1 T_1, U_2 T_2$ are represented as follows:

$$U_1 N_2 \setminus \{U_1\} : \{n_2\} \cup \{n, n_2\} : n \parallel n_2 \text{ and distinct from } n_2,$$

$$U_2 N_1 \setminus \{U_2\} : \{n_1\} \cup \{n, n_1\} : n \parallel n_1 \text{ and distinct from } n_1,$$

$$U_1 T_1 \setminus \{U_1\} : \{t_1\} \cup \{t, t_1\} : t \parallel t_1 \text{ and distinct from } t_1,$$

$$U_2 T_2 \setminus \{U_2\} : \{t_2\} \cup \{t, t_2\} : t \parallel t_2 \text{ and distinct from } t_2,$$

Since the hyperbolic quadrics are all equivalent in $PG(3, q)$, it follows that for any hyperbolic quadric $\mathcal{H}$ of $PG(3, q)$ there is an $R(U_1, U_2, \pi, 3)$-representation which allows us to represent $\mathcal{H}$ as in Theorem 3.
Those three different representations of a hyperbolic quadric of $PG(3, q)$ in $AG(2, q)$ show a further application of the representation called also "crashing" cited in the bibliography.

Bibliography


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