RELATED FIXED POINT THEOREM FOR SIX MAPPINGS
ON THREE MODIFIED INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. Related fixed point theorems on two or three metric spaces have been proved in different ways. Sharma, Deshpande and Thakur were the first who have established related fixed point theorem for four mappings on two complete fuzzy metric spaces. Their work was maiden in this line. In this paper we obtain a related fixed point theorem for six mappings on three complete modified intuitionistic fuzzy metric spaces. Of course this is a new result on this line.

AMS Subject Classification (2000): 47H10, 54H25.
Keywords: modified intuitionistic fuzzy metric space, common fixed point, Cauchy sequence.

1. Introduction

Motivated by the potential applicability of fuzzy topology to quantum particle physics particularly in connection with both string and $e^{(\infty)}$ theory developed by El Naschie [10], [11], Park introduced and discussed in [24] a notion of intuitionistic fuzzy metric spaces which is based on the idea of intuitionistic fuzzy sets due to Atanassov [3] and the concept of fuzzy metric space given by George and Veeramani [18]. Actually, Park’s notion is useful in modelling some phenomena where it is necessary to study the relationship between two probability functions. It has direct physics motivation in the context of the two-slit experiment as the foundation of E-infinity of high energy physics, recently studied by El Naschie [12], [13].
Alaca et al. [2] using the idea of intuitionistic fuzzy sets, they defined the notion of intuitionistic fuzzy metric space as Park [24] with the help of continuous $t$-norms and continuous $t$-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [22]. Further, they introduced the notion of Cauchy sequences in intuitionistic fuzzy metric spaces and proved the well known fixed point theorems of Banach [4] and Edelstein [9] extended to intuitionistic fuzzy metric spaces with the help of Grabiec [13]. Turkoglu et al. [30] introduced the concept of compatible maps and compatible maps of types $(\alpha)$ and $(\beta)$ in intuitionistic fuzzy metric spaces and gave some relations between the concepts of compatible maps and compatible maps of types $(\alpha)$ and $(\beta)$.

Since the intuitionistic fuzzy metric space has extra conditions, Saadati, Sedghi and Shobe [28] modified the idea of intuitionistic fuzzy metric spaces and gave the new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous $t$-representable.

Related fixed point theorems on two or three metric spaces were proved by Fisher [14], [15], Nung [23], Popa [24], Jain, Sahu and Fisher [19], Jain, Shrivastava and Fisher [20], Cho, Kang and Kim [5], Fisher and Murthy [16] and many others. Sharma, Deshpande and Thakur [29] established a related fixed point theorem for four mappings on two complete fuzzy metric spaces. Deshpande and Pathak [8] intuitionistically fuzzified the results of Sharma, Deshpande and Thakur [29] and proved a related fixed point theorem for two pairs of mappings on two intuitionistic fuzzy metric spaces. In this paper, we extend the results of Deshpande and Pathak [8] and prove a related fixed point theorem for six mappings on three complete modified intuitionistic fuzzy metric spaces.

2. Preliminaries

**Definition 2.1.** ([26]) A binary operation $*$ : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous $t$-norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0, 1]$,

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

**Definition 2.2.** ([26]) A binary operation $\diamond : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous $t$-conorm if $\diamond$ is satisfying the following conditions:

(i) $\diamond$ is commutative and associative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a$ for all $a \in [0, 1]$,

(iv) $a \diamond b = c \diamond d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$. 
Lemma 2.1. ([7]) Consider the set \( L^* \) and operation \( \leq_{L^*} \) defined by
\[
L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\}
\]
\[
(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1 \text{ and } x_2 \geq y_2, \text{ for every } (x_1, x_2), (y_1, y_2) \in L^*.
\]
Then \( (L^*, \leq_{L^*}) \) is a complete lattice.

Definition 2.3. ([3]) An intuitionistic fuzzy set \( A_{\zeta,\eta} \) in a universe \( U \) is an object
\[
A_{\zeta,\eta} = \{(\zeta_A(u), \eta_A(u)) \mid u \in U\},
\]
where, for all \( u \in U \), \( \zeta_A(u) \in [0, 1] \) and \( \eta_A(u) \in [0, 1] \) are called the membership degree and the non-membership degree, respectively, of \( u \) in \( A_{\zeta,\eta} \), and, furthermore, they satisfy \( \zeta_A(u) + \eta_A(u) \leq 1 \).

For every \( z_i = (x_i, y_i) \in L^* \), if \( c_i \in [0, 1] \) such that \( \sum_{j=1}^n c_j = 1 \), then it is easy that
\[
(2.1) \quad \sum_{j=1}^n c_j (x_j, y_j) = \left( \sum_{j=1}^n c_j x_j, \sum_{j=1}^n c_j y_j \right) \in L^*.
\]
We denote its units by \( 0_{L^*} = (0, 1) \) and \( 1_{L^*} = (1, 0) \). Classically, a triangular norm \( * = T \) on \([0, 1]\) is defined as an increasing, commutative, associative mapping \( T : [0, 1]^2 \to [0, 1] \) satisfying \( T(1, x) = 1 \leftrightarrow x = x \), for all \( x \in [0, 1] \). A triangular conorm \( S = \circ \) is defined as an increasing, commutative, associative mapping \( S : [0, 1]^2 \to [0, 1] \) satisfying \( S(0, x) = 0 \leftrightarrow x = x \), for all \( x \in [0, 1] \). Using the lattice \((L^*, \leq_{L^*})\) these definitions can be straightforwardly extended.

Definition 2.4. ([6]) A triangular norm \( (t\text{-norm}) \) on \( L^* \) is a mapping \( \tau : (L^*)^2 \to L^* \) satisfying the following conditions:
\[
(\forall x \in L^*) (\tau(x, 1_{L^*}) = x) \text{ (boundary condition)},
\]
\[
(\forall (x, y) \in (L^*)^2) (\tau(x, y) = \tau(y, x)) \text{ (commutativity)},
\]
\[
(\forall (x, y, z) \in (L^*)^3) (\tau(x, \tau(y, z)) = \tau(\tau(x, y), z)) \text{ (associativity)},
\]
\[
(\forall (x, x', y, y') \in (L^*)^4) (x \leq_{L^*} x') \text{ and } (y \leq_{L^*} y') \to \tau(x, y) \leq_{L^*} \tau(x', y') \text{ (monotonicity)}.
\]

Definition 2.5. ([6], [7]) A continuous \( t\text{-norm} \) \( \tau \) on \( L^* \) is called continuous \( t\text{-representable} \) if and only if there exist a continuous \( t\text{-norm} * \) and a continuous \( t\text{-conorm} \circ \) on \([0, 1]\) such that, for all \( x = (x_1, x_2), y = (y_1, y_2) \in L^* \),
\[
\tau(x, y) = (x_1 \circ y_1, x_2 \circ y_2).
\]

Now, define a sequence \( \tau^n \) recursively by \( \tau^1 = \tau \) and
\[
\tau^n(x^{(1)}, ..., x^{(n+1)}) = \tau(\tau^{n-1}(x^{(1)}, ..., x^{(n)}), x^{(n+1)}) \text{ for } n \geq 2 \text{ and } x^{(i)} \in L^*.
\]

Definition 2.6. ([28]) Let \( M, N \) are fuzzy sets from \( X^2 \times (0, +\infty) \) to \([0, 1]\) such that \( M(x, y, t) + N(x, y, t) \leq 1 \) for all \( x, y \in X \) and \( t > 0 \). The 3-tuple \((X, M_{M, N}, \tau)\) is said to be an intuitionistic fuzzy metric space if \( X \) is an arbitrary (non-empty) set, \( \tau \) is a continuous \( t\text{-representable} \) and \( M_{M, N} \) is a mapping \( X^2 \times (0, +\infty) \to L^* \) (an intuitionistic fuzzy set, see Definition 2.3) satisfying the following conditions for every \( x, y \in X \) and \( t, s > 0 \):
(a) $\mathcal{M}_{M,N}(x, y, t) >_{L^*} 0_{L^*}$;
(b) $\mathcal{M}_{M,N}(x, y, t) = 1_{L^*}$ if and only if $x = y$;
(c) $\mathcal{M}_{M,N}(x, y, t) = \mathcal{M}_{M,N}(y, x, t)$;
(d) $\mathcal{M}_{M,N}(x, y, t + s) \geq_{L^*} \tau(\mathcal{M}_{M,N}(x, z, t), \mathcal{M}_{M,N}(z, y, s))$;
(e) $\mathcal{M}_{M,N}(x, y, \cdot) : (0, \infty) \to L^*$ is continuous.

In this case, $\mathcal{M}_{M,N}$ is called an intuitionistic fuzzy metric.

Here, $\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t))$.

**Example 2.1.** ([28]) Let $(X, d)$ be a metric space. Denote

$$\tau(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$$

for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let $M$ and $N$ be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left( \frac{ht^n}{ht^n + md(x, y)}, \frac{md(x, y)}{ht^n + md(x, y)} \right)$$

for all $t, h, m, n \in R^+$.

Then, $(X, \mathcal{M}_{M,N}, \tau)$ is an intuitionistic fuzzy metric space.

**Example 2.2.** ([28]) Let $X = N$. Define

$$\tau(a, b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2 b_2)$$

for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let $M$ and $N$ be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \begin{cases} \left( \frac{x}{y}, \frac{y - x}{y} \right) & \text{if } x \leq y, \\ \left( \frac{y}{x}, \frac{x - y}{x} \right) & \text{if } y \leq x, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{M,N}, \tau)$ is an intuitionistic fuzzy metric space.

**Definition 2.7.** ([28]) A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \tau)$ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that

$$\mathcal{M}_{M,N}(x_n, y_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon)$$
and for each $n, m \geq n_0$, here $N_s$ is the standard negator. The sequence $\{x_n\}$ is said to be convergent to $x \in X$ in the intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \tau)$ and denoted by $x_n \xrightarrow{\mathcal{M}_{M,N}} x$ if $\mathcal{M}_{M,N}(x_n, x, t) \rightarrow 1_{L^*}$ whenever $n \rightarrow \infty$ for every $t > 0$. An intuitionistic fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Lemma 2.2. ([27]) Let $\mathcal{M}_{M,N}$ be an intuitionistic fuzzy metric space. Then, for any $t > 0$, $\mathcal{M}_{M,N}(x, y, t)$ is non-decreasing with respect to $t$, in $(L^*, \leq_{L^*})$, for all $x, y \in X$.

Lemma 2.3. ([1]) Let $(X, \mathcal{M}_{M,N}, \tau)$ be a modified intuitionistic fuzzy metric space. For each $\lambda \in (0, 1)$, define the map $E_{\lambda} : X^2 \rightarrow R^+ \cup \{0\}$ by

$$E_{\lambda}(x, y) = \inf \{t > 0 : \mathcal{M}_{M,N}(x, y, t) > L^*(1 - \lambda, \lambda)\},$$

then

(a) For each $\lambda \in (0, 1)$, we have a $\mu \in (0, 1)$ such that

$$E_{\lambda}(x_1, x_n) \leq E_{\mu}(x_1, x_2) + E_{\mu}(x_2, x_3) + \cdots + E_{\mu}(x_{n-1}, x_n),$$

for any $x_1, x_2, x_3, ..., x_n \in X$.

(b) The sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ is convergent to $x$ if and only if $E_{\lambda}(x_n, x) \rightarrow 0$.

Also, the sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $X$ if and only if it is a Cauchy sequence with respect to $E_{\lambda}$.

Lemma 2.4. ([21]) Let $(X, \mathcal{M}_{M,N}, \tau)$ be an intuitionistic fuzzy metric space. If for a sequence $\{x_n\}$ in $X$, there exists $k \in (0, 1)$ such that

$$\mathcal{M}_{M,N}(x_n, x_{n+1}, kt) \geq L^* \mathcal{M}_{M,N}(x_{n-1}, x_n, t), \text{ for all } n \text{ and for all } t,$$

then $\{x_n\}$ is a Cauchy sequence in $X$.

Proof. Let $(X, \mathcal{M}_{M,N}, \tau)$ be an intuitionistic fuzzy metric space. Let for a sequence $\{x_n\}$ in $X$, there exists $k \in (0, 1)$ such that

$$\mathcal{M}_{M,N}(x_n, x_{n+1}, kt) \geq L^* \mathcal{M}_{M,N}(x_{n-1}, x_n, t), \text{ for all } n \text{ and for all } t,$$

then

$$\mathcal{M}_{M,N}(x_n, x_{n+1}, t) \geq L^* \mathcal{M}_{M,N} \left(x_{n-1}, x_n, \frac{t}{k}\right) \geq L^* \mathcal{M}_{M,N} \left(x_{n-2}, x_{n-1}, \frac{t}{k^2}\right) \cdots \geq L^* \mathcal{M}_{M,N} \left(x_0, x_1, \frac{t}{k^n}\right), \text{ for all } n.$$
Now
\[ E_\lambda(x_{n+1}, x_n) = \inf \{ t > 0 : M_{M,N}(x_{n+1}, x_n, t) \geq_L (1 - \lambda, \lambda) \} \]
\[ \leq \inf \{ t > 0 : M_{M,N}(x_1, x_0, \frac{t}{k^n}) \geq_L (1 - \lambda, \lambda) \} \]
\[ = \inf \{ k^n t > 0 : M_{M,N}(x_1, x_0, t) \geq_L (1 - \lambda, \lambda) \} \]
\[ = k^n \inf \{ t > 0 : M_{M,N}(x_1, x_0, t) \geq_L (1 - \lambda, \lambda) \} \]
\[ = k^n E_\lambda(x_0, x_1). \]

Again from Lemma 2.3, for \( \lambda \in (0, 1) \), there exists \( \mu \in (0, 1) \) such that
\[ E_\lambda(x_{n+1}, x_n) \leq k^n E_\lambda(x_0, x_1) \ldots (A) \]

which tends to 0, as \( n \to \infty \). Hence \( \{x_n\} \) is a Cauchy sequence in \( X \).

Lemma 2.5. ([21]) In an intuitionistic fuzzy metric space \( (X, M_{M,N}, \tau) \), if for some \( x, y \) in \( X \) there exists \( k \in (0, 1) \) such that
\[ M_{M,N}(x, y, kt) \geq_L, M_{M,N}(x, y, t), \text{for all } t, \]
then \( x = y \).

Proof. Let for \( \lambda \in (0, 1) \)
\[ E_\lambda(x, y) = \inf \{ t > 0 : M_{M,N}(x, y, t) \geq_L (1 - \lambda, \lambda) \} \]
\[ \leq \inf \{ t > 0 : M_{M,N}(x, y, t/k) \geq_L (1 - \lambda, \lambda) \} \]
\[ = \inf \{ kt > 0 : M_{M,N}(x, y, t) \geq_L (1 - \lambda, \lambda) \} \]
\[ = k \inf \{ t > 0 : M_{M,N}(x, y, t) \geq_L (1 - \lambda, \lambda) \} \]
\[ = k E_\lambda(x, y). \]

Therefore, \( E_\lambda(x, y) = 0 \). Hence \( x = y \).

Sharma, Deshpande and Thakur [29] established the following related fixed point theorem for four mappings on two complete fuzzy metric spaces.

Theorem A. Let \( (X, M_1, *) \) and \( (Y, M_2, *) \) be two complete fuzzy metric spaces. Let \( A, B \) be mappings from \( X \) into \( Y \) and \( S, T \) be mappings from \( Y \) into \( X \) satisfying the inequalities:
(i) \( M_1(SAx, T Bx', kt) \geq M_1(x, x', t) \ast M_1(x, SAx, t) \ast M_1(x', T Bx', t) \ast M_1(SAx, T Bx', t) \)

(ii) \( M_2(BSy, AT y', kt) \geq M_2(y, y', t) \ast M_2(y, BSy, t) \ast M_2(y', AT y', t) \ast M_2(BSy, AT y', kt) \)

for all \( x, x' \) in \( X \) and \( y, y' \) in \( Y \). If one of the mappings \( A, B, S, T \) is continuous, then \( SA \) and \( TB \) have a unique common fixed point \( z \) in \( X \) and \( BS \) and \( AT \) have a unique common fixed point \( w \) in \( Y \). Further, \( Az = Bz = w \) and \( Sw = Tw = z \).

We extend the results of Deshpande and Pathak [8] and prove a related fixed point theorem for six mappings on three complete modified intuitionistic fuzzy metric spaces.

3. Main result

**Theorem 3.1.** Let \((X, M_{M_1}, N_1, \tau)\), \((Y, M_{M_2}, N_2, \tau)\) and \((Z, M_{M_3}, N_3, \tau)\) be three complete intuitionistic fuzzy metric spaces. Let \(A, B\) be continuous mappings from \(X\) into \(Y\), let \(S, T\) be continuous mappings from \(Y\) into \(Z\) and let \(P, Q\) be continuous mappings from \(Z\) into \(X\) satisfying the inequalities:
(3.1) \( \mathcal{M}_{M_1,N_1}(PSAx,QTBx',kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x,x',t) * \mathcal{M}_{M_1,N_1}(PSAx,t) * \mathcal{M}_{M_1,N_1}(x',QTBx',t) \)

(3.2) \( \mathcal{M}_{M_2,N_2}(APSy,BQTy',kt) \geq_{L^*} \mathcal{M}_{M_2,N_2}(y,y',t) * \mathcal{M}_{M_2,N_2}(APSy,t) * \mathcal{M}_{M_2,N_2}(BQTy',t) \)

(3.3) \( \mathcal{M}_{M_3,N_3}(SAPz,TBQz',kt) \geq_{L^*} \mathcal{M}_{M_3,N_3}(z,z',t) * \mathcal{M}_{M_3,N_3}(SAPz,t) * \mathcal{M}_{M_3,N_3}(TBQz',t) \)

for all \( x, x' \) in \( X \), \( y, y' \) in \( Y \) and \( z, z' \) in \( Z \), \( t > 0 \) and \( k \in (0,1) \), then \( PSA \) and \( QTB \) have a unique common fixed point \( u \) in \( X \), \( APS \) and \( BQT \) have a unique common fixed point \( v \) in \( Y \) and \( SAP \) and \( TBQ \) have a unique common fixed point \( w \) in \( Z \). Further, \( Au = Bu = v, Sv = T v = w \) and \( Pw = Qw = u \).

**Proof.** Let \( x = x_0 \) be an arbitrary point in \( X \) and define sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) in \( X, Y \) and \( Z \) respectively as follows:

Choose a point \( z_1 = S y_1 \), a point \( y_1 = A x_0 \), a point \( x_1 = P z_1 \), a point \( z_2 = T y_2 \), a point \( y_2 = B x_1 \) and a point \( x_2 = Q z_2 \). In general, having chosen \( x_{2n-2} \) in \( X \), choose a point \( y_{2n-1} = A x_{2n-2} \), a point \( y_{2n} = B x_{2n-1} \), a point \( z_{2n-1} = S y_{2n-1} \), a point \( z_{2n} = T y_{2n} \), a point \( x_{2n-1} = P z_{2n-1} \) and a point \( x_{2n} = Q z_{2n} \) for all \( n = 1, 2, ... \)

Applying inequality (3.1), we have

\[
\mathcal{M}_{M_1,N_1}(x_{2n+1},x_{2n},kt) = \mathcal{M}_{M_1,N_1}(PSAx_{2n},QTBx_{2n-1},kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n},x_{2n-1},t) * \mathcal{M}_{M_1,N_1}(x_{2n},PSAx_{2n},t) * \mathcal{M}_{M_1,N_1}(x_{2n-1},QTBx_{2n-1},t) \]

(3.4)

Similarly, we have

\[
\mathcal{M}_{M_2,N_2}(x_{2n+2},x_{2n+1},kt) \geq_{L^*} \mathcal{M}_{M_2,N_2}(x_{2n+1},x_{2n},t) * \mathcal{M}_{M_2,N_2}(x_{2n+1},PSAx_{2n+1},t) * \mathcal{M}_{M_2,N_2}(x_{2n},QTBx_{2n},t) \]

(3.5)

Thus, from (3.4) and (3.5), it follows that

\[
\mathcal{M}_{M_1,N_1}(x_{n+1},x_{n+2},kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{n},x_{n+1},t) * \mathcal{M}_{M_1,N_1}(x_{n+1},x_{n+2},t),
\]

for \( n = 1, 2, ... \).

Consequently, for positive integers \( n, p \) we have

\[
\mathcal{M}_{M_1,N_1}(x_{n+1},x_{n+2},kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{n},x_{n+1},t) * \mathcal{M}_{M_1,N_1}(x_{n+1},x_{n+2},t/k^p).
\]

Thus, since \( \mathcal{M}_{M_1,N_1}(x_{n+1},x_{n+2},kt) \to 1_{L^*} \) as \( p \to \infty \), we have

\[
\mathcal{M}_{M_1,N_1}(x_{n+1},x_{n+2},kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{n},x_{n+1},t)
\]

(3.6)
Similarly, applying inequality (3.2) and (3.3), we have

\begin{align}
(3.7) & \quad M_{M_2, N_2}(y_{n+1}, y_{n+2}, kt) \geq L^* \cdot M_{M_2, N_2}(y_n, y_{n+1}, t) \\
(3.8) & \quad M_{M_3, N_3}(z_{n+1}, z_{n+2}, kt) \geq L^* \cdot M_{M_3, N_3}(z_n, z_{n+1}, t)
\end{align}

By Lemma 2.4, \( \{x_n\} \) is a Cauchy sequence in a complete intuitionistic fuzzy metric space \( X \) and so has a limit \( u \) in \( X \). It follows similarly that the sequences \( \{y_n\} \) and \( \{z_n\} \) are also Cauchy sequences in complete intuitionistic fuzzy metric space \( Y \) and \( Z \) and so have limits \( v \) in \( Y \) and \( w \) in \( Z \).

Using (3.1), we have

\[
M_{M_1, N_1}(PSAx_{2n}, u, kt) \geq L^* \cdot M_{M_1, N_1}(PSAx_{2n}, x_{2n}, \frac{kt}{2}) \cdot M_{M_1, N_1}(x_{2n}, u, \frac{kt}{2}) \\
\geq L^* \cdot M_{M_1, N_1}(x_{2n}, x_{2n-1}, \frac{1}{2}) \cdot M_{M_1, N_1}(x_{2n}, PSpAx_{2n}, \frac{1}{2}) \\
\geq L^* \cdot M_{M_1, N_1}(x_{2n-1}, QTBx_{2n-1}, \frac{1}{2}) \cdot M_{M_1, N_1}(x_{2n}, u, \frac{kt}{2}) \\
\geq L^* \cdot M_{M_1, N_1}(x_{2n-1}, x_{2n-1}, \frac{1}{2}) \cdot M_{M_1, N_1}(x_{2n}, x_{2n+1}, \frac{1}{2}) \\
\geq L^* \cdot M_{M_1, N_1}(x_{2n-1}, x_{2n}, \frac{1}{2}) \cdot M_{M_1, N_1}(x_{2n+1}, x_{2n}, \frac{1}{2}) \cdot M_{M_1, N_1}(x_{2n}, u, \frac{kt}{2})
\]

Taking limit \( n \to \infty \), we have

\[
\lim_{n \to \infty} M_{M_1, N_1}(PSAx_{2n}, u, kt) \to 1_{L^*}.
\]

Thus, we have

\[
\lim_{n \to \infty} PSAx_{2n} = u = \lim_{n \to \infty} PSpAz_{2n+1}
\]

Similarly, we can prove that

\begin{align}
(3.9) & \quad \lim_{n \to \infty} QTBx_{2n-1} = u = \lim_{n \to \infty} QTz_{2n} \\
(3.10) & \quad \lim_{n \to \infty} APz_{2n-1} = v = \lim_{n \to \infty} APSz_{2n} \\
(3.11) & \quad \lim_{n \to \infty} BQz_{2n} = v = \lim_{n \to \infty} BQTz_{2n} \\
(3.12) & \quad \lim_{n \to \infty} SAPz_{2n} = w = \lim_{n \to \infty} SAz_{2n} \\
(3.13) & \quad \lim_{n \to \infty} TBQz_{2n-1} = w = \lim_{n \to \infty} TBx_{2n-1}
\end{align}

Since \( A \) and \( B \) are continuous, we have

\[
\lim_{n \to \infty} Ax_{2n} = Au = v, \quad \lim_{n \to \infty} Bx_{2n-1} = Bu = v.
\]
Using inequality (3.1), we have
\[ M_{M_1,N_1}(PSAu, QT_{xn_{1}}(u, x_{2n_{1}} - 1, t)) \geq L\ M_{M_1,N_1}(u, PSAu, t) \]
\[ *M_{M_1,N_1}(u, QT_{xn_{1}}(u, PSAu, t)) \] 
\[ \geq L\ M_{M_1,N_1}(PSAu, QT_{xn_{1}}(u, PSAu, t)). \]

Letting \( n \to \infty \) and using (3.10), we have
\[ M_{M_1,N_1}(PSAu, u, kt) \geq L\ M_{M_1,N_1}(u, PSAu, t). \]

Therefore, by Lemma 2.5 and using (3.15), we have \( PSAu = u = PSv \).

Using inequality (3.1), we have
\[ M_{M_1,N_1}(PSAx_{2n_{1}}, QT_{Bu}, kt) \geq L\ M_{M_1,N_1}(x_{2n_{1}}, PSAx_{2n_{1}}, t) \]
\[ *M_{M_1,N_1}(u, QT_{Bu}, t) \]
\[ \geq L\ M_{M_1,N_1}(PSAx_{2n_{1}}, QT_{Bu}, t). \]

Letting \( n \to \infty \) and using (3.9), we have
\[ M_{M_1,N_1}(u, QT_{Bu}, kt) \geq L\ M_{M_1,N_1}(u, QT_{Bu}, t). \]

Therefore, by Lemma 2.5 and using (3.15), we have \( QT_{Bu} = u = QT_{v} \).

Since \( S \) and \( T \) are continuous, we have
\[ (3.16) \lim_{n \to \infty} S_{y_{2n_{1}} - 1} = Sv = w, \quad \lim_{n \to \infty} T_{y_{2n}} = Tv = w. \]

Using inequality (3.2), we have
\[ M_{M_2,N_2}(APSv, BQT_{y_{2n_{1}}}, kt) \geq L\ M_{M_2,N_2}(v, y_{2n_{1}}, t) \]
\[ *M_{M_2,N_2}(v, BQT_{y_{2n_{1}}}, t) \]
\[ \geq L\ M_{M_2,N_2}(APSv, BQT_{y_{2n_{1}}}, t). \]

Letting \( n \to \infty \) and using (3.12), we have
\[ M_{M_2,N_2}(APSv, v, kt) \geq L\ M_{M_2,N_2}(v, APSv, t). \]

Therefore, by Lemma 2.5 and using (3.15), we have \( APSv = v = APw \).

Using inequality (3.2), we have
\[ M_{M_2,N_2}(APS_{y_{2n_{1}}}, BQT_{v}, kt) \geq L\ M_{M_2,N_2}(y_{2n_{1}}, APS_{y_{2n_{1}}}, t) \]
\[ *M_{M_2,N_2}(v, BQT_{v}, t) \]
\[ \geq L\ M_{M_2,N_2}(APS_{y_{2n_{1}}}, BQT_{v}, t). \]

Letting \( n \to \infty \) and using (3.11), we have
\[ M_{M_2,N_2}(v, BQT_{v}, kt) \geq L\ M_{M_2,N_2}(v, BQT_{v}, t). \]

Therefore, by Lemma 2.5 and using (3.16), we have \( BQT_{v} = v = BQw \).

Since \( P \) and \( S \) are continuous, we have
\[ (3.17) \lim_{n \to \infty} P_{z_{2n}} = Pw = u, \quad \lim_{n \to \infty} Q_{z_{2n_{1}} - 1} = Qw = u. \]
Using inequality (3.3), we have
\[
\mathcal{M}_{M_3, N_3} (SAPw, TBQ_{z_{2n-1}}, kt) \geq L^* \mathcal{M}_{M_3, N_3} (w, z_{2n-1}, t) \ast \mathcal{M}_{M_3, N_3} (w, SAPw, t)
\ast \mathcal{M}_{M_3, N_3} (z_{2n-1}, TBQ_{2n-1}, t) \ast \mathcal{M}_{M_3, N_3} (SAPw, TBQ_{2n-1}, t).
\]

Letting \( n \to \infty \) and using (3.14), we have
\[
\mathcal{M}_{M_3, N_3} (SAPw, w, kt) \geq L^* \mathcal{M}_{M_3, N_3} (w, SAPw, t).
\]
Therefore, by Lemma 2.5 and using (3.17), we have
\[
SAPw = w = SAu.
\]

Using inequality (3.3), we have
\[
\mathcal{M}_{M_3, N_3} (SAPz_{2n}, TBQw, kt) \geq L^* \mathcal{M}_{M_3, N_3} (z_{2n}, w, t) \ast \mathcal{M}_{M_3, N_3} (z_{2n}, SAPz_{2n}, t)
\ast \mathcal{M}_{M_3, N_3} (w, TBQw, t) \ast \mathcal{M}_{M_3, N_3} (SAPz_{2n}, TBQw, t).
\]

Letting \( n \to \infty \) and using (3.13), we have
\[
\mathcal{M}_{M_3, N_3} (w, TBQw, kt) \geq L^* \mathcal{M}_{M_3, N_3} (w, TBQw, t).
\]
Therefore, by Lemma 2.5 and using (3.17), we have
\[
TBQw = w = TBu.
\]

Thus, we have
\[
\begin{align*}
PSAu &= QT Bu = PSv = QT v = Pw = Qw = u, \\
APSv &= BQT v = APw = BQw = Au = Bu = v, \\
SAPw &= TBQw = SAu = TBu = Su = T v = w.
\end{align*}
\]

To prove the uniqueness of the fixed point, suppose that \( PSA \) and \( QT B \) have a common fixed point \( u' \) also.

Using inequality (3.1), we have
\[
\mathcal{M}_{M_1, N_1} (PSAu, QTBu', kt) \geq L^* \mathcal{M}_{M_1, N_1} (u, u', t) \ast \mathcal{M}_{M_1, N_1} (u, PSAu, t)
\ast \mathcal{M}_{M_1, N_1} (u', QTBu', t) \ast \mathcal{M}_{M_1, N_1} (PSAu, QTBu', t).
\]

Therefore, we have
\[
\mathcal{M}_{M_1, N_1} (u, u', kt) \geq L^* \mathcal{M}_{M_1, N_1} (u, u', t).
\]
By Lemma 2.5, we have \( u = u' \). Similarly we can prove that \( v \) and \( w \) are unique common fixed point of \( APS \) and \( BQT \) and of \( SAP \) and \( TBQ \). This completes the proof.

If we put \( M_1 = M_2 = M_3 = M \) and \( N_1 = N_2 = N_3 = N \) in Theorem 3.1, we get the following:

**Corollary 1.** Let \((X, \mathcal{M}_M, N, \tau), (Y, \mathcal{M}_M, N, \tau)\) and \((Z, \mathcal{M}_M, N, \tau)\) be three complete intuitionistic fuzzy metric spaces. Let \( A, B \) be continuous mappings from \( X \) into \( Y \), let \( S, T \) be continuous mappings from \( Y \) into \( Z \) and let \( P, Q \) be continuous mappings from \( Z \) into \( X \) satisfying the inequalities:
(3.1) \( \mathcal{M}_{M,N}(PSAx, QT B x', k t) \geq_{L'} \mathcal{M}_{M,N}(x, x', t) * \mathcal{M}_{M,N}(x, PSAx, t) * \mathcal{M}_{M,N}(x', QT B x', t) \)

(3.2) \( \mathcal{M}_{M,N}(APSy, B QT y', k t) \geq_{L'} \mathcal{M}_{M,N}(y, y', t) * \mathcal{M}_{M,N}(y, APSy, t) * \mathcal{M}_{M,N}(y', B QT y', t) \)

(3.3) \( \mathcal{M}_{M,N}(SAP z, TBQ z', k t) \geq_{L'} \mathcal{M}_{M,N}(z, z', t) * \mathcal{M}_{M,N}(z, SAP z, t) * \mathcal{M}_{M,N}(z', TBQ z', t) \)

for all \( x, x' \) in \( X \), \( y, y' \) in \( Y \) and \( z, z' \) in \( Z \), \( t > 0 \) and \( k \in (0, 1) \), then \( PSA \) and \( QT B \) have a unique common fixed point \( u \) in \( X \), \( APS \) and \( B QT \) have a unique common fixed point \( v \) in \( Y \) and \( SAP \) and \( TBQ \) have a unique common fixed point \( w \) in \( Z \). Further, \( Au = Bu = v, Sv = Tv = w \) and \( Pw = Qw = u \).

If we put \( A = B, S = T \) and \( P = Q \) in Theorem 3.1, we get the following:

**Corollary 2.** Let \( (X, \mathcal{M}_{M,N_1}, \tau), (Y, \mathcal{M}_{M_2,N_2}, \tau) \) and \( (Z, \mathcal{M}_{M_3,N_3}, \tau) \) be three complete intuitionistic fuzzy metric spaces. Let \( A \) be continuous mapping from \( X \) into \( Y \), let \( S \) be continuous mapping from \( Y \) into \( Z \) and let \( P \) be continuous mapping from \( Z \) into \( X \) satisfying the inequalities:

(3.4) \( \mathcal{M}_{M_1,N_1}(PSAx, PSAx', k t) \geq_{L'} \mathcal{M}_{M_1,N_1}(x, x', t) * \mathcal{M}_{M_1,N_1}(x, PSAx, t) * \mathcal{M}_{M_1,N_1}(PSAx, PSAx', t) \)

(3.5) \( \mathcal{M}_{M_2,N_2}(APSy, APSy', k t) \geq_{L'} \mathcal{M}_{M_2,N_2}(y, y', t) * \mathcal{M}_{M_2,N_2}(y, APSy, t) * \mathcal{M}_{M_2,N_2}(APSy, APSy', t) \)

(3.6) \( \mathcal{M}_{M_3,N_3}(SAP z, SAP z', k t) \geq_{L'} \mathcal{M}_{M_3,N_3}(z, z', t) * \mathcal{M}_{M_3,N_3}(z, SAP z, t) * \mathcal{M}_{M_3,N_3}(SAP z, SAP z', t) \)

for all \( x, x' \) in \( X \), \( y, y' \) in \( Y \) and \( z, z' \) in \( Z \), \( t > 0 \) and \( k \in (0, 1) \), then \( PSA \) has a unique common fixed point \( u \) in \( X \), \( APS \) has a unique common fixed point \( v \) in \( Y \) and \( SAP \) has a unique common fixed point \( w \) in \( Z \). Further, \( Au = v, Sv = w \) and \( Pw = Qw = u \).

**Remark 3.1.** From Theorem 3.1, with \( P = Q = Ix \) (the identity mapping on \( X \)), we obtain modified intuitionistic version of the results of Sharma, Deshpande and Thakur [29] and Deshpande and Pathak [8].

**References**


Accepted: 03.07.2012