# GEOMETRIC EQUIVALENCE BETWEEN THE VEBLEN AND DESARGUES THEOREMS AND BETWEEN THE PAPPUS–PASCAL AND THE "THREE STARS THEOREMS"

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**Summary.** Let $P(r,k)$ and $A(2,k)$ be the projective $r$-dimensional space over the field $k$ and the projective plane over the same field $k$, respectively. Let $PG(3,q)$ be the three-dimensional projective space over the Galois field $GF(q)$ and $AG(2,q)$ be the affine plane over $GF(q)$. Referring to the representation of $P(r,k)$ over $A(2,k)$ called also "Crashing" (see [1]), we prove the equivalence, from the geometric point of view, between the Veblen axiom in $PG(3,q)$ and the Desargues theorem in $AG(2,q)$. Moreover, we get a representation in $PG(3,q)$ of the Pappus-Pascal theorem in $AG(2,q)$, consisting of a suitable configuration of planes, called the "Three stars theorem", which turns out to be a geometric equivalence between those two theorems. For the notations and theorems about the representation of $P(r,k)$ over $A(2,k)$ (and therefore in particular of $PG(3,q)$ over $AG(2,q)$), we refer to the paper [1], cited in the bibliography, which the reader must know before reading this text.

## 1. Geometric equivalence between the Veblen and Desargues configurations

It is known that in $PG(3,q)$, the projective three dimensional space over the field $k$, the following *Veblen axiom* holds:

*For any two lines $z$ and $t$ of $PG(3,q)$ meeting at $O$, if $r_1$ and $r_2$ are two lines each meeting both $z$ and $t$ at two distinct points, distinct from $O$, also $r_1$ and $r_2$ are incident.*

Let $z$ and $t$ be two lines of $PG(3,q)$ meeting at $Z$ (see Fig. 1). Let $Z_1$ and $Z_2$ be two distinct points of $z$, both different from $Z$. Let $T_1$ and $T_2$ be two distinct points of $t$, both distinct from $Z$. Let $r_1$ be the line $Z_1T_1$ and $r_2$ the line $Z_2T_2$. Since in $PG(3,q)$ the Veblen axiom holds, the lines $r_1$ and $r_2$ meet at a point $X$. The lines $z, t, r_1, r_2$ belong to the same plane $\alpha$. Now, let $\pi$ be a plane of $PG(3,q)$ through $z$ and distinct from $\alpha$. Let $Y$ be a point of $z - \{Z, Z_1, Z_2\}$. Finally, let $v$ be a line through $Y$ and not belonging either to $\pi$, or to $\alpha$ (see Fig. 1).

Let us represent (using the crashing [1]) the Veblen configuration of $PG(3, q)$ in the affine plane $AG(2, q)$. The points $Z$, $Z_1$ and $Z_2$ are represented by three distinct lines $z, z_1, z_2$ of $AG(2, q)$. The line $t$ belongs to the class a) of [1] and therefore is represented in $AG(2, q)$ by an ordered pair of distinct points $A$ and $B$ of $z'$. The point $T_1$ of $t$ is transformed in an ordered pair of parallel and distinct lines through $A$ and $B$, let they be $a$ and $b$ respectively. The point $T_2$ is transformed in the ordered pair of parallel and distinct lines $a'$ and $b'$ through $A$ and $B$, respectively.

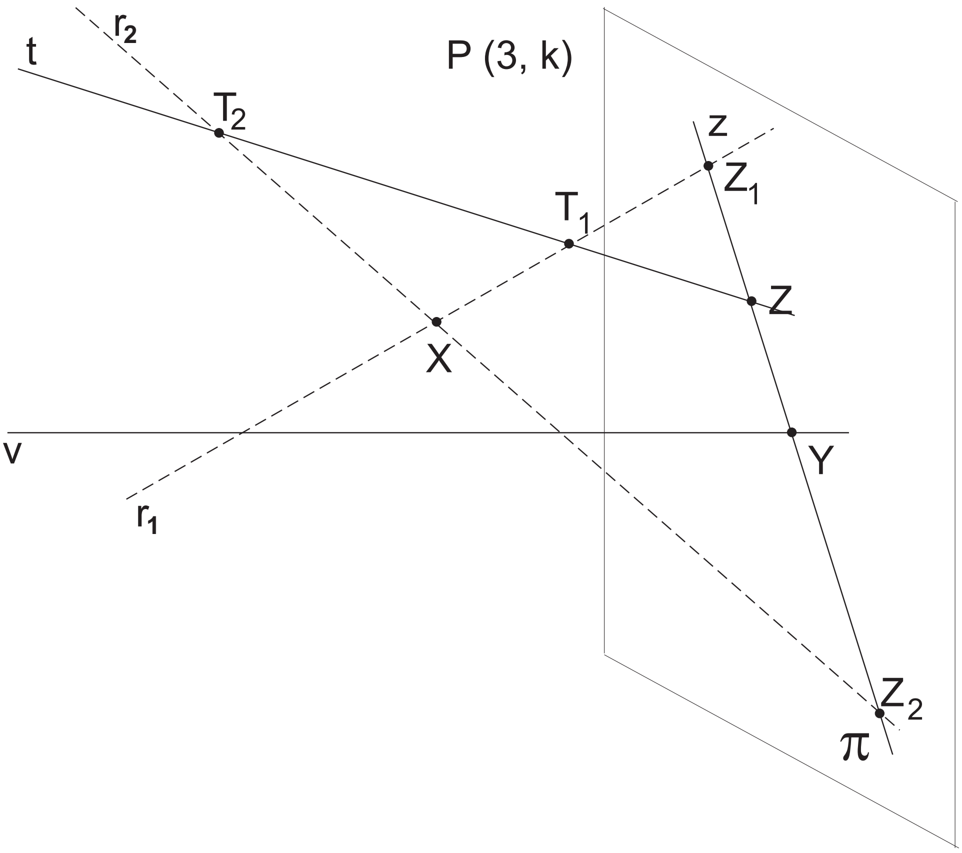


**Figure 1. The Veblen configuration.**

The line $Z_1T_1$ is represented in $AG(2, q)$ by the ordered pair of distinct points $(A', B')$, with $A' = \alpha \cap z$, $B' = b \cap z_1$. The line $T_2Z_2$ is represented in $AG(2, q)$ by the ordered pair of distinct points $(A'', B'')$, with $A'' = a' \cap z_2$, $B'' = b' \cap z_2$. By the Veblen axiom in $PG(3, q)$, the lines $Z_1T_1$ and $Z_2T_2$ meet at a point $X$ which does not belong either to $v$, or to $\pi$, therefore such a point is represented by an ordered pair of parallel and distinct lines of $AG(2, q)$. It follows that the line $A'A''$ and the line $B'B''$ of $AG(2, q)$ are parallel (see Fig. 2), since the line represented by the ordered pair $(A', B')$ and the line represented by the ordered pair $(A'', B'')$ must have a point in common, which is necessarily represented by an ordered pair of parallel and distinct lines.

The configuration obtained in this way in $AG(2,q)$ is the affine plane Desargues configuration. Conversely, let us consider the affine Desargues configuration in $AG(2,q)$, as in Fig. 2. The line $t$ of $PG(3,q)$ represented by the ordered pair of distinct points $A$ and $B$ of $z'$ and the line $z$ of $\pi$ containing the points represented by the lines $z$, $z'$, $z_2$ meet at the point $Z$, represented by the line $z'$. The line $r_1$ of $PG(3,q)$ (see Fig. 1) represented in $AG(2,q)$ by the ordered pair $(A', B')$ and the line $r_2$ of $PG(3,q)$, represented by the ordered pair $(A'', B'')$ both meet $z$ and $t$. By the Desargues theorem the lines $B'B''$ and $A'A''$ of $AG(2,q)$ are parallel. It follows that the lines $r_1$ and $r_2$ meet at $X$, represented by the ordered pair of parallel and distinct lines $A'A''$ and $B'B''$. Therefore, the Desargues configuration (see Fig. 2) changes to the Veblen configuration of $PG(3,q)$ (see Fig. 1). So, the geometric equivalence of Veblen and Desargues configurations is proved. ∎

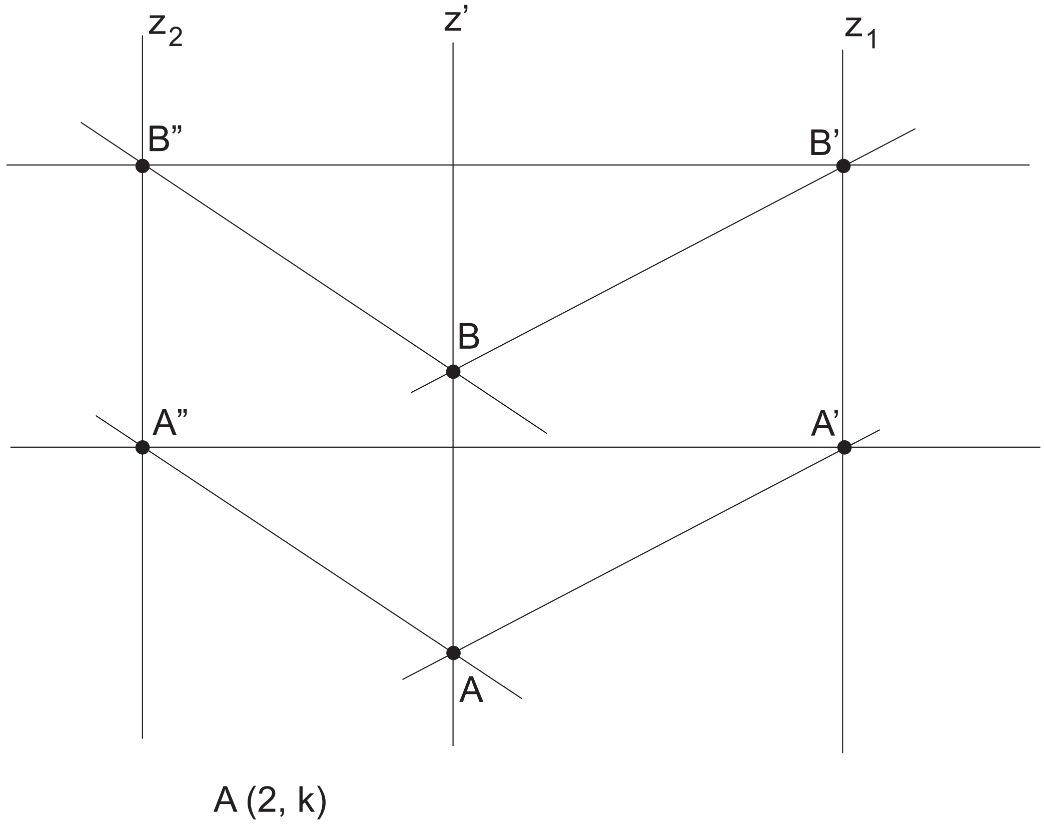


**Figure 2. The Desargues affine configuration.**

## 2. The three stars theorem in $PG(3,q)$ and its equivalence with the Pappus–Pascal theorem in the affine plane $AG(2,q)$

**Theorem 1. The "three stars theorem".** *Let $PG(3,q)$ be the projective three dimensional space over the field $k$. Let $v$ be a line of $PG(3,q)$ and let $\pi$ be a plane not through $v$ meeting $v$ at a point $Y$. Let $U_1$ and $U_2$ be two points of $v$, distinct between them and from $Y$. Let $A_1$ and $A_2$ be two distinct points of $\pi$ such that the line $A_1A_2 = d$ does not contain $Y$. Let $s_1, s_2, s_3$ be three lines of $\pi$ through $A_1$ distinct between them and from $d$ and not through $Y$. Let $s'_1, s'_2, s'_3$ be three lines of $\pi$ not through $A_2$, distinct between them and from $d$ and not through $Y$. Denoting by $\gamma(U_i, s_j)$ the plane through $U_i$ and the line $s_j$ $(i = 1,2,\ j = 1,2,3)$, assume that the following two conditions are satisfied:*

1) *The planes $\gamma(U_1, s_1)$, $\gamma(U_2, s_2)$, $\gamma(U_1, s'_2)$ and $\gamma(U_2, s'_1)$ belong to a star.*
2) *The planes $\gamma(U_1, s_2)$, $\gamma(U_2, s_3)$, $\gamma(U_1, s'_3)$ and $\gamma(U_2, s'_2)$ belong to a star* (see Fig. 3).

*Then, the four planes $\gamma(U_1, s_3)$, $\gamma(U_2, s_1)$, $\gamma(U_1, s'_1)$ and $\gamma(U_2, s'_3)$ belong to a star.*

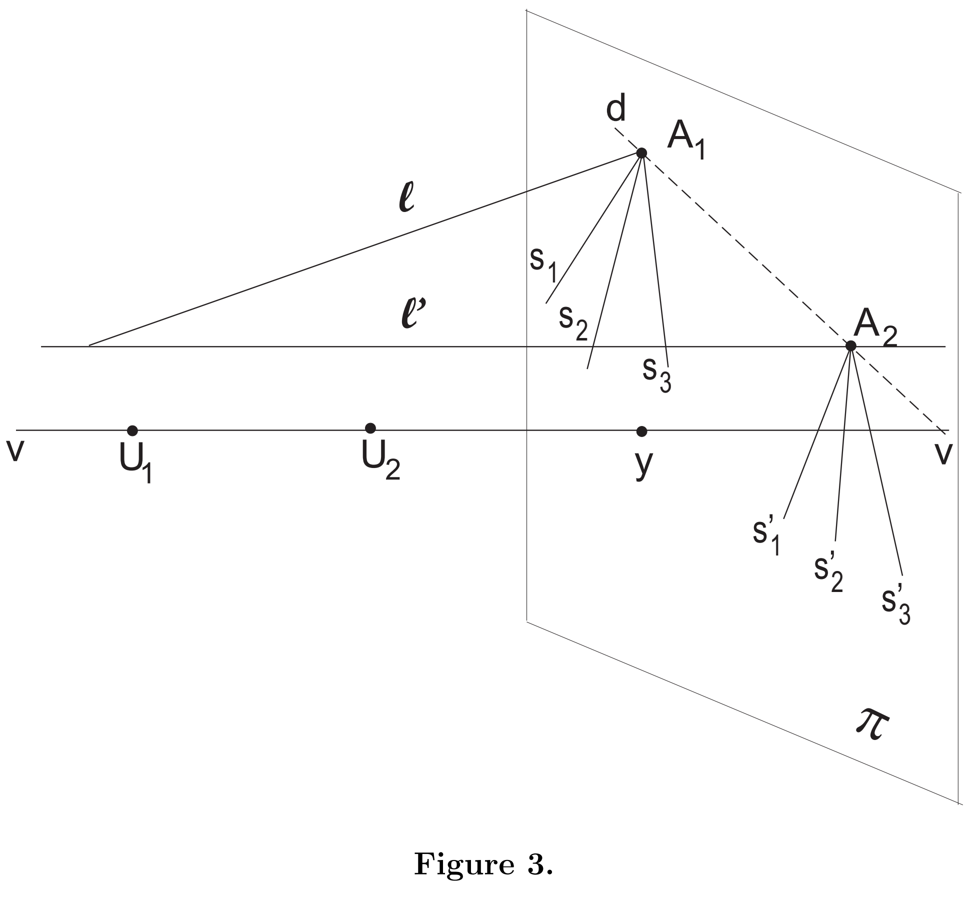


**Figure 3.**

**Proof.** Let $r_1$ and $r_2$ be the distinct lines of $AG(2,q)$ representing the points $A_1$ and $A_2$, respectively (see [1]). Such lines are not parallel, since the line $d$ does not pass through $Y$ and so they meet at a point $D$. The lines $s_1, s_2, s_3$ are represented in $AG(2,q)$, by three pencils of lines with centres at $D_1, D_2, D_3$, respectively, which are distinct between them and from $D$ and all contained in $r_1$.

Similarly, the lines $s'_1, s'_2, s'_3$ are represented in $AG(2,q)$ by three pencils of lines with centres $D'_1, D'_2, D'_3$, respectively, all contained in $r_2$. We remark that condition 1) is equivalent to the incidence of the lines $\ell = \gamma(U_1, s_1) \cap \gamma(U_2, s_2)$ and $\ell' = \gamma(U_1, s'_2) \cap \gamma(U_2, s'_1)$. Now, let us prove that the line $\ell$, which belongs to the class a) of [1], in the crashing is represented by the ordered pair of distinct points $(D_1, D_2)$. For, let $S_1$ be a point of $s_1$, distinct from $s_1 \cap s_2$ and let $t$ be the line $U_1S_1$. Such a line $t$ meets the line $f$ of the class a), represented by the pair $(D_1, D_2)$, since the dotted line $t$ (which belongs to the class e) meets $f$ at the point represented by the ordered pair $(h_1, h'_1)$, where $h_1$ is the line representing $S_1$ and $h'_1$ is the line through $D_2$ and parallel to $h_1$.

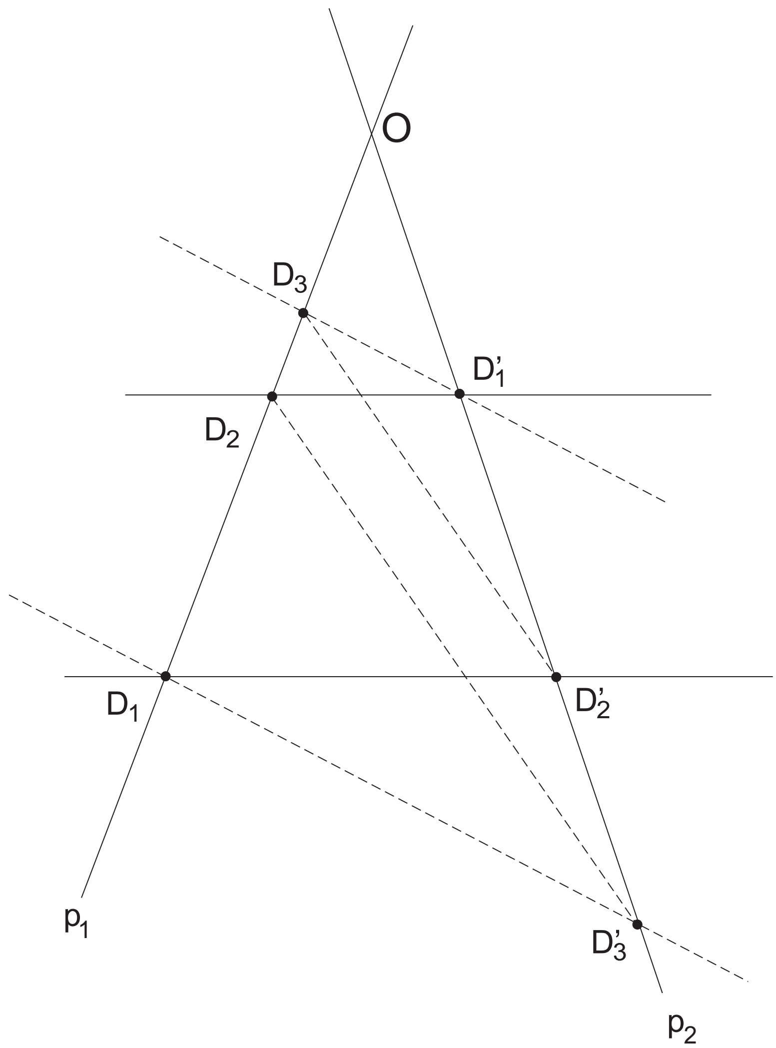


**Figure 4.**

By varying $S_1$ in $s_1 - (s_1 \cap s_2)$, the line $t$ meets the line $f$. It follows that $f$ is contained in the plane $\gamma(U_1, s_1)$. Similarly, we prove that $f$ is contained in the plane $\gamma(U_2, s_2)$ and then $f = \gamma(U_1, s_1) \cap \gamma(U_2, s_2)$. Therefore, $f$ coincides with $\ell$. Similarly, $\ell'$ is represented by the ordered pair $(D'_2, D'_1)$. Since $\ell$ and $\ell'$ meet and since $A_1 = \ell \cap \pi$, $A_2 = \ell' \cap \pi$ are two distinct points, by the representation of the lines of the class c) to which $\ell$ and $\ell'$ belong, it follows that $D_1D'_2$ and $D_2D'_1$ are parallel, since the ordered pair $(D_1D'_2, D_2D'_1)$ represents the point $\ell \cap \ell'$. In a similar way we prove that the condition 2) implies the parallelism between the lines $D'_3D_2$ and $D_2D'_3$ (see Fig. 4). The conditions 1) and 2) of Theorem 1 in $AG(2, q)$ by using the crashing become the conditions of the Pappus–Pascal configuration, which are the following (see Fig. 4): given two lines $p_1, p_2$ of $AG(2, q)$ meeting at $O$, let $D_1, D_2, D_3$ be three distinct points different from $O$ of $p_1$. Let $D'_1, D'_2, D'_3$ be three distinct points not coincident with $O$ of $p_2$. Let the lines $D_2D'_1$, $D_1D'_2$ be parallel like the lines $D_3D'_2$, $D_2D'_3$. Then, $D_3D'_1$ and $D_1D'_3$ are parallel. From such a parallelism it follows that the line $h_1$ of $PG(3, q)$ represented by the ordered pair $(D_3, D_1)$ meets the line $h_2$, represented by the ordered pair $(D'_1, D'_3)$.

Moreover, $h_1 = \gamma(U_1, s_3) \cap \gamma(U_2, s_1)$ and $h_2 = \gamma(U_1, s'_1) \cap \gamma(U_2, s'_3)$ similarly to the case of the lines $s_1$ and $s_2$. By the incidence of the lines $h_1$ and $h_2$ the thesis follows, that is, the fact that the planes $\gamma(U'_1, s_3)$, $\gamma(U_2, s_1)$, $\gamma(U_1, s'_1)$, $\gamma(U_2, s''_3)$ belong to the same star.

## References

[1] Scafati Tallini, M., *Representation of the projective space $P(r, k)$ in the affine plane $A(2, k)$*, Proc. Conference on Error-Correcting Codes, Cryptography and Finite Geometries, Amer. Math. Soc. (Eds. A. Bruen and D. Wehlan) (2010), 109-122.

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