A RADICAL PROPERTY OF HYPERRINGS

A. Asokkumar
M. Velrajan

Department of Mathematics
Aditanar College of Arts and Science
Tiruchendur – 628216, Tamilnadu
India
e-mail: ashok_a58@yahoo.co.in

Abstract. In this paper we prove that Von Neumann regularity is a radical property on hyperrings.

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1. Introduction

The theory of hyperstructures was introduced in 1934 by Marty [11] at the 8th Congress of Scandinavian Mathematicians. This theory has been subsequently developed by Corsini [5], [6], Mittas [12], [13], Stratigopoulos [17], Vougiouklis [20] and by various authors. Basic definitions and propositions about the hyperstructures are found in [5], [6] and [20]. Krasner [10] has studied the notion of hyperfields, hyperrings and then many researchers like Davvaz [7], Massouros [14] and others followed him.

There are different notions of hyperrings $(R, +, \cdot)$. If in a hyperring the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a binary operation, then the hyperring is called a Krasner (additive) hyperring [10]. The monograph [8] of Davvaz and Leoreanu-Fotea contains many results about various hyperrings. Asokkumar and Velrajan [1], [4] have studied Von Neumann regularity in Krasner hyperrings. Rota [16] introduced multiplicative hyperrings, where the additions are binary operations and multiplications are hyperoperations. De Salvo [9] introduced hyperrings in which the additions and the multiplications are hyperoperations. These hyperrings are studied by Rahnamani Barghi [15] and also by Asokkumar and Velrajan [2], [3], [19].
In this paper we prove that regularity (Von Neumann) is a radical property on hyperrings, where the additions and the multiplications are hyperoperations. We also prove that if a hyperring \( R \) is regular, then for a hyperideal \( I \) of \( R \) both \( I \) and \( R/I \) are regular. Conversely, if \( R \) is a hyperring and if there exists a hyperideal \( I \) of \( R \) such that both \( I \) and \( R/I \) are regular, then \( R \) is regular.

2. Basic definitions and notations

This section explains some basic definitions that have been used in the sequel. A hyperoperation \( \circ \) on a nonempty set \( H \) is a mapping of \( H \times H \) into the family of nonempty subsets of \( H \) (i.e., \( x \circ y \subseteq H \) for every \( x, y \in H \)). A hypergroupoid is a nonempty set \( H \) equipped with a hyperoperation \( \circ \). For any two subsets \( A, B \) of a hypergroupoid \( H \), the set \( A \circ B \) means \( \bigcup_{a \in A} a \circ b \) for every \( a \in A \) and \( b \in B \). A hypergroupoid \( (H, \circ) \) is called a semihypergroup if \( x \circ (y \circ z) = (x \circ y) \circ z \) for every \( x, y, z \in H \) (the associative axiom). A hypergroupoid \( (H, \circ) \) is called a quasihypergroup if \( x \circ H = H \circ x = H \) for every \( x \in H \) (the reproductive axiom). A reproductive semihypergroup is called a hypergroup (Marty). A comprehensive review of the theory of hypergroups appears in [5].

A nonempty set \( H \) with a hyperoperation \( + \) is said to be a canonical hypergroup if the following conditions hold:

(i) for every \( x, y, z \in H \), \( x + (y + z) = (x + y) + z \),

(ii) for every \( x, y \in H \), \( x + y = y + x \),

(iii) there exists \( 0 \in H \) such that \( 0 + x = x \) for all \( x \in H \),

(iv) for every \( x \in H \) there exists an unique element denoted by \( -x \in H \) such that \( 0 \in x + (-x) \),

(v) for every \( x, y, z \in H \), \( z \in x + y \) implies \( y \in -x + z \) and \( x \in z - y \).

A nonempty subset \( N \) of a canonical hypergroup of \( H \) is called a subcanonical hypergroup of \( H \) if \( N \) itself is a canonical hypergroup under the same hyperoperation as that of \( H \). Equivalently, for every \( x, y \in N \), \( x - y \subseteq N \). Moreover, for any subset \( A \) of \( H \), \( -A \) denotes the set \( \{-a : a \in A\} \).

The following elementary facts in a canonical hypergroup easily follow from the axioms.

(i) \( -(a) = a \) for every \( a \in R \);

(ii) \( 0 \) is the unique element such that for every \( a \in R \), there is an element \( -a \in R \) with the property \( 0 \in a + (-a) \);

(iii) \( -0 = 0 \);

(iv) \( -(a + b) = -b - a \) for all \( a, b \in R \).
Theorem 2.1 [19] Let $H$ be a canonical hypergroup and $N$ be a subcanonical hypergroup of $H$. For any two elements $a, b \in H$, if we define a relation $a \sim b$ if and only if $a \in b + N$, then $\sim$ is an equivalence relation on $H$.

Let $\overline{x}$ be the equivalence class determined by the element $x \in H$ and $H/N$ be the collection of all equivalence classes.

Theorem 2.2 [19] Let $H$ be a canonical hypergroup and $N$ be a subcanonical hypergroup of $H$. Then $\overline{x} = x + N$ for any $x \in H$.

Theorem 2.3 [19] Let $H$ be a canonical hypergroup, $N$ be a subcanonical hypergroup of $H$. If we define $\overline{x} \oplus \overline{y} = \{z : z \in x + y\}$ for all $\overline{x}, \overline{y} \in H/N$, then $H/N$ is a canonical hypergroup.

A nonempty set $R$ with two hyperoperations $+$ and $\cdot$ is said to be a hyperring if $(R, +)$ is a canonical hypergroup, $(R, \cdot)$ is a semihypergroup with $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$ (0 as a bilaterally absorbing element) and the hyperoperation $\cdot$ is distributive over $+$, i.e., for every $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. The hyperoperation $+$ is usually called hyperaddition and the hyperoperation $\cdot$ is called hypermultiplication.

Definition 2.4 Let $R$ be a hyperring and $I$ be a nonempty subset of $R$. Then $I$ is called a left (resp. right) hyperideal of $R$ if $(I, +)$ is a canonical subhypergroup of $R$ and for every $a \in I$ and $r \in R$, $ra \subseteq I$ (resp. $ar \subseteq I$). A hyperideal of $R$ is one which is a left as well as a right hyperideal of $R$.

If $I, J$ are left (resp. right) hyperideals of a hyperring $R$, then $I + J$ is a left (resp. right) hyperideal of $R$. If $I, J$ are hyperideals of a hyperring $R$, then $I + J$ is a hyperideal of $R$. Let $R$ be a hyperring, $I$ a hyperideal of $R$ and $R/I$ be the set of all distinct equivalence classes of $I$ in $R$ obtained by considering $I$ as a subcanonical hypergroup of $R$. Then $R/I$ is a canonical hypergroup under the hyperaddition defined in the Theorem 2.3.

Theorem 2.5 [19] If we define $\overline{x} \otimes \overline{y} = \{z : z \in xy\}$ for all $\overline{x}, \overline{y} \in R/I$, then $R/I$ is a hyperring.

Definition 2.6 Let $R_1$ and $R_2$ be two hyperrings. A mapping $\phi$ from $R_1$ into $R_2$ is called a homomorphism if the following conditions hold for all $a, b \in R_1$:

(i) $\phi(a + b) \subseteq \phi(a) + \phi(b)$;

(ii) $\phi(ab) \subseteq \phi(a)\phi(b)$, and

(iii) $\phi(0) = 0$.

The mapping $\phi$ is called a good homomorphism or a strong homomorphism if
(i) $\phi(a + b) = \phi(a) + \phi(b)$;
(ii) $\phi(ab) = \phi(a)\phi(b)$, and
(iii) $\phi(0) = 0$ for all $a, b \in R_1$.

**Definition 2.7** A homomorphism (resp. strong homomorphism) $\phi$ from a hyperring $R_1$ into a hyperring $R_2$ is said to be an isomorphism (resp. strong isomorphism) if $\phi$ is one to one and onto. In this case we say $R_1$ is isomorphic (resp. strongly isomorphic) to $R_2$ and is denoted by $R_1 \cong R_2$.

**Definition 2.8** Let $\phi$ be a homomorphism from a hyperring $R_1$ into another hyperring $R_2$. Then the set $\{x \in R_1 : \phi(x) = 0\}$ is called the kernel of $\phi$ and is denoted by $Ker\phi$ and the set $\{\phi(x) : x \in R_1\}$ is called Image of $\phi$ and is denoted by $Im\phi$.

It is clear that $Ker\phi$ is a hyperideal of $R_1$ and $Im\phi$ is a subcanonical hypergroup of $R_2$ and $R_1/Ker\phi$ is a hyperring.

**Theorem 2.9**[19] (First Isomorphism Theorem) Let $\phi$ be a strong homomorphism from a hyperring $R_1$ onto a hyperring $R_2$ with kernel $K$. Then $R_1/K$ is strongly isomorphic to $R_2$.

**Theorem 2.10**[19] (Second Isomorphism Theorem) If $I$ and $J$ are hyperideals of a hyperring $R$ then $J/(I \cap J) \cong (I + J)/I$.

## 3. Regular hyperring

First, let us recall the definition of a regular ring. An element $a$ in a ring $R$ is said to be regular if $a \in aRa$. A ring $R$ is called regular if every element of $R$ is regular. We define a regular hyperring as follows.

**Definition 3.1**[2] An element $a \in R$ is said to be regular if $a \in aRa$. That is, there exists an element $b \in R$ such that $a \in aba$. A hyperring $R$ is said to be regular if every element of $R$ is regular.

**Proposition 3.2**[2] Strong homomorphic image of a regular hyperring is a regular hyperring.

**Proposition 3.3** If $I$ is a hyperideal of a regular hyperring $R$, then $I$ is regular.

**Proof.** Consider a hyperideal $I$ of $R$. Let $a \in I$. Since $R$ is regular, there exists $x \in R$ such that $a \in axa$. Then $a \in a(xa) \subseteq (axa)(xa) = a(xax)a$ where $xax \subseteq I$. Thus $I$ is regular. \[\blacksquare\]

**Theorem 3.4** If $I, J$ are regular hyperideals of a hyperring $R$, then $I \cap J$ is also a regular hyperideal of $R$. 

Proof. It is clear that $I \cap J$ is a hyperideal of $R$. Let $a \in I \cap J$. Then there exist $x \in I$ and $y \in J$ such that $a \in axa$ and $a \in aya$. Now,

$$a \in axa \subseteq (axa)x(aya) = a(xaxay)a.$$ 

Since $I, J$ are hyperideals of $R$, $xaxay \subseteq I \cap J$. Thus $I \cap J$ is regular. 

4. Regularity is a radical property on hyperrings

In this section, we show that regularity is a radical property on hyperrings. We also prove that if a hyperring $R$ is regular, then for a hyperideal $I$ of $R$ both $I$ and $R/I$ are regular. Conversely, if $R$ is a hyperring and if there exists a hyperideal $I$ of $R$ such that both $I$ and $R/I$ are regular, then $R$ is regular.

Definition 4.1 Let $P$ be a property of hyperrings. A hyperring with the property $P$ is called a $P$-hyperring. A hyperideal $I$ of a hyperring $R$ is called a $P$-hyperideal if the hyperideal $I$, as a hyperring, is a $P$-hyperring.

Definition 4.2 A $P$-hyperideal $P(R)$ of a hyperring $R$ which contains every $P$-hyperideal of $R$ is called the $P$-hyperradical of $R$.

Definition 4.3 A property $P$ of a hyperring is called a radical property (in the sense of Amitsur and Kurosh [18]) if $P$ satisfies the following conditions:

(i) Strong homomorphic image of a $P$-hyperring is a $P$-hyperring.

(ii) Every hyperring $R$ has a $P$-hyperradical $P(R)$.

(iii) The hyperring $R/P(R)$ has no non-zero $P$-hyperideals.

Lemma 4.4 Let $R$ be a hyperring and $a \in R$. If there exists $x \in R$ and $c \in axa - a$ such that $c$ is regular, then $a$ is regular.

Proof. Since $c \in axa - a$ is regular, there exists $d \in R$ such that $c \in cdc$. This means that

$$c \in (axa - a)d(axa - a) = (axad - ad)(axa - a) \subseteq axadaxa - axada - adaxa + ada = a(xadaxa - xada - daxa + da) = a(xadax - xad - dax + d)a$$

Hence $c \in aba$ for some $b \in xadax - xad - dax + d$. Since $c \in (axa - a)$, we get $a \in (axa - c) \subseteq axa - aba = a(x - b)a$. So $a \in aya$ for some $y \in x - b$. That is, $a$ is regular.
Let $a$ be a hyperideal of $R$. Then $I$ and $R/I$ are regular. Conversely, if $R$ is a hyperring and if there exists a hyperideal $I$ of $R$ such that both $I$ and $R/I$ are regular, then $R$ is regular.

Proof. Let $R$ be a regular hyperring and $I$ be a hyperideal of $R$. Then by the Proposition 3.3, $I$ is a regular hyperideal. Let $x + I \in R/I$. Since $R$ is regular, there exists $y \in R$ such that $x \in xyx$. Consider $\overline{y} = y + I$. Now, $\overline{x} \overline{y} \overline{x} = \{z : z \in xyx\}$. Since $x \in xyx$ we have $\overline{x} \in \{z : z \in xyx\}$. That is, $\overline{x} \in \overline{x} \overline{y} \overline{x}$. So $x + I$ is regular in $R/I$. Hence $R/I$ is regular.

Conversely, suppose $R$ is a hyperring and there exists a hyperideal $I$ of $R$ such that both $I$ and $R/I$ are regular. Let $a \in R$. Then $\overline{a} \in R/I$. Since $R/I$ is regular, there exists an element $\overline{b} \in R/I$ such that $\overline{a} = \overline{b} \overline{b} = \{z : z \in aba\}$. This means that $\overline{a} = \overline{z}$ for some $z \in aba$. That is, $a + I = z + I$ for some $z \in aba$. Since $z \in a + I$, we get $z \in a + i$ for some $i \in I$. Therefore, $i = -a + z = z - a \subseteq aba - a$. Thus $i \in aba - a$. Since $I$ is regular, $i$ is a regular element of $I$ and therefore $i$ is a regular element of $R$. Thus the set $aba - a$ contains a regular element $i$ of $R$. Then by the Lemma 4.4, the element $a$ is regular in $R$. Hence $R$ is regular.

Theorem 4.6 Let $R$ be a hyperring. If $I$ and $J$ are regular hyperideals of $R$, then $I + J$ is regular.

Proof. Since $J/(I \cap J)$ is a homomorphic image of a regular hyperideal $J$, it is regular. By the Theorem 2.10, $J/(I \cap J)$ is isomorphic to $(I + J)/I$. Therefore, $(I + J)/I$ is regular. Since both $I$ and $(I + J)/I$ are regular, by the Theorem 4.5, the hyperideal $I + J$ is regular.

Theorem 4.7 Any hyperring has a regular hyperradical.

Proof. Let $R$ be a hyperring. Consider the hyperideal $(0)$ of $R$. Clearly, $(0)$ is a regular hyperideal of $R$. If $(0)$ is the only regular hyperideal of $R$, then this is the regular hyperradical.

Otherwise, let $\{I_i\}$ be the collection of all regular hyperideals in a hyperring $R$. Their sum is given by $M = \bigcup \{\sum_{finit}\{a_i : a_i \in I_i\}\}$. Clearly, $M$ is a hyperideal of $R$. If $x \in M$, then $x \in a_1 + a_2 + a_3 + \cdots + a_i$, where $a_i \in I_i$. By Theorem 4.6, $I_i + I_j + I_k + \cdots + I_l$ is a regular hyperideal. Therefore, $x$ is regular. Hence, $M$ is regular. Since $M$ contains all regular hyperideals of $R$, we have $M$ is the regular hyperradical of $R$.

Theorem 4.8 Let $R$ be a hyperring and $M$ be the regular hyperradical of $R$. Then the hyperring $R/M$ has no non-zero regular hyperideals.

Proof. Let $J$ be a regular hyperideal of $R/M$. Then $J = I/M$ for some hyperideal $I$ of $R$ containing $M$. Since $M$ and $I/M$ are regular, by the Theorem 4.5, $I$ is regular. By the definition of $M$, we have $I \subseteq M$. Hence $I = M$. Therefore, $J$ is a zero hyperideal of $R/M$.

Theorem 4.9 The regularity is a radical property on hyperrings.
Proof. The proof follows from the Proposition 3.2, and the Theorems 4.7, 4.8.

References


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