

NUMBERS IN THE n DIMENSIONAL SPACE

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Abstract. This paper introduces the numbers in the n dimensional space. Namely, if in the first dimension we have the real numbers and in the second the complex numbers, in the next dimensions we have the complete numbers introduced here.

Keywords: complex numbers, complete numbers, real numbers, n dimensional space, extent of the numbers.

1. Introduction

Definition 1.1. We can define real number $r(a)$ as the position of the straight line R that can be reached starting from that unitary through operations of translation of positions.

We can observe, with regard to this, Figure 1.

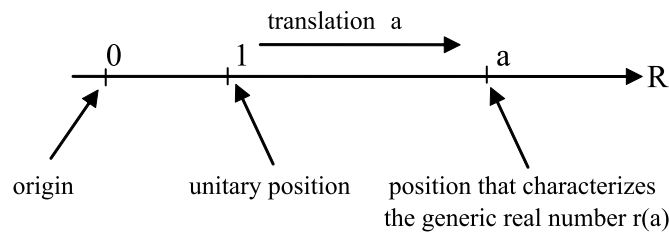


Figure 1: Cartesian representation of the real numbers

The straight line R that appears in the figure is defined line of the real numbers.

Theorem 1.2. *Real numbers can be expressed in the following way:*

$$r(a) = a$$

Proof. The proof is immediate and is a consequence of the bijection between translation operation of value (a) and the positions (a) on the line of the real numbers. ■

For more information on real numbers see [1], Chapter 1.

Definition 1.3. We can define complex number $c(t, \theta)$ as the position of the plane RI that can be reached starting from that unitary through operations of translation of positions and plane rotation of straight lines.

We can observe, with regard to this, Figure 2.

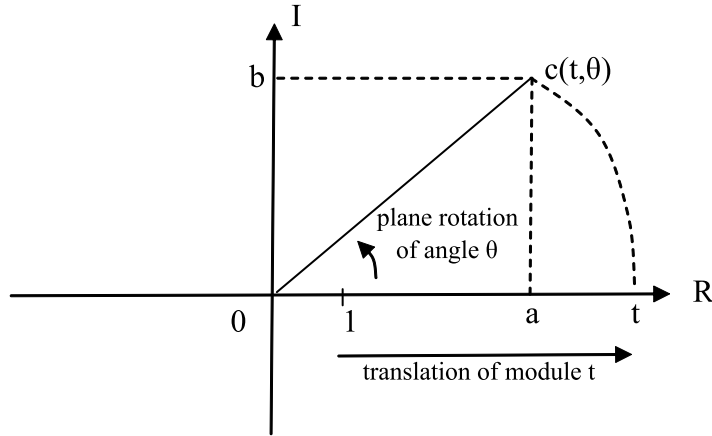


Figure 2: Cartesian representation of the complex numbers

We note that the position $c(t, \theta)$ is reached from that unitary of the line R before translating it of modulus t , and after making line R turn of the angle θ in the plane RI.

The straight line I that appears in the figure is defined line of the imaginary numbers and together with the line R of the real numbers identify the plan RI of the complex numbers.

Theorem 1.4. *Complex numbers can be expressed in the following way:*

$$c(t, \theta) = t \cdot [\cos(\theta) + i \cdot \sin(\theta)]$$

Proof. Making reference to trigonometric relations shown in Figure 3

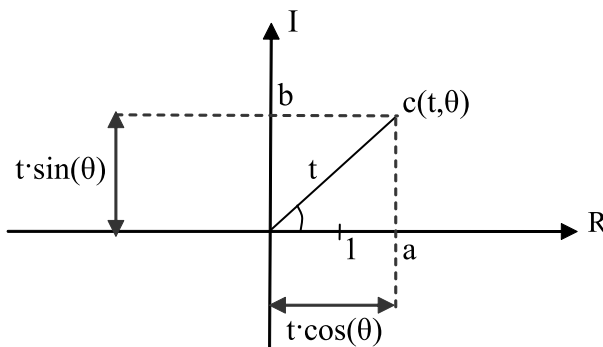


Figure 3: Trigonometric representation of complex numbers

we obtain just the result expected. ■

Definition 1.5. The symbol t that indicates the distance of a complex number $c(t, \theta)$ from the origin is defined modulus.

Theorem 1.6. *The modulus t has the following property:*

$$t = \sqrt{a^2 + b^2}$$

Proof. By using Pythagoras' theorem on the triangle identified in Figure 3 on the preceding page we can obtain the relation:

$$t^2 = a^2 + b^2$$

from which results the previous one. ■

Definition 1.7. The symbol θ that expresses the rotation that has to undergo the line R to align itself with the straight line that joins $c(t, \theta)$ to the origin is defined plane phase.

Theorem 1.8. *The plane phase θ has the following property:*

$$\theta = \arctan\left(\frac{b}{a}\right)$$

Proof. Making reference again to the same triangle of Figure 3 on the facing page we obtain the relation:

$$\frac{b}{a} = \tan(\theta)$$

from which results the previous one. ■

Theorem 1.9. *Complex numbers can be expressed in the following way:*

$$c(t, \theta) = t \cdot [\cos(\theta + k \cdot 360) + i \cdot \sin(\theta + k \cdot 360)] \quad \text{for } k = 0, \pm 1, \pm 2, \pm 3 \dots$$

Proof. The proof is immediate and is a consequence of the periodicity of the functions $\sin()$ and $\cos()$. ■

Theorem 1.10. *Complex numbers can be expressed in the following way:*

$$c(t, \theta) = c(a, b) = a + i \cdot b$$

Proof. The proof is immediate and is a consequence of the bijection between translation and rotation operations of values (t, θ) and the positions (a, b) of the plane RI . ■

For more information on complex numbers see [1], Chapter 3.

The transition from the first dimension of the real numbers to the second dimension of the complex numbers has required an operation of rotation. By further extending this procedure will be possible to introduce the n dimensional numbers and define their operations.

2. Numbers in three dimensional space

2.1. Introduction to the complete numbers

Definition 2.1. We can define complete number $o(t, \theta, \gamma)$ as the position of the space RIU that can be reached starting from that unitary through operations of translation of positions, of plane rotation of straight lines and spatial rotation of planes.

We can observe, with regard to this, Figure 4.

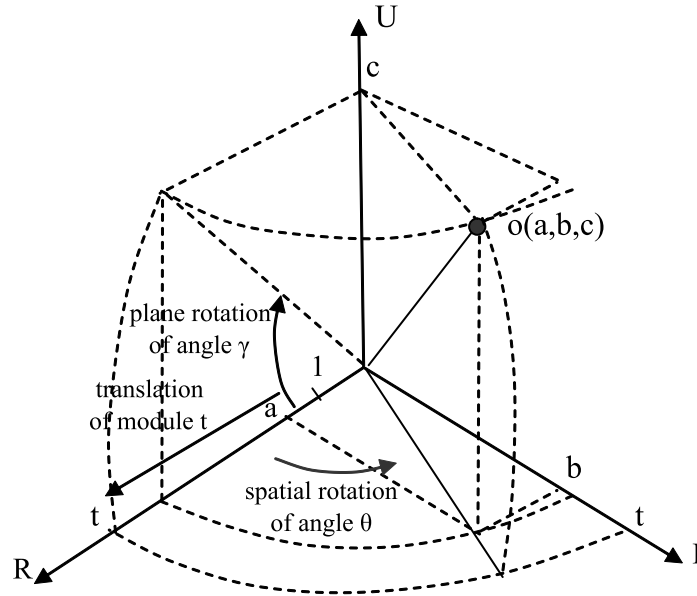


Figure 4: Cartesian representation of the complete numbers

We note that the position $o(t, \theta, \gamma)$ is reached from that unitary of the line R before translating it of modulus t , after making line R turn of the angle γ in the plane RI, and finally making the whole plane RU turn of the angle θ .

The straight line U that appears in the figure is defined line of the outgoing numbers and together with the line R of the real numbers and the line I of the imaginary numbers identify the space RIU of the complete numbers.

Theorem 2.2. Complete numbers can be expressed in the following way:

$$o(t, \theta, \gamma) = t \cdot \{[\cos(\gamma) \cdot \cos(\theta)] + i \cdot [\cos(\gamma) \cdot \sin(\theta)] + u \cdot [\sin(\gamma)]\}$$

Proof. Making reference to trigonometric relations shown in Figure 5 we obtain just the result expected. ■

Definition 2.3. The symbol t that indicates the distance of a complete number $o(t, \theta, \gamma)$ from the origin is defined modulus.

Theorem 2.4. The modulus t has the following property:

$$t = \sqrt{a^2 + b^2 + c^2}$$

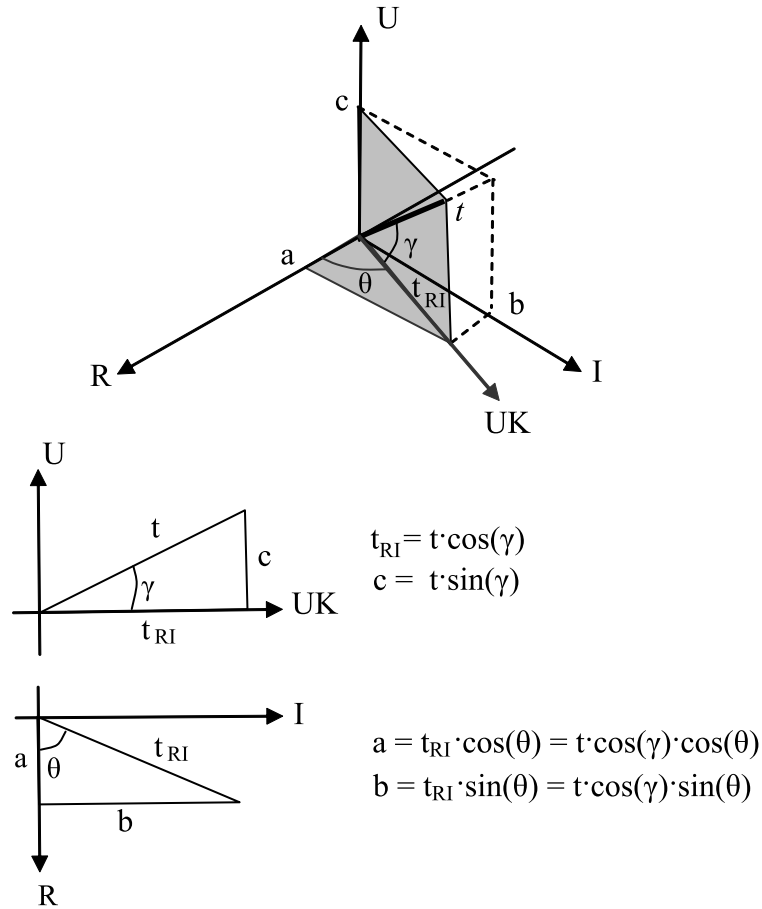


Figure 5: Trigonometric representation of the complete numbers

Proof. By using Pythagoras' theorem on the two triangles identified in Figure 5 we can obtain the following relations:

$$t^2 = t_{RI}^2 + c^2$$

$$t_{RI}^2 = a^2 + b^2$$

from which result the previous one. ■

Definition 2.5. The symbol γ that expresses the rotation that has to undergo the line R to align itself with the projection on the plane RU of the straight line that joins $o(t, \theta, \gamma)$ to the origin is defined plane phase.

Theorem 2.6. The plane phase γ has the following property:

$$\gamma = \arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right)$$

Proof. Making reference to the first triangle in Figure 5 we can write:

$$\gamma = \arctan \left(\frac{c}{t_{RI}} \right)$$

while making reference to the second one, we can write:

$$t_{RI}^2 = a^2 + b^2$$

from which results just the result expected. ■

Definition 2.7. The symbol θ that expresses the rotation that has to undergo the line R to align itself with the projection on the plane RI of the straight line that joins $o(t, \theta, \gamma)$ to the origin is defined spatial phase.

Theorem 2.8. *The spatial phase θ has the following property:*

$$\theta = \arctan\left(\frac{b}{a}\right)$$

Proof. Making reference to the second triangle in Figure 5 on the preceding page we obtain the following relation:

$$\frac{b}{a} = \tan(\theta)$$

from which results the previous one. ■

Theorem 2.9. *Complete numbers can be expressed in the following way:*

$$o(t, \theta, \gamma) = t \cdot \{[\cos(\gamma + j \cdot 360) \cdot \cos(\theta + k \cdot 360)] + \\ + i \cdot [\cos(\gamma + j \cdot 360) \cdot \sin(\theta + k \cdot 360)] + u \cdot [\sin(\gamma + j \cdot 360)]\}$$

$$\text{for } \begin{cases} j = 0, \pm 1, \pm 2, \pm 3 \dots \\ k = 0, \pm 1, \pm 2, \pm 3 \dots \end{cases}$$

Proof. The proof is immediate and is a consequence of the periodicity of the functions $\sin()$ and $\cos()$. ■

Definition 2.10. A complete numbers not belonging to the line U can be defined in standard representation if provided with phases θ and γ which satisfy the conventions introduced hereunder.

For positions $P(a, b, c)$ of the half-space R^+IU not belonging to the planes RI, RU, IU the phases of the standard representation will be those shown in Figure 6 on the next page.

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{|\sqrt{a^2 + b^2}|}\right)$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

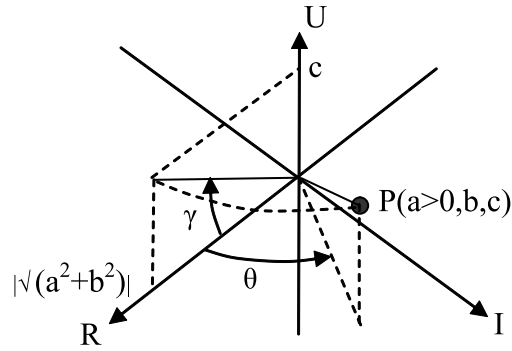


Figure 6: Phases that identify the positions of the half-space R^+IU according to the standard representation

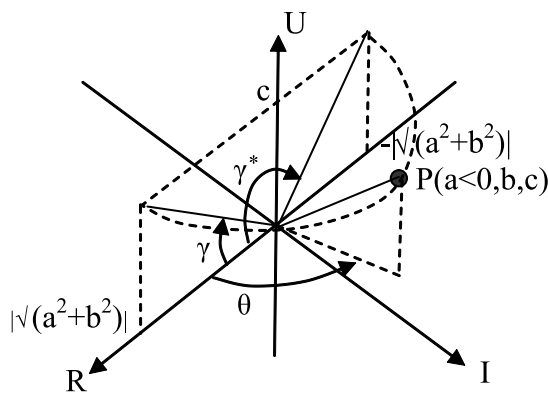


Figure 7: Phases that identify the positions of the half-space R^-IU according to the standard representation

For positions $P(a,b,c)$ of the half-space R^-IU not belonging to the planes RI , RU , IU the phases of the standard representation will be those shown in Figure 7. The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{|\sqrt{a^2 + b^2}|} \right)$$

$$\theta = \arctan \left(\frac{b}{a} \right)$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan \left(\frac{c}{-|\sqrt{a^2 + b^2}|} \right)$$

because it would correspond to the value γ^* .

For positions $P(a,b,c)$ of the plane RI not belonging to the lines R and I the phases of the standard representation will be those shown in Figure 8 on the next page.

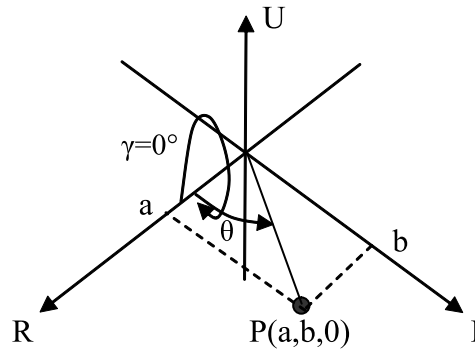


Figure 8: Phases that identify the positions of the plane RI according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right)$$

$$\theta = \arctan \left(\frac{b}{a} \right)$$

For positions $P(a,b,c)$ of the half-plane R^+U not belonging to the lines R and U the phases of the standard representation will be those shown in Figure 9.

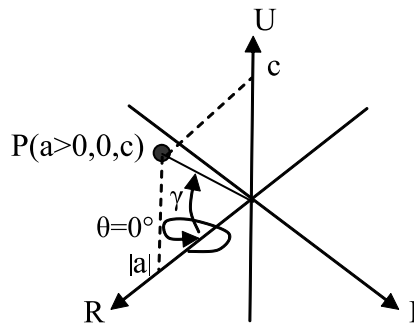


Figure 9: Phases that identify the positions of the half-plane R^+U according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{|a|} \right)$$

$$\theta = 0^\circ$$

For positions $P(a,b,c)$ of the half-plane R^-U not belonging to the lines R and U the phases of the standard representation will be those shown in Figure 10 on the next page.

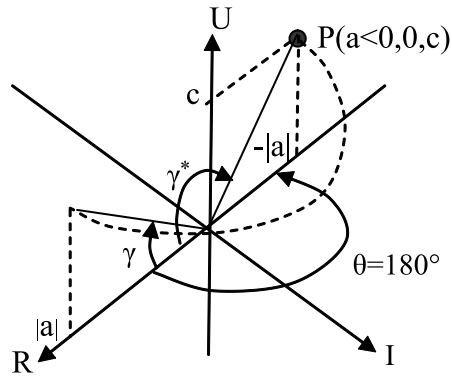


Figure 10: Phases that identify the positions of the half-plane RU according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{|a|} \right)$$

$$\theta = 180^\circ$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan \left(\frac{c}{-|a|} \right)$$

because it would correspond to the value γ^* .

For positions $P(a,b,c)$ of the half-plane I^+U not belonging to the lines I and U the phases of the standard representation will be those shown in Figure 11.

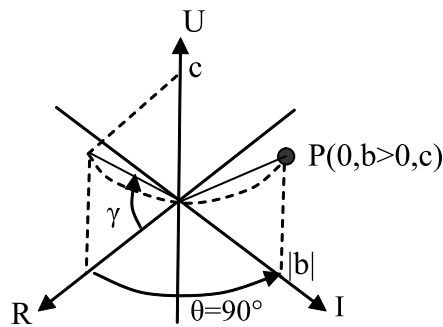


Figure 11: Phases that identify the positions of the half-plane I^+U according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{|b|} \right)$$

$$\theta = 90^\circ$$

For positions $P(a,b,c)$ of the half-plane I^-U not belonging to the lines I and U the phases of the standard representation will be those shown in Figure 12.

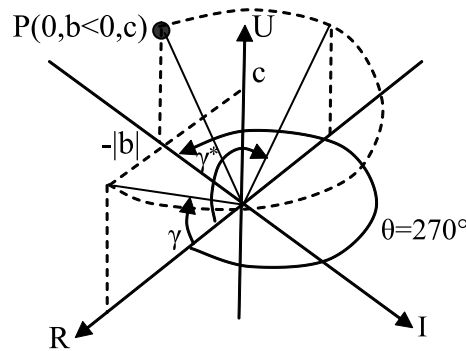


Figure 12: Phases that identify the positions of the half-plane I^-U according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{|b|} \right)$$

$$\theta = 270^\circ$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan \left(\frac{c}{-|b|} \right)$$

because it would correspond to the value γ^* .

For positions $P(a,b,c)$ of the half-line R^+ the phases of the standard representation will be those shown in Figure 13.

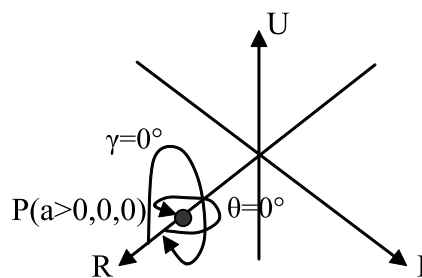


Figure 13: Phases that identify the positions of the half-line R^+ according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = 0^\circ$$

$$\theta = 0^\circ$$

For positions $P(a,b,c)$ of the half-line R^- the phases of the standard representation will be those shown in Figure 14.

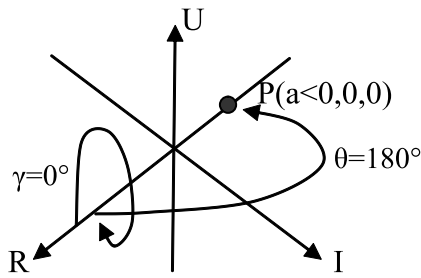


Figure 14: Phases that identify the positions of the half-line R^- according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\begin{aligned} \gamma &= 0^\circ \\ \theta &= 180^\circ \end{aligned}$$

For positions $P(a,b,c)$ of the half-line I^+ the phases of the standard representation will be those shown in Figure 15.

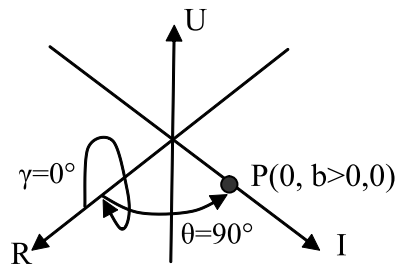


Figure 15: Phases that identify the positions of the half-line I^+ according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\begin{aligned} \gamma &= 0^\circ \\ \theta &= 90^\circ \end{aligned}$$

For positions $P(a,b,c)$ of the half-line I^- the phases of the standard representation will be those shown in Figure 16 on the next page.

The phases shown in the figure can be determined using the formulas:

$$\begin{aligned} \gamma &= 0^\circ \\ \theta &= 270^\circ \end{aligned}$$

Theorem 2.11. *The standard representation of a complete number of coordinates (a,b,c) not lying on the line U requires to give to the algebraic root $\sqrt{a^2 + b^2}$ the following positive solution:*

$$\sqrt{a^2 + b^2} = |\sqrt{a^2 + b^2}|$$

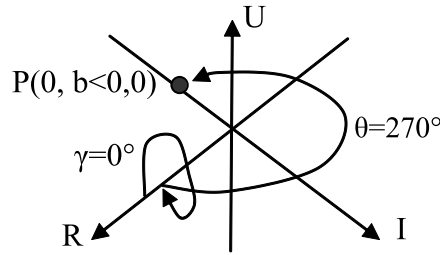


Figure 16: Phases that identify the positions of the half-line I^- according to the standard representation

Proof. In the case of the standard representations previously examined (that cover every region of the space RIU with the exception of the line U) the phase γ assumes the values provided by the formula:

$$\gamma = \arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right)$$

when we give to the algebraic root $\sqrt{a^2 + b^2}$ its positive solutions. And this immediately proves the thesis. ■

Definition 2.12. A complete numbers not belonging to the line U can be defined in complementary representation if provided with phases obtained by the values θ and γ of the standard representation through those substitutions which allow us to identify the same positions.

Theorem 2.13. *If we call θ and γ the phases that allow to a complete number not belonging to the line U and in standard representation to identify any position of the space RIU, an alternative set of phases able to individuate the same position has the following values: $(\theta + 180^\circ)$ and $(180^\circ - \gamma)$.*

Proof. Since the following relations are valid:

$$\begin{aligned} \cos(180^\circ - \gamma) \cdot \cos(\theta + 180^\circ) &= \cos(\gamma) \cdot \cos(\theta) \\ \cos(180^\circ - \gamma) \cdot \sin(\theta + 180^\circ) &= \cos(\gamma) \cdot \sin(\theta) \\ \sin(180^\circ - \gamma) &= \sin(\gamma) \end{aligned}$$

we can write:

$$o(t, \theta, \gamma) = o(t, \theta + 180^\circ, 180^\circ - \gamma)$$

proving the thesis. ■

Theorem 2.14. *Complete numbers not belonging to the line U are in complementary representation if provided with phases obtained by replacing the values θ and γ of the standard representation with the values $(\theta + 180^\circ)$ and $(180^\circ - \gamma)$.*

Proof. The definition of the complete numbers in complementary representation and the theorem 2.13 directly prove the thesis. ■

Making reference to what we saw for the standard representation, the conventions adopted for the phases of the complementary representation will be those introduced hereunder.

For positions $P(a,b,c)$ of the half-space R^+IU not belonging to the planes RI , RU , IU the phases of the complementary representation will be those shown in Figure 17.

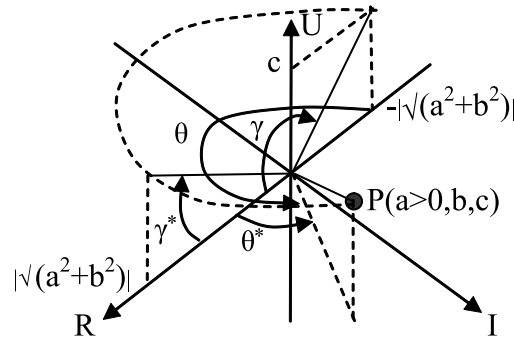


Figure 17: Phases that identify the positions of the half-space R^+IU according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{-|\sqrt{a^2 + b^2}|} \right)$$

$$\theta = \arctan \left(\frac{-b}{-a} \right)$$

We note that the plane phase γ and the spatial phase θ are not calculated by the formulas:

$$\gamma = \arctan \left(\frac{c}{|\sqrt{a^2 + b^2}|} \right)$$

$$\theta = \arctan \left(\frac{b}{a} \right)$$

because they would correspond to the values γ^* and θ^* .

For positions $P(a,b,c)$ of the half-space R^-IU not belonging to the planes RI , RU , IU the phases of the complementary representation will be those shown in Figure 18 on the next page.

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{-|\sqrt{a^2 + b^2}|} \right)$$

$$\theta = \arctan \left(\frac{-b}{-a} \right)$$

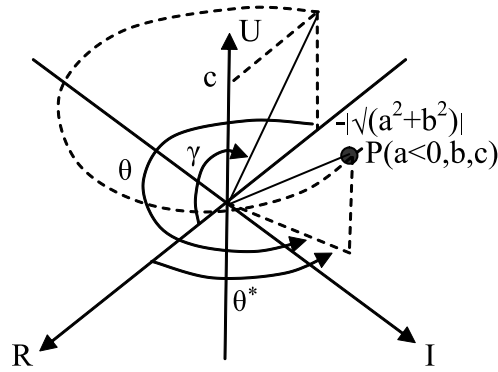


Figure 18: Phases that identify the positions of the half-space R^-IU according to the complementary representation

We note that the spatial phase θ is not calculated by the formula:

$$\theta = \arctan\left(\frac{b}{a}\right)$$

because it would correspond to the value θ^* .

For positions $P(a,b,c)$ of the plane RI not belonging to the lines R and I the phases of the complementary representation will be those shown in Figure 19.

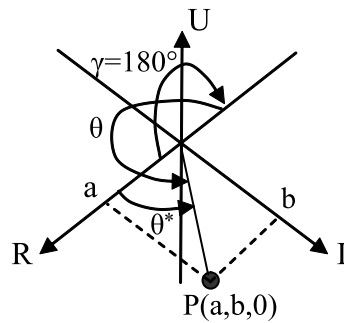


Figure 19: Phases that identify the positions of the plane RI according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\begin{aligned} \gamma &= 180^\circ \\ \theta &= \arctan\left(\frac{-b}{-a}\right) \end{aligned}$$

We note that the spatial phase θ is not calculated by the formula:

$$\theta = \arctan\left(\frac{b}{a}\right)$$

because it would correspond to the value θ^* .

For positions $P(a,b,c)$ of the half-plane R^+U not belonging to the lines R and U the phases of the complementary representation will be those shown in Figure 20.

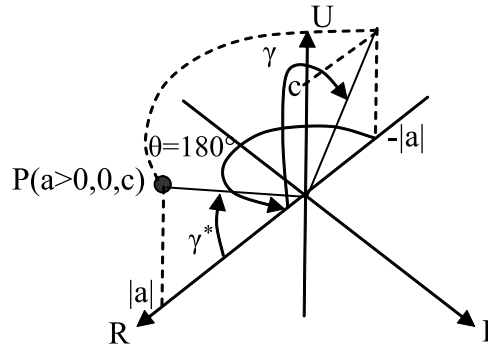


Figure 20: Phases that identify the positions of the half-plane R^+U according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{-|a|}\right)$$

$$\theta = 180^\circ$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan\left(\frac{c}{|a|}\right)$$

because it would correspond to the value γ^* .

For positions $P(a,b,c)$ of the half-plane R^-U not belonging to the lines R and U the phases of the complementary representation will be those shown in Figure 21.

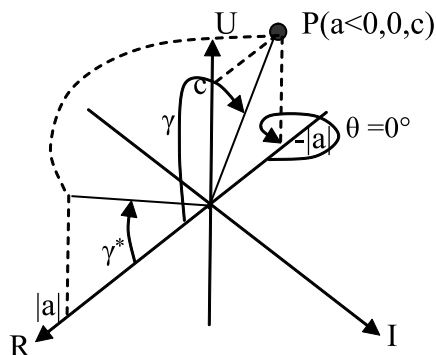


Figure 21: Phases that identify the positions of the half-plane R^-U according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{-|a|} \right)$$

$$\theta = 0^\circ$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan \left(\frac{c}{|a|} \right)$$

because it would correspond to the value γ^* .

For positions $P(a,b,c)$ of the half-plane I^+U not belonging to the lines I and U the phases of the complementary representation will be those shown in Figure 22.

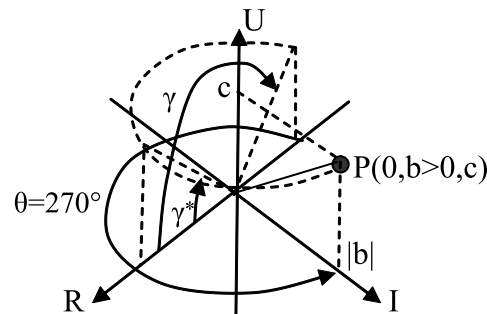


Figure 22: Phases that identify the positions of the half-plane I^+U according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{-|b|} \right)$$

$$\theta = 270^\circ$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan \left(\frac{c}{|b|} \right)$$

because it would correspond to the value γ^* .

For positions $P(a,b,c)$ of the half-plane I^-U not belonging to the lines I and U the phases of the complementary representation will be those shown in Figure 23 on the facing page.

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan \left(\frac{c}{-|b|} \right)$$

$$\theta = 90^\circ$$

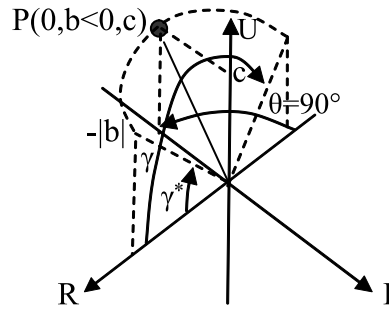


Figure 23: Phases that identify the positions of the half-plane I^-U according to the complementary representation

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan \left(\frac{c}{|b|} \right)$$

because it would correspond to the value γ^* .

For positions $P(a, b, c)$ of the half-line R^+ the phases of the complementary representation will be those shown in Figure 24.

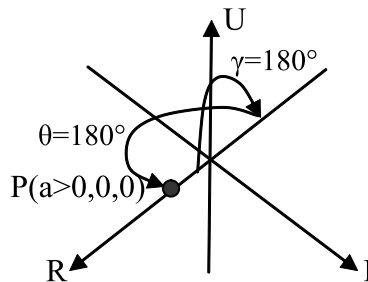


Figure 24: Phases that identify the positions of the half-line R^+ according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\begin{aligned} \gamma &= 180^\circ \\ \theta &= 180^\circ \end{aligned}$$

For positions $P(a, b, c)$ of the half-line R^- the phases of the complementary representation will be those shown in Figure 25 on the following page.

The phases shown in the figure can be determined using the formulas:

$$\begin{aligned} \gamma &= 180^\circ \\ \theta &= 0^\circ \end{aligned}$$

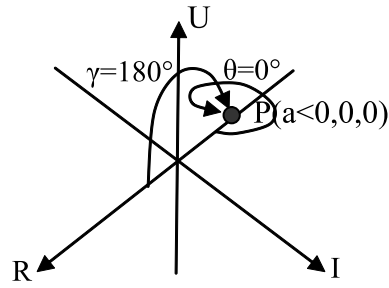


Figure 25: Phases that identify the positions of the half-line R^- according to the complementary representation

For positions $P(a,b,c)$ of the half-line I^+ the phases of the complementary representation will be those shown in Figure 26.

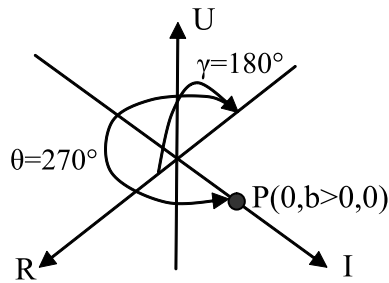


Figure 26: Phases that identify the positions of the half-line I^+ according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\begin{aligned} \gamma &= 180^\circ \\ \theta &= 270^\circ \end{aligned}$$

For positions $P(a,b,c)$ of the half-line I^- the phases of the complementary representation will be those shown in Figure 27.

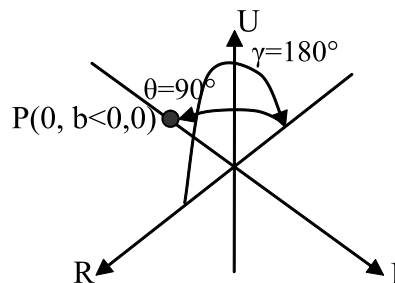


Figure 27: Phases that identify the positions of the half-line I^- according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\begin{aligned}\gamma &= 180^\circ \\ \theta &= 90^\circ\end{aligned}$$

Theorem 2.15. *The complementary representation of a complete number of coordinates (a,b,c) not lying on the line U requires to give to the algebraic root $\sqrt{a^2 + b^2}$ the following negative solution:*

$$\sqrt{a^2 + b^2} = -|\sqrt{a^2 + b^2}|$$

Proof. In the case of the complementary representations previously examined (that cover every region of the space RIU with the exception of the line U) the phase γ assumes the values provided by the formula:

$$\gamma = \arctan\left(\frac{c}{\sqrt{a^2 + b^2}}\right)$$

when we give to the algebraic root $\sqrt{a^2 + b^2}$ its negative solutions. And this immediately proves the thesis. ■

Theorem 2.16. *Each position of the line U corresponds to a complete number for each value assigned to the spatial phase θ .*

Proof. By assigning at the expression of the complete numbers the values $\gamma = \pm 90^\circ$ that characterize the outgoing numbers of the line U :

$$o(t, \theta, \pm 90^\circ) = t \cdot \{[\cos(\pm 90^\circ) \cdot \cos(\theta)] + i \cdot [\cos(\pm 90^\circ) \cdot \sin(\theta)] + u \cdot [\sin(\pm 90^\circ)]\}$$

we obtain the same result regardless of the value of the spatial phase θ :

$$o(t, \theta, \pm 90^\circ) = t \cdot u \cdot [\sin(\pm 90^\circ)] = \pm t \cdot u$$

proving the thesis. ■

Definition 2.17. A complete numbers belonging to the line U can be defined in standard representation if provided with spatial phase θ equal to zero.

Definition 2.18. A complete numbers belonging to the line U can be defined in complementary representation if provided with spatial phase θ different from zero.

Since the non zero values of the spatial phase are unlimited, unlimited will also be the complementary representation related to the complete numbers belonging to the line U .

Theorem 2.19. *Complex numbers cannot be expressed in the following way:*

$$o(a, b, c) = a + i \cdot b + u \cdot c$$

namely:

$$o(t, \theta, \gamma) \neq o(a, b, c) = a + i \cdot b + u \cdot c$$

Proof. The proof comes from the absence of bijection between translation and rotation operations of values (t, θ, γ) and the positions (a, b, c) of the space RIU, as confirmed by the existence of the complementary representation (Theorem 2.14). ■

Since it is impossible to associate the complete numbers to the individual positions of the space, we can always express them in terms of their coordinates (a, b, c) , provided that we make explicit the phases involved as well.

In other words we should use the following notation:

$$o(a, b, c)_{(t, \theta, \gamma)} = a_{(t)} + i \cdot b_{(\theta)} + u \cdot c_{(\gamma)}$$

where the values of t, θ, γ , if not yet given, should be reported to those which characterize the standard representation.

However it is even possible to introduce a more concise notation by indicating what representation is associate to the coordinates (a, b, c) or, in the case of the outgoing numbers, the value of the spatial phase θ . In practice for the standard representation we have:

$$o(a, b, c)_{(S)} = (a + i \cdot b + u \cdot c)_{(S)}$$

for the complementary representation:

$$o(a, b, c)_{(C)} = (a + i \cdot b + u \cdot c)_{(C)}$$

and finally for the outgoing numbers:

$$o(a, b, c)_{(\theta)} = u \cdot c_{(\theta)}$$

While any other notation of the following type:

$$o(a, b, c) = a + i \cdot b + u \cdot c$$

that is devoid of sufficient information to trace the values of the phases θ and γ , will be able to represent the positions of the space RIU, but not the complete numbers.

2.2. Addition

Definition 2.20. In the space RIU we can define addition between two positions $o_1(a_1, b_1, c_1)$ and $o_2(a_2, b_2, c_2)$ as the position $o_{1+2}(a_{1+2}, b_{1+2}, c_{1+2})$ represented also with the symbol $o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2)$ that satisfies the following condition:

$$o_{1+2}(a_{1+2}, b_{1+2}, c_{1+2}) = o_{1+2}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

This condition is equivalent to take the position of the space RIU provided with the following coordinates:

$$a_{1+2} = a_1 + a_2$$

$$b_{1+2} = b_1 + b_2$$

$$c_{1+2} = c_1 + c_2$$

We can observe, with regard to this, Figure 28.

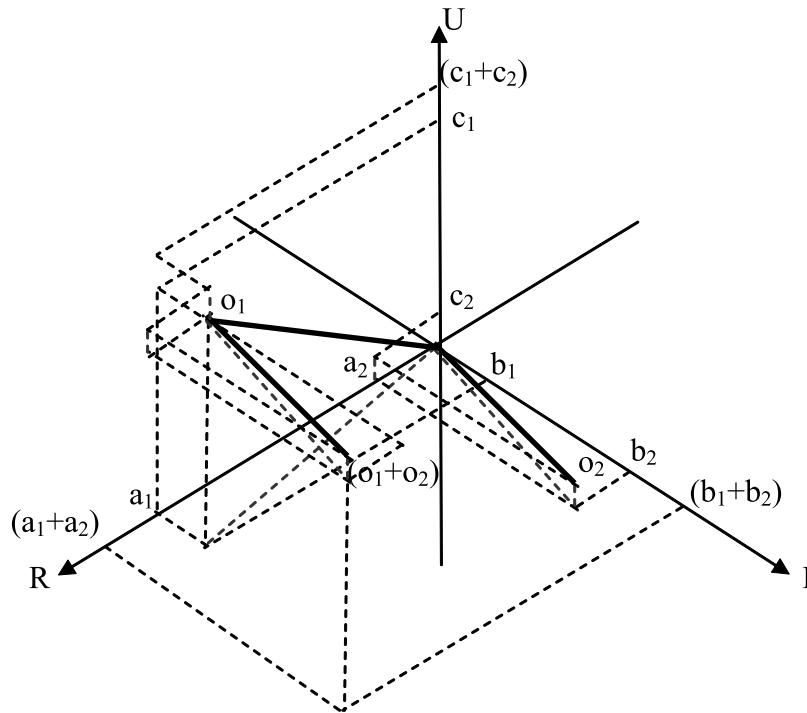


Figure 28: Representation of the addition between two complete numbers

It should be emphasized that the addition is not defined in terms of translations and rotations, and this means that it must be considered an operation that works on the positions and not on the complete numbers. If in one or two dimensions this does not happen is due to the fact that in such contexts there is a bijection between positions and numbers.

Since the addition works on the positions, the notation to use for the various terms involved will be the following:

$$o(a, b, c) = a + i \cdot b + u \cdot c$$

To integrate the operation of addition, working on the positions, with the others, working on the complete numbers, will be enough making reference to the complete number that we can obtain assigning to the sum the phases of the standard representation.

Theorem 2.21. *For the operation of addition is defined neuter the position 0, namely for:*

$$o_2(a_2, b_2, c_2) = 0$$

we have:

$$o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2) = o_1(a_1, b_1, c_1)$$

Proof. $a_1, b_1, c_1, a_2, b_2, c_2$ being real numbers, we can write:

$$\begin{aligned} a_{1+2} &= a_1 + a_2 = a_1 + 0 = a_1 \\ b_{1+2} &= b_1 + b_2 = b_1 + 0 = b_1 \\ c_{1+2} &= c_1 + a_2 = c_1 + 0 = c_1 \end{aligned}$$

proving the thesis. ■

Theorem 2.22. *For the operation of addition is defined opposite the position symmetric with respect to the origin, namely for:*

$$o_2(a_2, b_2, c_2) = o_2(-a_1, -b_1, -c_1)$$

we have:

$$o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2) = 0$$

Proof. $a_1, b_1, c_1, a_2, b_2, c_2$ being real numbers, we can write:

$$\begin{aligned} a_{1+2} &= a_1 + a_2 = a_1 - a_1 = 0 \\ b_{1+2} &= b_1 + b_2 = b_1 - b_1 = 0 \\ c_{1+2} &= c_1 + a_2 = c_1 - c_1 = 0 \end{aligned}$$

proving the thesis. ■

Theorem 2.23. *For the operation of addition is valid the commutative property, namely:*

$$o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2) = o_2(a_2, b_2, c_2) + o_1(a_1, b_1, c_1)$$

Proof. $a_1, b_1, c_1, a_2, b_2, c_2$ being real numbers, we can write:

$$\begin{aligned} a_{1+2} &= a_1 + a_2 \\ b_{1+2} &= b_1 + b_2 \\ c_{1+2} &= c_1 + c_2 \end{aligned}$$

$$\begin{aligned} a_{2+1} &= a_2 + a_1 = a_1 + a_2 \\ b_{2+1} &= b_2 + b_1 = b_1 + b_2 \\ c_{2+1} &= c_2 + c_1 = c_1 + c_2 \end{aligned}$$

proving the thesis. ■

Theorem 2.24. *For the operation of addition are valid the associative and dissociative properties, namely for:*

$$o_2(a_2, b_2, c_2) = o_3(a_3, b_3, c_3) + o_4(a_4, b_4, c_4)$$

we have:

$$\begin{aligned} o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2) &= [o_1(a_1, b_1, c_1) + o_3(a_3, b_3, c_3)] + o_4(a_4, b_4, c_4) \\ [o_1(a_1, b_1, c_1) + o_3(a_3, b_3, c_3)] + o_4(a_4, b_4, c_4) &= o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2) \end{aligned}$$

Proof. $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, a_4, b_4, c_4$ being real numbers, we can write:

$$\begin{aligned} a_{1+2} &= a_1 + a_2 = a_1 + (a_3 + a_4) = (a_1 + a_3) + a_4 = a_{(1+3)+4} \\ b_{1+2} &= b_1 + b_2 = b_1 + (b_3 + b_4) = (b_1 + b_3) + b_4 = b_{(1+3)+4} \\ c_{1+2} &= c_1 + c_2 = c_1 + (c_3 + c_4) = (c_1 + c_3) + c_4 = c_{(1+3)+4} \\ a_{(1+3)+4} &= (a_1 + a_3) + a_4 = a_1 + (a_3 + a_4) = a_1 + a_2 = a_{1+2} \\ b_{(1+3)+4} &= (b_1 + b_3) + b_4 = b_1 + (b_3 + b_4) = b_1 + b_2 = b_{1+2} \\ c_{(1+3)+4} &= (c_1 + c_3) + c_4 = c_1 + (c_3 + c_4) = c_1 + a_2 = c_{1+2} \end{aligned}$$

proving the thesis. ■

2.3. Subtraction

Definition 2.25. In the space RIU we can define subtraction between two positions $o_1(a_1, b_1, c_1)$ and $o_2(a_2, b_2, c_2)$ as the position $o_{1-2}(a_{1-2}, b_{1-2}, c_{1-2})$ represented also with the symbol $o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2)$ that satisfies the following condition:

$$o_{1-2}(a_{1-2}, b_{1-2}, c_{1-2}) + o_2(a_2, b_2, c_2) = o_1(a_1, b_1, c_1)$$

This condition defines the subtraction as the inverse operation of addition, and it is equivalent to require:

$$\begin{aligned} a_{1-2} &= a_1 - a_2 \\ b_{1-2} &= b_1 - b_2 \\ c_{1-2} &= c_1 - c_2 \end{aligned}$$

It should be emphasized that the subtraction is not defined in terms of translations and rotations, and this means that it must be considered an operation that works on the positions and not on the complete numbers. If in one or two dimensions this does not happen is due to the fact that in such contexts there is a bijection between positions and numbers.

Since the subtraction works on the positions, the notation to use for the various terms involved will be the following:

$$o(a, b, c) = a + i \cdot b + u \cdot c$$

To integrate the operation of subtraction, working on the positions, with the others, working on the complete numbers, will be enough making reference to the complete number that we can obtain assigning to the difference the phases of the standard representation.

Theorem 2.26. *For the operation of subtraction is defined neuter the position 0, namely for:*

$$o_2(a_2, b_2, c_2) = 0$$

we have:

$$o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2) = o_1(a_1, b_1, c_1)$$

Proof. $a_1, b_1, c_1, a_2, b_2, c_2$ being real numbers, we can write:

$$\begin{aligned} a_{1-2} &= a_1 - a_2 = a_1 - 0 = a_1 \\ b_{1-2} &= b_1 - b_2 = b_1 - 0 = b_1 \\ c_{1-2} &= c_1 - a_2 = c_1 - 0 = c_1 \end{aligned}$$

proving the thesis. ■

Theorem 2.27. *For the operation of subtraction is defined identical, the same position with respect to the origin, namely for:*

$$o_2(a_2, b_2, c_2) = o_2(a_1, b_1, c_1)$$

we have:

$$o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2) = 0$$

Proof. $a_1, b_1, c_1, a_2, b_2, c_2$ being real numbers, we can write:

$$\begin{aligned} a_{1-2} &= a_1 - a_2 = a_1 - a_1 = 0 \\ b_{1-2} &= b_1 - b_2 = b_1 - b_1 = 0 \\ c_{1-2} &= c_1 - a_2 = c_1 - c_1 = 0 \end{aligned}$$

proving the thesis. ■

Theorem 2.28. *For the operation of subtraction is valid the invariantive property, namely:*

$$\begin{aligned} o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2) &= [o_1(a_1, b_1, c_1) + o_3(a_3, b_3, c_3)] + \\ &\quad - [o_2(a_2, b_2, c_2) + o_3(a_3, b_3, c_3)] \\ o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2) &= [o_1(a_1, b_1, c_1) - o_3(a_3, b_3, c_3)] + \\ &\quad - [o_2(a_2, b_2, c_2) - o_3(a_3, b_3, c_3)] \end{aligned}$$

Proof. $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ being real numbers, we can write:

$$\begin{aligned} a_{1-2} &= a_1 - a_2 \\ b_{1-2} &= b_1 - b_2 \\ c_{1-2} &= c_1 - c_2 \end{aligned}$$

$$\begin{aligned} a_{(1+3)-(2+3)} &= (a_1 + a_3) - (a_2 + a_3) = a_1 + a_3 - a_2 - a_3 = a_1 - a_2 \\ b_{(1+3)-(2+3)} &= (b_1 + b_3) - (b_2 + b_3) = b_1 + b_3 - b_2 - b_3 = b_1 - b_2 \\ c_{(1+3)-(2+3)} &= (c_1 + c_3) - (c_2 + c_3) = c_1 + c_3 - c_2 - c_3 = c_1 - c_2 \\ a_{(1-3)-(2-3)} &= (a_1 - a_3) - (a_2 - a_3) = a_1 - a_3 - a_2 + a_3 = a_1 - a_2 \\ b_{(1-3)-(2-3)} &= (b_1 - b_3) - (b_2 - b_3) = b_1 - b_3 - b_2 + b_3 = b_1 - b_2 \\ c_{(1-3)-(2-3)} &= (c_1 - c_3) - (c_2 - c_3) = c_1 - c_3 - c_2 + c_3 = c_1 - c_2 \end{aligned}$$

proving the thesis. ■

Theorem 2.29. *It is valid the equivalence between addition and subtraction, namely:*

$$\begin{aligned} o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2) &= o_1(a_1, b_1, c_1) - [-o_2(a_2, b_2, c_2)] \\ o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2) &= o_1(a_1, b_1, c_1) + [-o_2(a_2, b_2, c_2)] \end{aligned}$$

Proof. $a_1, b_1, c_1, a_2, b_2, c_2$ being real numbers, we can write:

$$a_{1+2} = a_1 + a_2$$

$$b_{1+2} = b_1 + b_2$$

$$c_{1+2} = c_1 + c_2$$

$$a_{1-(-2)} = a_1 - (-a_2) = a_1 + a_2$$

$$b_{1-(-2)} = b_1 - (-b_2) = b_1 + b_2$$

$$c_{1-(-2)} = c_1 - (-c_2) = c_1 + c_2$$

$$a_{1-2} = a_1 - a_2$$

$$b_{1-2} = b_1 - b_2$$

$$c_{1-2} = c_1 - c_2$$

$$a_{1+(-2)} = a_1 + (-a_2) = a_1 - a_2$$

$$b_{1+(-2)} = b_1 + (-b_2) = b_1 - b_2$$

$$c_{1+(-2)} = c_1 + (-c_2) = c_1 - c_2$$

proving the thesis. ■

2.4. Multiplication

Definition 2.30. In the space RIU we can define multiplication between two complete numbers $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ as the number $o_{1.2}(t_{1.2}, \theta_{1.2}, \gamma_{1.2})$ represented also with the symbol $o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2)$ that satisfies the following condition:

$$o_{1.2}(t_{1.2}, \theta_{1.2}, \gamma_{1.2}) = o_{1.2}(t_1 \cdot t_2, \theta_1 + \theta_2, \gamma_1 + \gamma_2)$$

This condition defines the multiplication and it is equivalent to require:

$$t_{1.2} = t_1 \cdot t_2$$

$$\theta_{1.2} = \theta_1 + \theta_2$$

$$\gamma_{1.2} = \gamma_1 + \gamma_2$$

We can observe, with regard to this, Figure 29 on the following page.

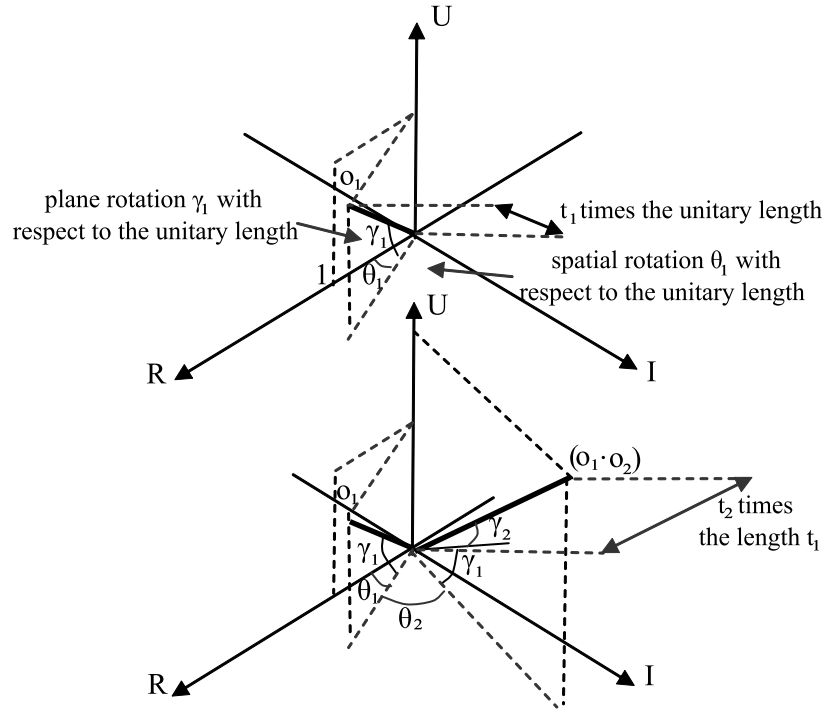


Figure 29: Representation of the multiplication between two complete numbers

Theorem 2.31. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, and both not belonging to the line U , their multiplication may be expressed in the following way:*

$$o_{1.2}(a_{1.2}, b_{1.2}, c_{1.2})_{(t_{1.2}, \theta_{1.2}, \gamma_{1.2})} = a_{1.2}(t_1 \cdot t_2) + i \cdot b_{1.2}(\theta_1 + \theta_2) + u \cdot c_{1.2}(\gamma_1 + \gamma_2)$$

where:

$$a_{1.2} = (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right)$$

$$b_{1.2} = (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right)$$

$$c_{1.2} = c_1 \cdot |\sqrt{a_2^2 + b_2^2}| + c_2 \cdot |\sqrt{a_1^2 + b_1^2}|$$

Proof. The multiplication between two complete numbers, as we know, satisfies the following formula:

$$o_{1.2}(t_{1.2}, \theta_{1.2}, \gamma_{1.2}) = t_1 \cdot t_2 \cdot \{ [\cos(\gamma_1 + \gamma_2) \cdot \cos(\theta_1 + \theta_2)] + i \cdot [\cos(\gamma_1 + \gamma_2) \cdot \sin(\theta_1 + \theta_2)] + u \cdot [\sin(\gamma_1 + \gamma_2)] \}$$

For the moduli and the phases involved will be valid the following relation as well:

$$t = \sqrt{a^2 + b^2 + c^2}$$

$$\gamma = \arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right)$$

$$\theta = \arctan \left(\frac{b}{a} \right)$$

This means that we can write the coordinates sought in the following way:

$$a_{1.2} = \sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \cos \left[\arctan \left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} \right) + \arctan \left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}} \right) \right] \cdot \cos \left[\arctan \left(\frac{b_1}{a_1} \right) + \arctan \left(\frac{b_2}{a_2} \right) \right]$$

$$b_{1.2} = \sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \cos \left[\arctan \left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} \right) + \arctan \left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}} \right) \right] \cdot \sin \left[\arctan \left(\frac{b_1}{a_1} \right) + \arctan \left(\frac{b_2}{a_2} \right) \right]$$

$$c_{1.2} = \sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \sin \left[\arctan \left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} \right) + \arctan \left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}} \right) \right]$$

To continue with the proof, we have to use the following trigonometric relations:

$$\cos(x + y) = \cos(x) \cdot \cos(y) - \sin(x) \cdot \sin(y)$$

$$\sin(x + y) = \sin(x) \cdot \cos(y) + \cos(x) \cdot \sin(y)$$

$$\cos \left[\arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right) \right] = \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + c^2}}$$

$$\sin \left[\arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right) \right] = \sqrt{\frac{c^2}{a^2 + b^2 + c^2}}$$

$$\cos \left[\arctan \left(\frac{b}{a} \right) \right] = \sqrt{\frac{a^2}{a^2 + b^2}}$$

$$\sin \left[\arctan \left(\frac{b}{a} \right) \right] = \sqrt{\frac{b^2}{a^2 + b^2}}$$

To determine the value of the coordinate $a_{1.2}$ the steps to perform will be the following:

$$\begin{aligned}
a_{1.2} &= \sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \left(\sqrt{\frac{a_1^2 + b_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{a_2^2 + b_2^2}{a_2^2 + b_2^2 + c_2^2}} + \right. \\
&\quad \left. - \sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \right) \cdot \left(\sqrt{\frac{a_1^2}{a_1^2 + b_1^2}} \cdot \sqrt{\frac{a_2^2}{a_2^2 + b_2^2}} + \right. \\
&\quad \left. - \sqrt{\frac{b_1^2}{a_1^2 + b_1^2}} \cdot \sqrt{\frac{b_2^2}{a_2^2 + b_2^2}} \right) = \\
&= \left(\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2} - \sqrt{c_1^2} \cdot \sqrt{c_2^2} \right) \cdot \left(\frac{\sqrt{a_1^2} \cdot \sqrt{a_2^2} - \sqrt{b_1^2} \cdot \sqrt{b_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right) = \\
&= \left(\sqrt{a_1^2} \cdot \sqrt{a_2^2} - \sqrt{b_1^2} \cdot \sqrt{b_2^2} \right) - \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \left(\frac{\sqrt{a_1^2} \cdot \sqrt{a_2^2} - \sqrt{b_1^2} \cdot \sqrt{b_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right) = \\
&= \left(\sqrt{a_1^2} \cdot \sqrt{a_2^2} - \sqrt{b_1^2} \cdot \sqrt{b_2^2} \right) \cdot \left(1 - \frac{\sqrt{c_1^2} \cdot \sqrt{c_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right)
\end{aligned}$$

To determine the value of the coordinate $b_{1.2}$ the steps to perform will be the following:

$$\begin{aligned}
b_{1.2} &= \sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \left(\sqrt{\frac{a_1^2 + b_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{a_2^2 + b_2^2}{a_2^2 + b_2^2 + c_2^2}} + \right. \\
&\quad \left. - \sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \right) \cdot \left(\sqrt{\frac{b_1^2}{a_1^2 + b_1^2}} \cdot \sqrt{\frac{a_2^2}{a_2^2 + b_2^2}} + \right. \\
&\quad \left. + \sqrt{\frac{a_1^2}{a_1^2 + b_1^2}} \cdot \sqrt{\frac{b_2^2}{a_2^2 + b_2^2}} \right) = \\
&= \left(\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2} - \sqrt{c_1^2} \cdot \sqrt{c_2^2} \right) \cdot \left(\frac{\sqrt{b_1^2} \cdot \sqrt{a_2^2} + \sqrt{a_1^2} \cdot \sqrt{b_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right) = \\
&= \left(\sqrt{b_1^2} \cdot \sqrt{a_2^2} + \sqrt{a_1^2} \cdot \sqrt{b_2^2} \right) - \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \left(\frac{\sqrt{b_1^2} \cdot \sqrt{a_2^2} + \sqrt{a_1^2} \cdot \sqrt{b_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right) = \\
&= \left(\sqrt{b_1^2} \cdot \sqrt{a_2^2} + \sqrt{a_1^2} \cdot \sqrt{b_2^2} \right) \cdot \left(1 - \frac{\sqrt{c_1^2} \cdot \sqrt{c_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right)
\end{aligned}$$

To determine the value of the coordinate $c_{1.2}$ the steps to perform will be the following:

$$\begin{aligned}
c_{1.2} &= \sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \left(\sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{a_2^2 + b_2^2}{a_2^2 + b_2^2 + c_2^2}} + \right. \\
&\quad \left. + \sqrt{\frac{a_1^2 + b_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \right) = \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2} + \sqrt{c_2^2} \cdot \sqrt{a_1^2 + b_1^2}
\end{aligned}$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a, b, c are zero (provided that we work with complete numbers not belonging in the line U). But their main peculiarity is that to contain many roots of the form $\sqrt{x^2}$.

Since the radicand x^2 is always positive we know that the operation of algebraic root considered here is permitted, and therefore it will be able to take as result two opposite values: one positive and one negative. This means that from mathematical point of view we obtain a relation able to satisfies the multiplication rule for each possible combination of signs attributable to the roots involved.

For example if we adopt the convention of attributing to the roots always the positive value, we obtain the following result:

$$\begin{aligned}\sqrt{a^2} &= |a| \\ \sqrt{b^2} &= |b| \\ \sqrt{c^2} &= |c|\end{aligned}$$

to which correspond relations able to satisfy the multiplication role as a function of the modulus of the coordinates involved. This means that distinct complete numbers will be able to give the same result of the multiplication if their coordinates will have the same modulus.

Wanting to find relations that satisfy the multiplication rule as a function of the effective coordinates of the complete numbers involved, we must assign to the roots the same sign of the coefficient located within them:

$$\begin{aligned}\sqrt{a^2} &= a \\ \sqrt{b^2} &= b \\ \sqrt{c^2} &= c\end{aligned}$$

The relations obtained will be the following:

$$\begin{aligned}(2.1) \quad a_{1.2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right) \\ b_{1.2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right) \\ c_{1.2} &= c_1 \cdot \sqrt{a_2^2 + b_2^2} + c_2 \cdot \sqrt{a_1^2 + b_1^2}\end{aligned}$$

Since the complete numbers involved are in standard representation, as determined by the theorem 2.11 we must consider the following relations:

$$\begin{aligned}\sqrt{a_1^2 + b_1^2} &= |\sqrt{a_1^2 + b_1^2}| \\ \sqrt{a_2^2 + b_2^2} &= |\sqrt{a_2^2 + b_2^2}|\end{aligned}$$

that combined with those indicated by the formulas (2.1), proving the thesis. ■

As an example of the theorem just proved, suppose you have to multiply the complete numbers in standard representation provided with coordinates:

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1.$$

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the standard representation:

$$\begin{aligned} \gamma_1 = \gamma_2 &= \arctan \left(\frac{c_1}{|\sqrt{a_1^2 + b_1^2}|} \right) = \arctan \left(\frac{c_2}{|\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= \arctan \left(\frac{1}{|\sqrt{2}|} \right) \simeq 35.26^\circ \\ \theta_1 = \theta_2 &= \arctan \left(\frac{b_1}{a_1} \right) = \arctan \left(\frac{b_2}{a_2} \right) = \arctan \left(\frac{1}{1} \right) = 45^\circ \end{aligned}$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 = 3 \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 \simeq 70.52^\circ \\ \theta_{1.2} &= \theta_1 + \theta_2 = 90^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \cos(\theta_{1.2}) = 3 \cdot \cos(\simeq 70.52^\circ) \cdot \cos(90^\circ) = 0 \\ b_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \sin(\theta_{1.2}) = 3 \cdot \cos(\simeq 70.52^\circ) \cdot \sin(90^\circ) = 1 \\ c_{1.2} &= t_{1.2} \cdot \sin(\gamma_{1.2}) = 3 \cdot \sin(\simeq 70.52^\circ) = 2 \cdot \sqrt{2} \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{1.2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= (1 - 1) \cdot \left(1 - \frac{1}{2} \right) = 0 \\ b_{1.2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= (1 + 1) \cdot \left(1 - \frac{1}{2} \right) = 1 \\ c_{1.2} &= c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| + c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| = 1 \cdot \sqrt{2} + 1 \cdot \sqrt{2} = 2 \cdot \sqrt{2} \end{aligned}$$

Theorem 2.32. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, and both not belonging to the line U , their multiplication may be expressed in the following way:*

$$o_{1.2}(a_{1.2}, b_{1.2}, c_{1.2})_{(t_{1.2}, \theta_{1.2}, \gamma_{1.2})} = a_{1.2}(t_1 \cdot t_2) + i \cdot b_{1.2}(\theta_1 + \theta_2) + u \cdot c_{1.2}(\gamma_1 + \gamma_2)$$

where:

$$a_{1.2} = (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right)$$

$$b_{1.2} = (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right)$$

$$c_{1.2} = -c_1 \cdot |\sqrt{a_2^2 + b_2^2}| - c_2 \cdot |\sqrt{a_1^2 + b_1^2}|$$

Proof. Since the complete numbers involved are in complementary representation, as determined by the theorem 2.15 we must consider the following relations:

$$\begin{aligned} \sqrt{a_1^2 + b_1^2} &= -|\sqrt{a_1^2 + b_1^2}| \\ \sqrt{a_2^2 + b_2^2} &= -|\sqrt{a_2^2 + b_2^2}| \end{aligned}$$

that combined with those indicated by the formulas (2.1), proving the thesis. ■

As an example of the theorem just proved, suppose you have to multiply the complete numbers in complementary representation provided with coordinates: $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the complementary representation:

$$\begin{aligned} \gamma_1 = \gamma_2 &= \arctan \left(\frac{c_1}{-|\sqrt{a_1^2 + b_1^2}|} \right) = \arctan \left(\frac{c_2}{-|\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= \arctan \left(\frac{1}{-|\sqrt{2}|} \right) \simeq 144.73^\circ \end{aligned}$$

$$\theta_1 = \theta_2 = \arctan \left(\frac{-b_1}{-a_1} \right) = \arctan \left(\frac{-b_2}{-a_2} \right) = \arctan \left(\frac{-1}{-1} \right) = 225^\circ$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 = 3 \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 \simeq 289.46^\circ \\ \theta_{1.2} &= \theta_1 + \theta_2 = 450^\circ = 90^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \cos(\theta_{1.2}) = 3 \cdot \cos(\simeq 289.46^\circ) \cdot \cos(90^\circ) = 0 \\ b_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \sin(\theta_{1.2}) = 3 \cdot \cos(\simeq 289.46^\circ) \cdot \sin(90^\circ) = 1 \\ c_{1.2} &= t_{1.2} \cdot \sin(\gamma_{1.2}) = 3 \cdot \sin(\simeq 289.46^\circ) = -2 \cdot \sqrt{2} \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{1.2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= (1 - 1) \cdot \left(1 - \frac{1}{2} \right) = 0 \\ b_{1.2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= (1 + 1) \cdot \left(1 - \frac{1}{2} \right) = 1 \\ c_{1.2} &= -c_1 \cdot |\sqrt{a_2^2 + b_2^2}| - c_2 \cdot |\sqrt{a_1^2 + b_1^2}| = -1 \cdot \sqrt{2} - 1 \cdot \sqrt{2} = -2 \cdot \sqrt{2} \end{aligned}$$

Theorem 2.33. *With $o_1(t_1, \theta_1, \gamma_1)$ in standard representation and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, and both not belonging to the line U , their multiplication may be expressed in the following way:*

$$o_{1.2}(a_{1.2}, b_{1.2}, c_{1.2})_{(t_{1.2}, \theta_{1.2}, \gamma_{1.2})} = a_{1.2}(t_1 \cdot t_2) + i \cdot b_{1.2}(\theta_1 + \theta_2) + u \cdot c_{1.2}(\gamma_1 + \gamma_2)$$

where:

$$\begin{aligned} a_{1.2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ b_{1.2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ c_{1.2} &= c_2 \cdot |\sqrt{a_1^2 + b_1^2}| - c_1 \cdot |\sqrt{a_2^2 + b_2^2}| \end{aligned}$$

Proof. Since the first factor is in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = |\sqrt{a_1^2 + b_1^2}|$$

while being the second factor in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = -|\sqrt{a_2^2 + b_2^2}|$$

that combined with those indicated by the formulas (2.1), proving the thesis. ■

As an example of the theorem just proved, suppose you have to multiply the complete number in standard representation provided with coordinates $a_1 = b_1 = c_1 = 1$ by that in complementary representation provided with the same coordinates coordinates: $a_2 = b_2 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the standard and complementary representations:

$$\begin{aligned} \gamma_1 &= \arctan \left(\frac{c_1}{|\sqrt{a_1^2 + b_1^2}|} \right) = \arctan \left(\frac{1}{|\sqrt{2}|} \right) \simeq 35.26^\circ \\ \gamma_2 &= \arctan \left(\frac{c_2}{-|\sqrt{a_2^2 + b_2^2}|} \right) = \arctan \left(\frac{1}{-|\sqrt{2}|} \right) \simeq 144.73^\circ \\ \theta_1 &= \arctan \left(\frac{b_1}{a_1} \right) = \arctan \left(\frac{1}{1} \right) = 45^\circ \\ \theta_2 &= \arctan \left(\frac{-b_2}{-a_2} \right) = \arctan \left(\frac{-1}{-1} \right) = 225^\circ \end{aligned}$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 = 3 \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 = 180^\circ \\ \theta_{1.2} &= \theta_1 + \theta_2 = 270^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \cos(\theta_{1.2}) = 3 \cdot \cos(180^\circ) \cdot \cos(270^\circ) = 0 \\ b_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \sin(\theta_{1.2}) = 3 \cdot \cos(180^\circ) \cdot \sin(270^\circ) = 3 \\ c_{1.2} &= t_{1.2} \cdot \sin(\gamma_{1.2}) = 3 \cdot \sin(180^\circ) = 0 \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{1.2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ &= (1 - 1) \cdot \left(1 + \frac{1}{2} \right) = 0 \\ b_{1.2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ &= (1 + 1) \cdot \left(1 + \frac{1}{2} \right) = 3 \\ c_{1.2} &= c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| - c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| = 1 \cdot \sqrt{2} - 1 \cdot \sqrt{2} = 0 \end{aligned}$$

Theorem 2.34. *With $o_1(t_1, \theta_1, \gamma_1)$ in complementary representation and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, and both not belonging to the line U , their multiplication may be expressed in the following way:*

$$o_{1.2}(a_{1.2}, b_{1.2}, c_{1.2})_{(t_{1.2}, \theta_{1.2}, \gamma_{1.2})} = a_{1.2}(t_1 \cdot t_2) + i \cdot b_{1.2}(\theta_1 + \theta_2) + u \cdot c_{1.2}(\gamma_1 + \gamma_2)$$

where:

$$\begin{aligned} a_{1.2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ b_{1.2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ c_{1.2} &= c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| - c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| \end{aligned}$$

Proof. Since the first factor is in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = -|\sqrt{a_1^2 + b_1^2}|$$

while being the second factor in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = |\sqrt{a_2^2 + b_2^2}|$$

that combined with those indicated by the formulas (2.1), proving the thesis. ■

As an example of the theorem just proved, suppose you have to multiply the complete number in complementary representation provided with coordinates $a_1 = b_1 = c_1 = 1$ by that in standard representation provided with the same coordinates: $a_2 = b_2 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the complementary and standard representations:

$$\begin{aligned} \gamma_1 &= \arctan \left(\frac{c_1}{-|\sqrt{a_1^2 + b_1^2}|} \right) = \arctan \left(\frac{1}{-|\sqrt{2}|} \right) \simeq 144.73^\circ \\ \gamma_2 &= \arctan \left(\frac{c_2}{|\sqrt{a_2^2 + b_2^2}|} \right) = \arctan \left(\frac{1}{|\sqrt{2}|} \right) \simeq 35.26^\circ \\ \theta_1 &= \arctan \left(\frac{-b_1}{-a_1} \right) = \arctan \left(\frac{-1}{-1} \right) = 225^\circ \\ \theta_2 &= \arctan \left(\frac{b_2}{a_2} \right) = \arctan \left(\frac{1}{1} \right) = 45^\circ \end{aligned}$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 = 3 \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 = 180^\circ \\ \theta_{1.2} &= \theta_1 + \theta_2 = 270^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \cos(\theta_{1.2}) = 3 \cdot \cos(180^\circ) \cdot \cos(270^\circ) = 0 \\ b_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \sin(\theta_{1.2}) = 3 \cdot \cos(180^\circ) \cdot \sin(270^\circ) = 3 \\ c_{1.2} &= t_{1.2} \cdot \sin(\gamma_{1.2}) = 3 \cdot \sin(180^\circ) = 0 \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{1.2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ &= (1 - 1) \cdot \left(1 + \frac{1}{2} \right) = 0 \\ b_{1.2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ &= (1 + 1) \cdot \left(1 + \frac{1}{2} \right) = 3 \\ c_{1.2} &= c_1 \cdot |\sqrt{a_2^2 + b_2^2}| - c_2 \cdot |\sqrt{a_1^2 + b_1^2}| = 1 \cdot \sqrt{2} - 1 \cdot \sqrt{2} = 0 \end{aligned}$$

Theorem 2.35. *With only $o_1(t_1, \theta_1, \gamma_1)$ belonging to the line U and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, their multiplication may be expressed in the following way:*

$$o_{1.2}(a_{1.2}, b_{1.2}, c_{1.2})_{(t_{1.2}, \theta_{1.2}, \gamma_{1.2})} = a_{1.2}(t_1 \cdot t_2) + i \cdot b_{1.2}(\theta_1 + \theta_2) + u \cdot c_{1.2}(\gamma_1 + \gamma_2)$$

where:

$$\begin{aligned} a_{1.2} &= -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) - b_2 \cdot \sin(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} \\ b_{1.2} &= -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) + b_2 \cdot \cos(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} \\ c_{1.2} &= c_1 \cdot |\sqrt{a_2^2 + b_2^2}| \end{aligned}$$

Proof. The multiplication between two complete numbers, as we know, satisfies the following formula:

$$\begin{aligned} o_{1.2}(t_{1.2}, \theta_{1.2}, \gamma_{1.2}) &= t_1 \cdot t_2 \cdot \{ [\cos(\gamma_1 + \gamma_2) \cdot \cos(\theta_1 + \theta_2)] + \\ &\quad + i \cdot [\cos(\gamma_1 + \gamma_2) \cdot \sin(\theta_1 + \theta_2)] + u \cdot [\sin(\gamma_1 + \gamma_2)] \} \end{aligned}$$

Since $o_1(t_1, \theta_1, \gamma_1)$ belongs to the line U will be provided with the following values of modulus and phases:

$$\begin{aligned} t_1 &= \sqrt{c_1^2} \\ \gamma_1 &= \text{sign}(c_1) \cdot 90^\circ \\ \theta_1 \text{ known} &\neq \arctan\left(\frac{b_1}{a_1}\right) \end{aligned}$$

unlike $o_2(t_2, \theta_2, \gamma_2)$ that will be provided with the following values:

$$\begin{aligned} t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} \\ \gamma_2 &= \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right) \\ \theta_2 &= \arctan\left(\frac{b_2}{a_2}\right) \end{aligned}$$

This means that we can write the coordinates sought in the following way:

$$\begin{aligned} a_{1.2} &= \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \cos\left[\text{sign}(c_1) \cdot 90^\circ + \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)\right] \\ &\quad \cdot \cos\left[\theta_1 + \arctan\left(\frac{b_2}{a_2}\right)\right] \\ b_{1.2} &= \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \cos\left[\text{sign}(c_1) \cdot 90^\circ + \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)\right] \\ &\quad \cdot \sin\left[\theta_1 + \arctan\left(\frac{b_2}{a_2}\right)\right] \\ c_{1.2} &= \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \sin\left[\text{sign}(c_1) \cdot 90^\circ + \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)\right] \end{aligned}$$

To continue with the proof, we have to use the following trigonometric relations:

$$\begin{aligned} \cos(x + y) &= \cos(x) \cdot \cos(y) - \sin(x) \cdot \sin(y) \\ \sin(x + y) &= \sin(x) \cdot \cos(y) + \cos(x) \cdot \sin(y) \\ \cos\left[\arctan\left(\frac{c}{\sqrt{a^2 + b^2}}\right)\right] &= \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + c^2}} \\ \sin\left[\arctan\left(\frac{c}{\sqrt{a^2 + b^2}}\right)\right] &= \sqrt{\frac{c^2}{a^2 + b^2 + c^2}} \\ \cos\left[\arctan\left(\frac{b}{a}\right)\right] &= \sqrt{\frac{a^2}{a^2 + b^2}} \\ \sin\left[\arctan\left(\frac{b}{a}\right)\right] &= \sqrt{\frac{b^2}{a^2 + b^2}} \\ \cos[\text{sign}(x) \cdot 90^\circ + y] &= -\text{sign}(x) \cdot \sin(y) \\ \sin[\text{sign}(x) \cdot 90^\circ + y] &= \text{sign}(x) \cdot \cos(y) \end{aligned}$$

To determine the value of the coordinate $a_{1,2}$ the steps to perform will be the following:

$$\begin{aligned} a_{1,2} &= - \operatorname{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \\ &\cdot \left[\sqrt{\frac{a_2^2}{a_2^2 + b_2^2}} \cdot \cos(\theta_1) - \sqrt{\frac{b_2^2}{a_2^2 + b_2^2}} \cdot \sin(\theta_1) \right] = \\ &= - \operatorname{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \frac{\sqrt{a_2^2} \cdot \cos(\theta_1) - \sqrt{b_2^2} \cdot \sin(\theta_1)}{\sqrt{a_2^2 + b_2^2}} \end{aligned}$$

To determine the value of the coordinate $b_{1,2}$ the steps to perform will be the following:

$$\begin{aligned} b_{1,2} &= - \operatorname{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \\ &\cdot \left[\sqrt{\frac{a_2^2}{a_2^2 + b_2^2}} \cdot \sin(\theta_1) + \sqrt{\frac{b_2^2}{a_2^2 + b_2^2}} \cdot \cos(\theta_1) \right] = \\ &= - \operatorname{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \frac{\sqrt{a_2^2} \cdot \sin(\theta_1) + \sqrt{b_2^2} \cdot \cos(\theta_1)}{\sqrt{a_2^2 + b_2^2}} \end{aligned}$$

To determine the value of the coordinate $c_{1,2}$ the steps to perform will be the following:

$$c_{1,2} = \operatorname{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \sqrt{\frac{a_2^2 + b_2^2}{a_2^2 + b_2^2 + c_2^2}} = \operatorname{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2}$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a_2, b_2, c_2 are zero (provided that $o_2(a_2, b_2, c_2)$ remains in the context of the complete numbers not belonging in the line U).

Wanting to find relations that satisfy the multiplication rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients a, b, c the convention $\sqrt{x^2} = x$, with the exception of c_1 for which we should adopt the convention $\sqrt{x^2} = |x|$. The reason is simple because if we adopt for c_1 the usual convention, we will have:

$$\operatorname{sign}(c_1) \cdot \sqrt{c_1^2} = |c_1|$$

and therefore a result of the multiplication that depends on the modulus of the coordinate c_1 . While adopting $\sqrt{x^2} = |x|$ we will have:

$$\operatorname{sign}(c_1) \cdot \sqrt{c_1^2} = c_1$$

and therefore a result of the multiplication that depends on the effective value of this coordinate.

The relations obtained will be the following:

$$(2.2) \quad \begin{aligned} a_{1.2} &= -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) - b_2 \cdot \sin(\theta_1)}{\sqrt{a_2^2 + b_2^2}} \\ b_{1.2} &= -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) + b_2 \cdot \cos(\theta_1)}{\sqrt{a_2^2 + b_2^2}} \\ c_{1.2} &= c_1 \cdot \sqrt{a_2^2 + b_2^2} \end{aligned}$$

Since the number $o_2(a_2, b_2, c_2)$ is in standard representation, as determined by the Theorem 2.11 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = |\sqrt{a_2^2 + b_2^2}|$$

that combined with those indicated by formulas (2.2), proving the thesis. ■

As an example of the theorem just proved, suppose you have to multiply the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30^\circ$ by a complete number in standard representation provided with coordinates: $a_2 = 1$, $b_2 = -1$, $c_2 = 1$.

Their modulus may be calculated in the following way:

$$\begin{aligned} t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1 \\ t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \end{aligned}$$

For their phases in the case of the outgoing number we have:

$$\begin{aligned} \gamma_1 &= \text{sign}(c_1) \cdot 90^\circ = 90^\circ \\ \theta_1 &= 30^\circ \end{aligned}$$

while in the case of the complete number we should refer to the formulas related to the standard representation:

$$\begin{aligned} \gamma_2 &= \arctan\left(\frac{c_2}{|\sqrt{a_2^2 + b_2^2}|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ \\ \theta_2 &= \arctan\left(\frac{b_2}{a_2}\right) = \arctan\left(\frac{-1}{1}\right) = -45^\circ \end{aligned}$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 = \sqrt{3} \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 \simeq 125.26^\circ \\ \theta_{1.2} &= \theta_1 + \theta_2 = -15^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \cos(\theta_{1.2}) = \sqrt{3} \cdot \cos(\simeq 125.26^\circ) \cdot \cos(-15^\circ) \simeq -0.97 \\ b_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \sin(\theta_{1.2}) = \sqrt{3} \cdot \cos(\simeq 125.26^\circ) \cdot \sin(-15^\circ) \simeq 0.26 \\ c_{1.2} &= t_{1.2} \cdot \sin(\gamma_{1.2}) = \sqrt{3} \cdot \sin(\simeq 125.26^\circ) = \sqrt{2} \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{1.2} &= -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) - b_2 \cdot \sin(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} = -\frac{\cos(30^\circ) + \sin(30^\circ)}{|\sqrt{2}|} \simeq -0.97 \\ b_{1.2} &= -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) + b_2 \cdot \cos(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} = -\frac{\sin(30^\circ) - \cos(30^\circ)}{|\sqrt{2}|} \simeq 0.26 \\ c_{1.2} &= c_1 \cdot |\sqrt{a_2^2 + b_2^2}| = \sqrt{2} \end{aligned}$$

Theorem 2.36. *With only $o_1(t_1, \theta_1, \gamma_1)$ belonging to the line U and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, their multiplication may be expressed in the following way:*

$$o_{1.2}(a_{1.2}, b_{1.2}, c_{1.2})_{(t_{1.2}, \theta_{1.2}, \gamma_{1.2})} = a_{1.2}(t_1 \cdot t_2) + i \cdot b_{1.2}(\theta_1 + \theta_2) + u \cdot c_{1.2}(\gamma_1 + \gamma_2)$$

where:

$$\begin{aligned} a_{1.2} &= (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) - b_2 \cdot \sin(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} \\ b_{1.2} &= (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) + b_2 \cdot \cos(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} \\ c_{1.2} &= -c_1 \cdot |\sqrt{a_2^2 + b_2^2}| \end{aligned}$$

Proof. Since the number $o_2(a_2, b_2, c_2)$ is in complementary representation, as determined by Theorem 2.15, we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = -|\sqrt{a_2^2 + b_2^2}|$$

that combined with those indicated by the formulas (2.2), proving the thesis. ■

As an example of the theorem just proved, suppose you have to multiply the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30^\circ$ by a complete number in complementary representation provided with coordinates: $a_2 = 1$, $b_2 = -1$, $c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1$$

$$t_2 = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

For their phases in the case of the outgoing number we have:

$$\gamma_1 = \text{sign}(c_1) \cdot 90^\circ = 90^\circ$$

$$\theta_1 = 30^\circ$$

while in the case of the complete number we should refer to the formulas related to the complementary representation:

$$\gamma_2 = \arctan\left(\frac{c_2}{-|\sqrt{a_2^2 + b_2^2}|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.74^\circ$$

$$\theta_2 = \arctan\left(\frac{-b_2}{-a_2}\right) = \arctan\left(\frac{1}{-1}\right) = 135^\circ$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{1.2} = t_1 \cdot t_2 = \sqrt{3}$$

$$\gamma_{1.2} = \gamma_1 + \gamma_2 \simeq 234.74^\circ$$

$$\theta_{1.2} = \theta_1 + \theta_2 = 165^\circ$$

and the following coordinates:

$$a_{1.2} = t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \cos(\theta_{1.2}) = \sqrt{3} \cdot \cos(\simeq 234.74^\circ) \cdot \cos(165^\circ) \simeq 0.97$$

$$b_{1.2} = t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \sin(\theta_{1.2}) = \sqrt{3} \cdot \cos(\simeq 234.74^\circ) \cdot \sin(165^\circ) \simeq -0.26$$

$$c_{1.2} = t_{1.2} \cdot \sin(\gamma_{1.2}) = \sqrt{3} \cdot \sin(\simeq 234.74^\circ) = -\sqrt{2}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$a_{1.2} = (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) - b_2 \cdot \sin(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} = \frac{\cos(30^\circ) + \sin(30^\circ)}{|\sqrt{2}|} \simeq 0.97$$

$$b_{1.2} = (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) + b_2 \cdot \cos(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} = \frac{\sin(30^\circ) - \cos(30^\circ)}{|\sqrt{2}|} \simeq -0.26$$

$$c_{1.2} = -c_1 \cdot |\sqrt{a_2^2 + b_2^2}| = -\sqrt{2}$$

Theorem 2.37. *With only $o_2(t_2, \theta_2, \gamma_2)$ belonging to the line U and $o_1(t_1, \theta_1, \gamma_1)$ in standard representation, their multiplication may be expressed in the following way:*

$$o_{1.2}(a_{1.2}, b_{1.2}, c_{1.2})_{(t_{1.2}, \theta_{1.2}, \gamma_{1.2})} = a_{1.2}(t_1 \cdot t_2) + i \cdot b_{1.2}(\theta_1 + \theta_2) + u \cdot c_{1.2}(\gamma_1 + \gamma_2)$$

where:

$$a_{1.2} = -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \cos(\theta_2) - b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|}$$

$$b_{1.2} = -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \sin(\theta_2) + b_1 \cdot \cos(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|}$$

$$c_{1.2} = c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right|$$

Proof. The multiplication between two complete numbers, as we know, satisfies the following formula:

$$o_{1.2}(t_{1.2}, \theta_{1.2}, \gamma_{1.2}) = t_1 \cdot t_2 \cdot \{[\cos(\gamma_1 + \gamma_2) \cdot \cos(\theta_1 + \theta_2)] + i \cdot [\cos(\gamma_1 + \gamma_2) \cdot \sin(\theta_1 + \theta_2)] + u \cdot [\sin(\gamma_1 + \gamma_2)]\}$$

Since $o_2(t_2, \theta_2, \gamma_2)$ belongs to the line U will be provided with the following values of modulus and phases:

$$t_2 = \sqrt{c_2^2}$$

$$\gamma_2 = \text{sign}(c_2) \cdot 90^\circ$$

$$\theta_2 \text{ known } \neq \arctan\left(\frac{b_2}{a_2}\right)$$

unlike $o_1(t_1, \theta_1, \gamma_1)$ that will be provided with the following values:

$$t_1 = \sqrt{a_1^2 + b_1^2 + c_1^2}$$

$$\gamma_1 = \arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right)$$

$$\theta_1 = \arctan\left(\frac{b_1}{a_1}\right)$$

This means that we can write the coordinates sought in the following way:

$$a_{1.2} = \sqrt{c_1^2 + b_1^2 + c_1^2} \cdot \sqrt{c_2^2} \cdot \cos\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right) + \text{sign}(c_2) \cdot 90^\circ\right] \cdot \cos\left[\arctan\left(\frac{b_1}{a_1}\right) + \theta_2\right]$$

$$b_{1.2} = \sqrt{c_1^2 + b_1^2 + c_1^2} \cdot \sqrt{c_2^2} \cdot \cos\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right) + \text{sign}(c_2) \cdot 90^\circ\right] \cdot \sin\left[\arctan\left(\frac{b_1}{a_1}\right) + \theta_2\right]$$

$$c_{1.2} = \sqrt{c_1^2 + b_1^2 + c_1^2} \cdot \sqrt{c_2^2} \cdot \sin\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right) + \text{sign}(c_2) \cdot 90^\circ\right]$$

To continue with the proof, we have to use the following trigonometric relations:

$$\begin{aligned}\cos(x+y) &= \cos(x) \cdot \cos(y) - \sin(x) \cdot \sin(y) \\ \sin(x+y) &= \sin(x) \cdot \cos(y) + \cos(x) \cdot \sin(y) \\ \cos\left[\arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)\right] &= \sqrt{\frac{a^2+b^2}{a^2+b^2+c^2}} \\ \sin\left[\arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)\right] &= \sqrt{\frac{c^2}{a^2+b^2+c^2}} \\ \cos\left[\arctan\left(\frac{b}{a}\right)\right] &= \sqrt{\frac{a^2}{a^2+b^2}} \\ \sin\left[\arctan\left(\frac{b}{a}\right)\right] &= \sqrt{\frac{b^2}{a^2+b^2}} \\ \cos[x + \text{sign}(y) \cdot 90^\circ] &= -\text{sign}(y) \cdot \sin(x) \\ \sin[x + \text{sign}(y) \cdot 90^\circ] &= \text{sign}(y) \cdot \cos(x)\end{aligned}$$

To determine the value of the coordinate $a_{1.2}$ the steps to perform will be the following:

$$\begin{aligned}a_{1.2} &= -\text{sign}(c_2) \cdot \sqrt{c_1^2 + b_1^2 + c_1^2} \cdot \sqrt{c_2^2} \cdot \sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \\ &\cdot \left[\sqrt{\frac{a_1^2}{a_1^2 + b_1^2}} \cdot \cos(\theta_2) - \sqrt{\frac{b_1^2}{a_1^2 + b_1^2}} \cdot \sin(\theta_2) \right] = \\ &= -\text{sign}(c_2) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \frac{\sqrt{a_1^2} \cdot \cos(\theta_2) - \sqrt{b_1^2} \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}}\end{aligned}$$

To determine the value of the coordinate $b_{1.2}$ the steps to perform will be the following:

$$\begin{aligned}b_{1.2} &= -\text{sign}(c_2) \cdot \sqrt{c_1^2 + b_1^2 + c_1^2} \cdot \sqrt{c_2^2} \cdot \sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \\ &\cdot \left[\sqrt{\frac{b_1^2}{a_1^2 + b_1^2}} \cdot \cos(\theta_2) + \sqrt{\frac{a_1^2}{a_1^2 + b_1^2}} \cdot \sin(\theta_2) \right] = \\ &= -\text{sign}(c_2) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \frac{\sqrt{b_1^2} \cdot \cos(\theta_2) + \sqrt{a_1^2} \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}}\end{aligned}$$

To determine the value of the coordinate $c_{1.2}$ the steps to perform will be the following:

$$c_{1.2} = \text{sign}(c_2) \cdot \sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{c_2^2} \cdot \sqrt{\frac{a_1^2 + b_1^2}{a_1^2 + b_1^2 + c_1^2}} = \text{sign}(c_2) \cdot \sqrt{c_2^2} \cdot \sqrt{a_1^2 + b_1^2}$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a_1, b_1, c_1 are zero (provided that $o_1(a_1, b_1, c_1)$ remains in the context of the complete numbers not belonging in the line U).

Wanting to find relations that satisfy the multiplication rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients a,b,c the convention $\sqrt{x^2} = x$, with the exception of c_2 for which we should adopt the convention $\sqrt{x^2} = |x|$. The reason is simple because if we adopt for c_2 the usual convention, we will have:

$$\text{sign}(c_2) \cdot \sqrt{c_2^2} = |c_2|$$

and therefore a result of the multiplication that depends on the modulus of the coordinate c_2 . While adopting $\sqrt{x^2} = |x|$ we will have:

$$\text{sign}(c_2) \cdot \sqrt{c_2^2} = c_2$$

and therefore a result of the multiplication that depends on the effective value of this coordinate.

The relations obtained will be the following:

$$(2.3) \quad \begin{aligned} a_{1.2} &= -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \cos(\theta_2) - b_1 \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}} \\ b_{1.2} &= -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \sin(\theta_2) + b_1 \cdot \cos(\theta_2)}{\sqrt{a_1^2 + b_1^2}} \\ c_{1.2} &= c_2 \cdot \sqrt{a_1^2 + b_1^2} \end{aligned}$$

Since the number $o_1(a_1, b_1, c_1)$ is in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = |\sqrt{a_1^2 + b_1^2}|$$

that combined with those indicated by the formulas (2.3), proving the thesis. ■

As an example of the theorem just proved, suppose you have to multiply the complete number in standard representation provided with coordinates: $a_1 = 1$, $b_1 = -1$, $c_1 = 1$ by an outgoing numbers of coordinate: $c_2 = 1$ and phase $\theta_2 = 30^\circ$.

Their modulus may be calculated in the following way:

$$\begin{aligned} t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \\ t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1 \end{aligned}$$

For their phases in the case of the outgoing number we have:

$$\begin{aligned} \gamma_2 &= \text{sign}(c_2) \cdot 90^\circ = 90^\circ \\ \theta_2 &= 30^\circ \end{aligned}$$

while in the case of the complete number we should refer to the formulas related to the standard representation:

$$\begin{aligned}\gamma_1 &= \arctan \left(\frac{c_1}{|\sqrt{a_1^2 + b_1^2}|} \right) = \arctan \left(\frac{1}{|\sqrt{2}|} \right) \simeq 35.26^\circ \\ \theta_1 &= \arctan \left(\frac{b_1}{a_1} \right) = \arctan \left(\frac{-1}{1} \right) = -45^\circ\end{aligned}$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned}t_{1.2} &= t_1 \cdot t_2 = \sqrt{3} \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 \simeq 125.26^\circ \\ \theta_{1.2} &= \theta_1 + \theta_2 = -15^\circ\end{aligned}$$

and the following coordinates:

$$\begin{aligned}a_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \cos(\theta_{1.2}) = \sqrt{3} \cdot \cos(\simeq 125.26^\circ) \cdot \cos(-15^\circ) \simeq -0.97 \\ b_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \sin(\theta_{1.2}) = \sqrt{3} \cdot \cos(\simeq 125.26^\circ) \cdot \sin(-15^\circ) \simeq 0.26 \\ c_{1.2} &= t_{1.2} \cdot \sin(\gamma_{1.2}) = \sqrt{3} \cdot \sin(\simeq 125.26^\circ) = \sqrt{2}\end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned}a_{1.2} &= -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \cos(\theta_2) - b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} = -\frac{\cos(30^\circ) + \sin(30^\circ)}{|\sqrt{2}|} \simeq -0.97 \\ b_{1.2} &= -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \sin(\theta_2) + b_1 \cdot \cos(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} = -\frac{\sin(30^\circ) - \cos(30^\circ)}{|\sqrt{2}|} \simeq 0.26 \\ c_{1.2} &= c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| = \sqrt{2}\end{aligned}$$

Theorem 2.38. *With only $o_2(t_2, \theta_2, \gamma_2)$ belonging to the line U and $o_1(t_1, \theta_1, \gamma_1)$ in complementary representation, their multiplication may be expressed in the following way:*

$$o_{1.2}(a_{1.2}, b_{1.2}, c_{1.2})_{(t_{1.2}, \theta_{1.2}, \gamma_{1.2})} = a_{1.2}(t_1 \cdot t_2) + i \cdot b_{1.2}(\theta_1 + \theta_2) + u \cdot c_{1.2}(\gamma_1 + \gamma_2)$$

where

$$\begin{aligned}a_{1.2} &= (c_1 \cdot c_2) \cdot \frac{a_1 \cdot \cos(\theta_2) - b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} \\ b_{1.2} &= (c_1 \cdot c_2) \cdot \frac{a_1 \cdot \sin(\theta_2) + b_1 \cdot \cos(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} \\ c_{1.2} &= -c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right|\end{aligned}$$

Proof. Since the number $o_1(a_1, b_1, c_1)$ is in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = -|\sqrt{a_1^2 + b_1^2}|$$

that combined with those indicated by the formulas (2.3), proving the thesis. ■

As an example of the theorem just proved, suppose you have to multiply a complete number in complementary representation provided with coordinates: $a_2 = 1, b_2 = -1, c_2 = 1$ by the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30^\circ$.

Their modulus may be calculated in the following way:

$$t_1 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

$$t_2 = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1$$

For their phases in the case of the outgoing number we have:

$$\gamma_2 = \text{sign}(c_2) \cdot 90^\circ = 90^\circ$$

$$\theta_2 = 30^\circ$$

while in the case of the complete number we should refer to the formulas related to the complementary representation:

$$\gamma_2 = \arctan\left(\frac{c_1}{-|\sqrt{a_1^2 + b_1^2}|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.74^\circ$$

$$\theta_2 = \arctan\left(\frac{-b_1}{-a_1}\right) = \arctan\left(\frac{1}{-1}\right) = 135^\circ$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{1.2} = t_1 \cdot t_2 = \sqrt{3}$$

$$\gamma_{1.2} = \gamma_1 + \gamma_2 \simeq 234.74^\circ$$

$$\theta_{1.2} = \theta_1 + \theta_2 = 165^\circ$$

and the following coordinates:

$$a_{1.2} = t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \cos(\theta_{1.2}) = \sqrt{3} \cdot \cos(\simeq 234.74^\circ) \cdot \cos(165^\circ) \simeq 0.97$$

$$b_{1.2} = t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \sin(\theta_{1.2}) = \sqrt{3} \cdot \cos(\simeq 234.74^\circ) \cdot \sin(165^\circ) \simeq -0.26$$

$$c_{1.2} = t_{1.2} \cdot \sin(\gamma_{1.2}) = \sqrt{3} \cdot \sin(\simeq 234.74^\circ) = -\sqrt{2}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{1.2} &= (c_1 \cdot c_2) \cdot \frac{a_1 \cdot \cos(\theta_2) - b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} = \frac{\cos(30^\circ) + \sin(30^\circ)}{|\sqrt{2}|} \simeq 0.97 \\ b_{1.2} &= (c_1 \cdot c_2) \cdot \frac{a_1 \cdot \sin(\theta_2) + b_1 \cdot \cos(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} = \frac{\sin(30^\circ) - \cos(30^\circ)}{|\sqrt{2}|} \simeq -0.26 \\ c_{1.2} &= -c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| = -\sqrt{2} \end{aligned}$$

Theorem 2.39. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ both belonging to the line U , their multiplication may be expressed in the following way:*

$$o_{1.2}(a_{1.2}, b_{1.2}, c_{1.2})_{(t_{1.2}, \theta_{1.2}, \gamma_{1.2})} = a_{1.2}(t_1 \cdot t_2) + i \cdot b_{1.2}(\theta_1 + \theta_2) + u \cdot c_{1.2}(\gamma_1 + \gamma_2)$$

where

$$\begin{aligned} a_{1.2} &= -(c_1 \cdot c_2) \cdot \cos(\theta_1 + \theta_2) \\ b_{1.2} &= -(c_1 \cdot c_2) \cdot \sin(\theta_1 + \theta_2) \\ c_{1.2} &= 0 \end{aligned}$$

Proof. The multiplication between two complete numbers, as we know, satisfies the following formula:

$$\begin{aligned} o_{1.2}(t_{1.2}, \theta_{1.2}, \gamma_{1.2}) &= t_1 \cdot t_2 \cdot \{[\cos(\gamma_1 + \gamma_2) \cdot \cos(\theta_1 + \theta_2)] + \\ &+ i \cdot [\cos(\gamma_1 + \gamma_2) \cdot \sin(\theta_1 + \theta_2)] + u \cdot [\sin(\gamma_1 + \gamma_2)]\} \end{aligned}$$

Since $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ belong to the line U will be provided with the following values of modulus and phases:

$$\begin{aligned} t_1 &= \sqrt{c_1^2} \\ t_2 &= \sqrt{c_2^2} \\ \gamma_1 &= \text{sign}(c_1) \cdot 90^\circ \\ \gamma_2 &= \text{sign}(c_2) \cdot 90^\circ \\ \theta_1 \text{ known} &\neq \arctan\left(\frac{b_1}{a_1}\right) \\ \theta_2 \text{ known} &\neq \arctan\left(\frac{b_2}{a_2}\right) \end{aligned}$$

This means that we can write the coordinates sought in the following way:

$$\begin{aligned} a_{1.2} &= \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \cos[\text{sign}(c_1) \cdot 90^\circ + \text{sign}(c_2) \cdot 90^\circ] \cdot \cos(\theta_1 + \theta_2) \\ b_{1.2} &= \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \cos[\text{sign}(c_1) \cdot 90^\circ + \text{sign}(c_2) \cdot 90^\circ] \cdot \sin(\theta_1 + \theta_2) \\ c_{1.2} &= \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \sin[\text{sign}(c_1) \cdot 90^\circ + \text{sign}(c_2) \cdot 90^\circ] \end{aligned}$$

Considering that when c_1 and c_2 have the same sign we obtained:

$$\begin{aligned}\cos [\text{sign}(c_1) \cdot 90^\circ + \text{sign}(c_2) \cdot 90^\circ] &= \cos(\pm 180^\circ) = -1 = -\text{sign}(c_1) \cdot \text{sign}(c_2) \\ \sin [\text{sign}(c_1) \cdot 90^\circ + \text{sign}(c_2) \cdot 90^\circ] &= \sin(\pm 180^\circ) = 0\end{aligned}$$

and that when they have the opposite sign we obtained:

$$\begin{aligned}\cos [\text{sign}(c_1) \cdot 90^\circ + \text{sign}(c_2) \cdot 90^\circ] &= \cos(\pm 0^\circ) = 1 = -\text{sign}(c_1) \cdot \text{sign}(c_2) \\ \sin [\text{sign}(c_1) \cdot 90^\circ + \text{sign}(c_2) \cdot 90^\circ] &= \sin(\pm 0^\circ) = 0\end{aligned}$$

we can write:

$$\begin{aligned}a_{1.2} &= -\text{sign}(c_1) \cdot \text{sign}(c_2) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \cos(\theta_1 + \theta_2) \\ b_{1.2} &= -\text{sign}(c_1) \cdot \text{sign}(c_2) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \sin(\theta_1 + \theta_2) \\ c_{1.2} &= 0\end{aligned}$$

Wanting to find relations that satisfy the multiplication rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients $c_{1,2}$ the convention $\sqrt{x^2} = |x|$. In fact in this way we obtain:

$$\begin{aligned}\text{sign}(c_1) \cdot \sqrt{c_1^2} &= c_1 \\ \text{sign}(c_2) \cdot \sqrt{c_2^2} &= c_2\end{aligned}$$

and therefore a result of the multiplication that depends on the effective value of this coordinate. The relation that we obtain following these conventions proves the thesis. ■

As an example of the theorem just proved, suppose you have to multiply the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30^\circ$ by the outgoing number of coordinate: $c_2 = 1$ and phase $\theta_2 = 30^\circ$.

Their modulus may be calculated in the following way:

$$\begin{aligned}t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1 \\ t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1\end{aligned}$$

For their phases we have:

$$\begin{aligned}\gamma_1 &= \text{sign}(c_1) \cdot 90^\circ = 90^\circ \\ \gamma_2 &= \text{sign}(c_2) \cdot 90^\circ = 90^\circ \\ \theta_1 &= 30^\circ \\ \theta_2 &= 30^\circ\end{aligned}$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned}t_{1.2} &= t_1 \cdot t_2 = 1 \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 = 180^\circ \\ \theta_{1.2} &= \theta_1 + \theta_2 = 60^\circ\end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \cos(\theta_{1.2}) = 1 \cdot \cos(180^\circ) \cdot \cos(60^\circ) = -\frac{1}{2} \\ b_{1.2} &= t_{1.2} \cdot \cos(\gamma_{1.2}) \cdot \sin(\theta_{1.2}) = 1 \cdot \cos(180^\circ) \cdot \sin(60^\circ) = -\frac{\sqrt{3}}{2} \\ c_{1.2} &= t_{1.2} \cdot \sin(\gamma_{1.2}) = 1 \cdot \sin(180^\circ) = 0 \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{1.2} &= -(c_1 \cdot c_2) \cdot \cos(\theta_1 + \theta_2) = -\cos(60^\circ) = -\frac{1}{2} \\ b_{1.2} &= -(c_1 \cdot c_2) \cdot \sin(\theta_1 + \theta_2) = -\sin(60^\circ) = -\frac{\sqrt{3}}{2} \\ c_{1.2} &= 0 \end{aligned}$$

Theorem 2.40. *For the operation of multiplication is defined null the complete number 0, namely for:*

$$o_2(t_2, \theta_2, \gamma_2) = 0$$

we have:

$$o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) = 0$$

Proof. $t_1, \theta_1, \gamma_1, t_2, \theta_2, \gamma_2$ being real numbers, we can write:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 = t_1 \cdot 0 = 0 \\ \theta_{1.2} &= \theta_1 + \theta_2 = \theta_1 + \textit{indeterminate} = \textit{indeterminate} \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 = \gamma_1 + \textit{indeterminate} = \textit{indeterminate} \end{aligned}$$

proving the thesis. ■

Theorem 2.41. *For the operation of multiplication is defined neuter the complete number $1_{(S)}$, namely for:*

$$o_2(a_2, b_2, c_2)_{(S)} = 1_{(S)}$$

we have:

$$o_1(a_1, b_1, c_1)_{(t_1, \theta_1, \gamma_1)} \cdot o_2(a_2, b_2, c_2)_{(S)} = o_1(a_1, b_1, c_1)_{(t_1, \theta_1, \gamma_1)}$$

Proof. $t_1, \theta_1, \gamma_1, t_2, \theta_2, \gamma_2$ being real numbers, we can write:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 = t_1 \cdot 1 = t_1 \\ \theta_{1.2} &= \theta_1 + \theta_2 = \theta_1 + 0 = \theta_1 \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 = \gamma_1 + 0 = \gamma_1 \end{aligned}$$

proving the thesis. ■

Theorem 2.42. *For the operation of multiplication is defined inverse the complete number that identifies the inverse position with respect the origin, namely for:*

$$o_2(t_2, \theta_2, \gamma_2) = o_2\left(\frac{1}{t_1}, -\theta_1, -\gamma_1\right)$$

we have:

$$o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) = 1_{(S)}$$

Proof. $t_1, \theta_1, \gamma_1, t_2, \theta_2, \gamma_2$ being real numbers, we can write:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 = t_1 \cdot \frac{1}{t_1} = 1 \\ \theta_{1.2} &= \theta_1 + \theta_2 = \theta_1 - \theta_1 = 0 \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 = \gamma_1 - \gamma_1 = 0 \end{aligned}$$

proving the thesis. ■

Theorem 2.43. *For the operation of multiplication is valid the commutative property, namely:*

$$o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) = o_2(t_2, \theta_2, \gamma_2) \cdot o_1(t_1, \theta_1, \gamma_1)$$

Proof. $t_1, \theta_1, \gamma_1, t_2, \theta_2, \gamma_2$ being real numbers, we can write:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 \\ \theta_{1.2} &= \theta_1 + \theta_2 \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 \\ \\ t_{2.1} &= t_2 \cdot t_1 = t_1 \cdot t_2 \\ \theta_{2.1} &= \theta_2 + \theta_1 = \theta_1 + \theta_2 \\ \gamma_{2.1} &= \gamma_2 + \gamma_1 = \gamma_1 + \gamma_2 \end{aligned}$$

proving the thesis. ■

Theorem 2.44. *For the operation of multiplication are valid the associative and dissociative properties, namely for:*

$$o_2(t_2, \theta_2, \gamma_2) = o_3(t_3, \theta_3, \gamma_3) + o_4(t_4, \theta_4, \gamma_4)$$

we have:

$$\begin{aligned} [o_1(t_1, \theta_1, \gamma_1) \cdot o_3(t_3, \theta_3, \gamma_3)] \cdot o_4(t_4, \theta_4, \gamma_4) &= o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) \\ o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) &= [o_1(t_1, \theta_1, \gamma_1) \cdot o_3(t_3, \theta_3, \gamma_3)] \cdot o_4(t_4, \theta_4, \gamma_4) \end{aligned}$$

Proof. $t_1, \theta_1, \gamma_1, t_2, \theta_2, \gamma_2, t_3, \theta_3, \gamma_3, t_4, \theta_4, \gamma_4$ being real numbers, we can write:

$$\begin{aligned} t_{(1.3).4} &= (t_1 \cdot t_3) \cdot t_4 = t_1 \cdot (t_3 \cdot t_4) \\ \theta_{(1.3).4} &= (\theta_1 + \theta_3) + \theta_4 = \theta_1 + (\theta_3 + \theta_4) \\ \gamma_{(1.3).4} &= (\gamma_1 + \gamma_3) + \gamma_4 = \gamma_1 + (\gamma_3 + \gamma_4) \end{aligned}$$

$$\begin{aligned}
t_{1.2} &= t_1 \cdot t_2 = t_1 \cdot (t_3 \cdot t_4) \\
\theta_{1.2} &= \theta_1 + \theta_2 = \theta_1 + (\theta_3 + \theta_4) \\
\gamma_{1.2} &= \gamma_1 + \gamma_2 = \gamma_1 + (\gamma_3 + \gamma_4)
\end{aligned}$$

proving the thesis. ■

Theorem 2.45. *It is not valid the distributive property of multiplication over addition, namely for:*

$$o_2(t_2, \theta_2, \gamma_2) = o_3(t_3, \theta_3, \gamma_3) + o_4(t_4, \theta_4, \gamma_4)$$

we have:

$$o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) \neq [o_1(t_1, \theta_1, \gamma_1) \cdot o_3(t_3, \theta_3, \gamma_3)] + [o_1(t_1, \theta_1, \gamma_1) \cdot o_4(t_4, \theta_4, \gamma_4)]$$

Proof. Referring to the situation described by theorem 2.31 and considering that $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, a_4, b_4, c_4$ are real numbers, we can write:

$$\begin{aligned}
c_{1.2} &= c_1 \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| + c_2 \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \\
c_{(1.3)+(1.4)} &= \left[c_1 \cdot \left| \sqrt{(a_3^2 + b_3^2)} \right| + c_3 \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \right] \\
&\quad + \left[c_1 \cdot \left| \sqrt{(a_4^2 + b_4^2)} \right| + c_4 \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \right] \\
&= c_1 \cdot \left[\left| \sqrt{(a_3^2 + b_3^2)} \right| + \left| \sqrt{(a_4^2 + b_4^2)} \right| \right] + (c_3 + c_4) \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \\
&= c_1 \cdot \left[\left| \sqrt{(a_3^2 + b_3^2)} \right| + \left| \sqrt{(a_4^2 + b_4^2)} \right| \right] + c_2 \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \neq c_{1.2}
\end{aligned}$$

proving the thesis. ■

Theorem 2.46. *It is not valid the distributive property of multiplication over subtraction, namely for:*

$$o_2(t_2, \theta_2, \gamma_2) = o_3(t_3, \theta_3, \gamma_3) - o_4(t_4, \theta_4, \gamma_4)$$

we have:

$$o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) \neq [o_1(t_1, \theta_1, \gamma_1) \cdot o_3(t_3, \theta_3, \gamma_3)] - [o_1(t_1, \theta_1, \gamma_1) \cdot o_4(t_4, \theta_4, \gamma_4)]$$

Proof. Referring to the situation described by theorem 2.31 and considering that $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, a_4, b_4, c_4$ are real numbers, we can write:

$$\begin{aligned}
c_{1.2} &= c_1 \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| + c_2 \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \\
c_{(1.3)-(1.4)} &= \left[c_1 \cdot \left| \sqrt{(a_3^2 + b_3^2)} \right| + c_3 \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \right] \\
&\quad - \left[c_1 \cdot \left| \sqrt{(a_4^2 + b_4^2)} \right| + c_4 \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \right] \\
&= c_1 \cdot \left[\left| \sqrt{(a_3^2 + b_3^2)} \right| - \left| \sqrt{(a_4^2 + b_4^2)} \right| \right] + (c_3 - c_4) \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \\
&= c_1 \cdot \left[\left| \sqrt{(a_3^2 + b_3^2)} \right| - \left| \sqrt{(a_4^2 + b_4^2)} \right| \right] + c_2 \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \neq c_{1.2}
\end{aligned}$$

proving the thesis. ■

2.5. Divisiton

Definition 2.47. In the space RIU we can define division between two complete numbers $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ as the number $o_{\frac{1}{2}}(a_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})$ represented also with the symbol $\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)}$ that satisfies the following conditions:

1. $o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}) \cdot o_2(t_2, \theta_2, \gamma_2) = o_1(t_1, \theta_1, \gamma_1)$
2. $o_2(t_2, \theta_2, \gamma_2) \neq 0$

The first condition defines the division as the inverse operation of multiplication, and it is equivalent to require that:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \end{aligned}$$

The second condition gets its own justification by the necessity of defining the divisions in an univocal way. In fact when that condition is not valid, the expression:

$$o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}) \cdot 0 = 0$$

besides to require a zero dividend $o_1(t_1, \theta_1, \gamma_1)$ as well, would be satisfied by more values of $o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})$.

Theorem 2.48. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, and both not belonging to the line U , their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}}(\frac{t_1}{t_2}) + i \cdot b_{\frac{1}{2}}(\theta_1 - \theta_2) + u \cdot c_{\frac{1}{2}}(\gamma_1 - \gamma_2)$$

where

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| - c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| \right] \end{aligned}$$

Proof. The division between two complete numbers, as we know, satisfies the following formula:

$$\begin{aligned} o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}) &= \frac{t_1}{t_2} \cdot \{ [\cos(\gamma_1 - \gamma_2) \cdot \cos(\theta_1 - \theta_2)] \\ &+ i \cdot [\cos(\gamma_1 - \gamma_2) \cdot \sin(\theta_1 - \theta_2)] + u \cdot [\sin(\gamma_1 - \gamma_2)] \} \end{aligned}$$

For the moduli and the phases involved will be valid the following relation as well:

$$\begin{aligned} t &= \sqrt{a^2 + b^2 + c^2} \\ \gamma &= \arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right) \\ \theta &= \arctan \left(\frac{b}{a} \right) \end{aligned}$$

This means that we can write the coordinates sought in the following way:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{\sqrt{a_1^2 + b_1^2 + c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \cos \left[\arctan \left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} \right) - \arctan \left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}} \right) \right] \\ &\quad \cdot \cos \left[\arctan \left(\frac{b_1}{a_1} \right) - \arctan \left(\frac{b_2}{a_2} \right) \right] \\ b_{\frac{1}{2}} &= \frac{\sqrt{a_1^2 + b_1^2 + c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \cos \left[\arctan \left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} \right) - \arctan \left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}} \right) \right] \\ &\quad \cdot \sin \left[\arctan \left(\frac{b_1}{a_1} \right) - \arctan \left(\frac{b_2}{a_2} \right) \right] \\ c_{\frac{1}{2}} &= \frac{\sqrt{a_1^2 + b_1^2 + c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \sin \left[\arctan \left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} \right) - \arctan \left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}} \right) \right] \end{aligned}$$

To continue with the proof, we have to use the following trigonometric relations:

$$\begin{aligned} \cos(x - y) &= \cos(x) \cdot \cos(y) + \sin(x) \cdot \sin(y) \\ \sin(x - y) &= \sin(x) \cdot \cos(y) - \cos(x) \cdot \sin(y) \\ \cos \left[\arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right) \right] &= \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + c^2}} \\ \sin \left[\arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right) \right] &= \sqrt{\frac{c^2}{a^2 + b^2 + c^2}} \\ \cos \left[\arctan \left(\frac{b}{a} \right) \right] &= \sqrt{\frac{a^2}{a^2 + b^2}} \\ \sin \left[\arctan \left(\frac{b}{a} \right) \right] &= \sqrt{\frac{b^2}{a^2 + b^2}} \end{aligned}$$

To determine the value of the coordinate $a_{\frac{1}{2}}$ the steps to perform will be the following:

$$\begin{aligned}
 a_{\frac{1}{2}} &= \frac{\sqrt{a_1^2 + b_1^2 + c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \left(\sqrt{\frac{a_1^2 + b_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{a_2^2 + b_2^2}{a_2^2 + b_2^2 + c_2^2}} \right. \\
 &\quad \left. + \sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \right) \cdot \left(\sqrt{\frac{a_1^2}{a_1^2 + b_1^2}} \cdot \sqrt{\frac{a_2^2}{a_2^2 + b_2^2}} \right. \\
 &\quad \left. + \sqrt{\frac{b_1^2}{a_1^2 + b_1^2}} \cdot \sqrt{\frac{b_2^2}{a_2^2 + b_2^2}} \right) \\
 &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left(\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2} + \sqrt{c_1^2} \cdot \sqrt{c_2^2} \right) \cdot \\
 &\quad \cdot \left(\frac{\sqrt{a_1^2} \cdot \sqrt{a_2^2} + \sqrt{b_1^2} \cdot \sqrt{b_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right) \\
 &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[\left(\sqrt{a_1^2} \cdot \sqrt{a_2^2} + \sqrt{b_1^2} \cdot \sqrt{b_2^2} \right) \right. \\
 &\quad \left. + \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \left(\frac{\sqrt{a_1^2} \cdot \sqrt{a_2^2} + \sqrt{b_1^2} \cdot \sqrt{b_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right) \right] \\
 &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left(\sqrt{a_1^2} \cdot \sqrt{a_2^2} + \sqrt{b_1^2} \cdot \sqrt{b_2^2} \right) \cdot \left(1 + \frac{\sqrt{c_1^2} \cdot \sqrt{c_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right)
 \end{aligned}$$

To determine the value of the coordinate $b_{\frac{1}{2}}$ the steps to perform will be the following:

$$\begin{aligned}
 b_{\frac{1}{2}} &= \frac{\sqrt{a_1^2 + b_1^2 + c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \left(\sqrt{\frac{a_1^2 + b_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{a_2^2 + b_2^2}{a_2^2 + b_2^2 + c_2^2}} \right. \\
 &\quad \left. + \sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \right) \cdot \left(\sqrt{\frac{b_1^2}{a_1^2 + b_1^2}} \cdot \sqrt{\frac{a_2^2}{a_2^2 + b_2^2}} \right. \\
 &\quad \left. - \sqrt{\frac{a_1^2}{a_1^2 + b_1^2}} \cdot \sqrt{\frac{b_2^2}{a_2^2 + b_2^2}} \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left(\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2} + \sqrt{c_1^2} \cdot \sqrt{c_2^2} \right) \cdot \\
&\quad \cdot \left(\frac{\sqrt{b_1^2} \cdot \sqrt{a_2^2} - \sqrt{a_1^2} \cdot \sqrt{b_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right) \\
&= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[\left(\sqrt{b_1^2} \cdot \sqrt{a_2^2} - \sqrt{a_1^2} \cdot \sqrt{b_2^2} \right) \right. \\
&\quad \left. + \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \left(\frac{\sqrt{b_1^2} \cdot \sqrt{a_2^2} - \sqrt{a_1^2} \cdot \sqrt{b_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right) \right] \\
&= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left(\sqrt{b_1^2} \cdot \sqrt{a_2^2} - \sqrt{a_1^2} \cdot \sqrt{b_2^2} \right) \cdot \left(1 + \frac{\sqrt{c_1^2} \cdot \sqrt{c_2^2}}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}} \right)
\end{aligned}$$

To determine the value of the coordinate $c_{\frac{1}{2}}$ the steps to perform will be the following:

$$\begin{aligned}
c_{\frac{1}{2}} &= \frac{\sqrt{a_1^2 + b_1^2 + c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \left(\sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{a_2^2 + b_2^2}{a_2^2 + b_2^2 + c_2^2}} + \right. \\
&\quad \left. - \sqrt{\frac{a_1^2 + b_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \right) = \\
&= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[\sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2} - \sqrt{c_2^2} \cdot \sqrt{a_1^2 + b_1^2} \right]
\end{aligned}$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a,b,c are zero (provided that we work with complete numbers not belonging in the line U). The only limitation in this regard is the need to avoid the following situation:

$$a_2^2 + b_2^2 + c_2^2 = 0$$

which confirms the impossibility to divide a complete number $o(t, \theta, \gamma)$ for zero (characterized by the values a_2, b_2, c_2 that make the above mentioned condition true).

Wanting to find relations that satisfy the division rule as a function of the effective coordinates of the complete numbers involved, we must assign to the roots the same sign of the coefficient located within them:

$$\begin{aligned}\sqrt{a^2} &= a \\ \sqrt{b^2} &= b \\ \sqrt{c^2} &= c\end{aligned}$$

The relations obtained will be the following:

$$(2.4) \quad \begin{aligned}a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}}\right) \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}}\right) \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot [c_1 \cdot \sqrt{a_2^2 + b_2^2} + c_2 \cdot \sqrt{a_1^2 + b_1^2}]\end{aligned}$$

Since the complete numbers involved are in standard representation, as determined by the theorem 2.11 we must consider the following relations:

$$\begin{aligned}\sqrt{a_1^2 + b_1^2} &= |\sqrt{a_1^2 + b_1^2}| \\ \sqrt{a_2^2 + b_2^2} &= |\sqrt{a_2^2 + b_2^2}|\end{aligned}$$

that combined with those indicated by the formulas (2.4), proving the thesis. ■

As an example of the theorem just proved, suppose you have to divide the complete numbers in standard representation provided with coordinates: $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the standard representation:

$$\begin{aligned}\gamma_1 = \gamma_2 &= \arctan\left(\frac{c_1}{|\sqrt{a_1^2 + b_1^2}|}\right) = \arctan\left(\frac{c_2}{|\sqrt{a_2^2 + b_2^2}|}\right) = \\ &= \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ \\ \theta_1 = \theta_2 &= \arctan\left(\frac{b_1}{a_1}\right) = \arctan\left(\frac{b_2}{a_2}\right) = \arctan\left(\frac{1}{1}\right) = 45^\circ\end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned}t_{\frac{1}{2}} &= \frac{t_1}{t_2} = 1 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 = 0^\circ \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = 0^\circ\end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = 1 \cdot \cos(0^\circ) \cdot \cos(0^\circ) = 1 \\ b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = 1 \cdot \cos(0^\circ) \cdot \sin(0^\circ) = 0 \\ c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = 1 \cdot \sin(0^\circ) = 0 \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= \frac{1}{3} \cdot (1 + 1) \cdot \left(1 + \frac{1}{2} \right) = 1 \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= \frac{1}{3} \cdot (1 - 1) \cdot \left(1 + \frac{1}{2} \right) = 0 \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| - c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| \right] = \\ &= \frac{1}{3} \cdot [1 \cdot \sqrt{2} - 1 \cdot \sqrt{2}] = 0 \end{aligned}$$

Theorem 2.49. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, and both not belonging to the line U , their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}}(\frac{t_1}{t_2}) + i \cdot b_{\frac{1}{2}}(\theta_1 - \theta_2) + u \cdot c_{\frac{1}{2}}(\gamma_1 - \gamma_2)$$

where

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| - c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| \right] \end{aligned}$$

Proof. Since the complete numbers involved are in complementary representation, as determined by the theorem 2.15 we must consider the following relations:

$$\begin{aligned} \sqrt{a_1^2 + b_1^2} &= -\left| \sqrt{a_1^2 + b_1^2} \right| \\ \sqrt{a_2^2 + b_2^2} &= -\left| \sqrt{a_2^2 + b_2^2} \right| \end{aligned}$$

that combined with those indicated by the formulas (2.4), proving the thesis. ■

As an example of the theorem just proved, suppose you have to divide the complete numbers in complementary representation provided with coordinates: $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the complementary representation:

$$\begin{aligned} \gamma_1 = \gamma_2 &= \arctan \left(\frac{c_1}{-|\sqrt{a_1^2 + b_1^2}|} \right) = \arctan \left(\frac{c_2}{-|\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= \arctan \left(\frac{1}{-|\sqrt{2}|} \right) \simeq 144.73^\circ \\ \theta_1 = \theta_2 &= \arctan \left(\frac{-b_1}{-a_1} \right) = \arctan \left(\frac{-b_2}{-a_2} \right) = \arctan \left(\frac{-1}{-1} \right) = 225^\circ \end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} = 1 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 = 0^\circ \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = 0^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = 1 \cdot \cos(0^\circ) \cdot \cos(0^\circ) = 1 \\ b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = 1 \cdot \cos(0^\circ) \cdot \sin(0^\circ) = 0 \\ c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = 1 \cdot \sin(0^\circ) = 0 \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ &= \frac{1}{3} \cdot (1 + 1) \cdot \left(1 + \frac{1}{2} \right) = 1 \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ &= \frac{1}{3} \cdot (1 - 1) \cdot \left(1 + \frac{1}{2} \right) = 0 \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| - c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| \right] \\ &= \frac{1}{3} \cdot [1 \cdot \sqrt{2} - 1 \cdot \sqrt{2}] = 0 \end{aligned}$$

Theorem 2.50. *With $o_1(t_1, \theta_1, \gamma_1)$ in standard representation and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, and both not belonging to the line U , their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}} \left(\frac{t_1}{t_2} \right) + i \cdot b_{\frac{1}{2}} (\theta_1 - \theta_2) + u \cdot c_{\frac{1}{2}} (\gamma_1 - \gamma_2)$$

where

$$a_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right)$$

$$b_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right)$$

$$c_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[-c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| - c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| \right]$$

Proof. Since the dividend is in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = \left| \sqrt{a_1^2 + b_1^2} \right|$$

while being the divisor in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = - \left| \sqrt{a_2^2 + b_2^2} \right|$$

that combined with those indicated by the formulas (2.4), proving the thesis. ■

As an example of the theorem just proved, suppose you have to divide the complete number in standard representation provided with coordinates $a_1 = b_1 = c_1 = 1$ by that in complementary representation provided with the same coordinates: $a_2 = b_2 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the standard and complementary representations:

$$\gamma_1 = \arctan \left(\frac{c_1}{|\sqrt{a_1^2 + b_1^2}|} \right) = \arctan \left(\frac{1}{|\sqrt{2}|} \right) \simeq 35.26^\circ$$

$$\gamma_2 = \arctan \left(\frac{c_2}{-|\sqrt{a_2^2 + b_2^2}|} \right) = \arctan \left(\frac{1}{-|\sqrt{2}|} \right) \simeq 144.73^\circ$$

$$\theta_1 = \arctan \left(\frac{b_1}{a_1} \right) = \arctan \left(\frac{1}{1} \right) = 45^\circ$$

$$\theta_2 = \arctan \left(\frac{-b_2}{-a_2} \right) = \arctan \left(\frac{-1}{-1} \right) = 225^\circ$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} = 1 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \simeq -109.47^\circ \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = -180^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = 1 \cdot \cos(\simeq -109.47^\circ) \cdot \cos(-180^\circ) = \frac{1}{3} \\ b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = 1 \cdot \cos(\simeq -109.47^\circ) \cdot \sin(-180^\circ) = 0 \\ c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = 1 \cdot \sin(\simeq -109.47^\circ) = \frac{-2 \cdot \sqrt{2}}{3} \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= \frac{1}{3} \cdot (1 + 1) \cdot \left(1 - \frac{1}{2} \right) = \frac{1}{3} \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) = \\ &= \frac{1}{3} \cdot (1 - 1) \cdot \left(1 - \frac{1}{2} \right) = 0 \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[-c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| - c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| \right] = \\ &= \frac{1}{3} \cdot [-1 \cdot \sqrt{2} - 1 \cdot \sqrt{2}] = \frac{-2 \cdot \sqrt{2}}{3} \end{aligned}$$

Theorem 2.51. *With $o_1(t_1, \theta_1, \gamma_1)$ in complementary representation and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, and both not belonging to the line U , their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}}(\frac{t_1}{t_2}) + i \cdot b_{\frac{1}{2}}(\theta_1 - \theta_2) + u \cdot c_{\frac{1}{2}}(\gamma_1 - \gamma_2)$$

where:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| + c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| \right] \end{aligned}$$

Proof. Since the dividend is in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = -|\sqrt{a_1^2 + b_1^2}|$$

while being the divisor in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = |\sqrt{a_2^2 + b_2^2}|$$

that combined with those indicated by the formulas (2.4), proving the thesis. ■

As an example of the theorem just proved, suppose you have to divide the complete number in complementary representation provided with coordinates $a_1 = b_1 = c_1 = 1$ by that in standard representation provided with the same coordinates: $a_2 = b_2 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the complementary and standard representations:

$$\begin{aligned}\gamma_1 &= \arctan\left(\frac{c_1}{-|\sqrt{a_1^2 + b_1^2}|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.73^\circ \\ \gamma_2 &= \arctan\left(\frac{c_2}{|\sqrt{a_2^2 + b_2^2}|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ \\ \theta_1 &= \arctan\left(\frac{-b_1}{-a_1}\right) = \arctan\left(\frac{-1}{-1}\right) = 225^\circ \\ \theta_2 &= \arctan\left(\frac{b_2}{a_2}\right) = \arctan\left(\frac{1}{1}\right) = 45^\circ\end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned}t_{\frac{1}{2}} &= \frac{t_1}{t_2} = 1 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \simeq 109.47^\circ \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = 180^\circ\end{aligned}$$

and the following coordinates:

$$\begin{aligned}a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = 1 \cdot \cos(\simeq 109.47^\circ) \cdot \cos(180^\circ) = \frac{1}{3} \\ b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = 1 \cdot \cos(\simeq 109.47^\circ) \cdot \sin(180^\circ) = 0 \\ c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = 1 \cdot \sin(\simeq 109.47^\circ) = \frac{2 \cdot \sqrt{2}}{3}\end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ &= \frac{1}{3} \cdot (1 + 1) \cdot \left(1 - \frac{1}{2} \right) = \frac{1}{3} \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{|\sqrt{a_1^2 + b_1^2}| \cdot |\sqrt{a_2^2 + b_2^2}|} \right) \\ &= \frac{1}{3} \cdot (1 - 1) \cdot \left(1 - \frac{1}{2} \right) = 0 \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| + c_2 \cdot \left| \sqrt{a_1^2 + b_1^2} \right| \right] \\ &= \frac{1}{3} \cdot [1 \cdot \sqrt{2} + 1 \cdot \sqrt{2}] = \frac{2 \cdot \sqrt{2}}{3} \end{aligned}$$

Theorem 2.52. *With only $o_1(t_1, \theta_1, \gamma_1)$ belonging to the line U and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}} \left(\frac{t_1}{t_2} \right) + i \cdot b_{\frac{1}{2}} (\theta_1 - \theta_2) + u \cdot c_{\frac{1}{2}} (\gamma_1 - \gamma_2)$$

where:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) + b_2 \cdot \sin(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) - b_2 \cdot \cos(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| \end{aligned}$$

Proof. The division between two complete numbers, as we know, satisfies the following formula:

$$\begin{aligned} o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}) &= \frac{t_1}{t_2} \cdot \{ [\cos(\gamma_1 - \gamma_2) \cdot \cos(\theta_1 - \theta_2)] \\ &\quad + i \cdot [\cos(\gamma_1 - \gamma_2) \cdot \sin(\theta_1 - \theta_2)] + u \cdot [\sin(\gamma_1 - \gamma_2)] \} \end{aligned}$$

Since $o_1(t_1, \theta_1, \gamma_1)$ belongs to the line U will be provided with the following values of modulus and phases:

$$\begin{aligned} t_1 &= \sqrt{c_1^2} \\ \gamma_1 &= \text{sign}(c_1) \cdot 90^\circ \\ \theta_1 \text{ known} &\neq \arctan\left(\frac{b_1}{a_1}\right) \end{aligned}$$

unlike $o_2(t_2, \theta_2, \gamma_2)$ that will be provided with the following values:

$$\begin{aligned} t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} \\ \gamma_2 &= \arctan \left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}} \right) \\ \theta_2 &= \arctan \left(\frac{b_2}{a_2} \right) \end{aligned}$$

This means that we can write the coordinates sought in the following way:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{\sqrt{c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \cos \left[\text{sign}(c_1) \cdot 90^\circ - \arctan \left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}} \right) \right] \\ &\quad \cdot \cos \left[\theta_1 - \arctan \left(\frac{b_2}{a_2} \right) \right] \\ b_{\frac{1}{2}} &= \frac{\sqrt{c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \cos \left[\text{sign}(c_1) \cdot 90^\circ - \arctan \left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}} \right) \right] \\ &\quad \cdot \sin \left[\theta_1 - \arctan \left(\frac{b_2}{a_2} \right) \right] \\ c_{\frac{1}{2}} &= \frac{\sqrt{c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \sin \left[\text{sign}(c_1) \cdot 90^\circ - \arctan \left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}} \right) \right] \end{aligned}$$

To continue with the proof, we have to use the following trigonometric relations:

$$\begin{aligned} \cos(x - y) &= \cos(x) \cdot \cos(y) + \sin(x) \cdot \sin(y) \\ \sin(x - y) &= \sin(x) \cdot \cos(y) - \cos(x) \cdot \sin(y) \\ \cos \left[\arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right) \right] &= \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + c^2}} \\ \sin \left[\arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right) \right] &= \sqrt{\frac{c^2}{a^2 + b^2 + c^2}} \\ \cos \left[\arctan \left(\frac{b}{a} \right) \right] &= \sqrt{\frac{a^2}{a^2 + b^2}} \\ \sin \left[\arctan \left(\frac{b}{a} \right) \right] &= \sqrt{\frac{b^2}{a^2 + b^2}} \\ \cos[\text{sign}(x) \cdot 90^\circ - y] &= \text{sign}(x) \cdot \sin(y) \\ \sin[\text{sign}(x) \cdot 90^\circ - y] &= \text{sign}(x) \cdot \cos(y) \end{aligned}$$

To determine the value of the coordinate $a_{\frac{1}{2}}$ the steps to perform will be the following:

$$\begin{aligned}
 a_{\frac{1}{2}} &= \text{sign}(c_1) \cdot \frac{\sqrt{c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \\
 &\cdot \left[\sqrt{\frac{a_2^2}{a_2^2 + b_2^2}} \cdot \cos(\theta_1) + \sqrt{\frac{b_2^2}{a_2^2 + b_2^2}} \cdot \sin(\theta_1) \right] \\
 &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \text{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \frac{\sqrt{a_2^2} \cdot \cos(\theta_1) + \sqrt{b_2^2} \cdot \sin(\theta_1)}{\sqrt{a_2^2 + b_2^2}}
 \end{aligned}$$

To determine the value of the coordinate $b_{\frac{1}{2}}$ the steps to perform will be the following:

$$\begin{aligned}
 b_{\frac{1}{2}} &= \text{sign}(c_1) \cdot \frac{\sqrt{c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \\
 &\cdot \left[\sqrt{\frac{a_2^2}{a_2^2 + b_2^2}} \cdot \sin(\theta_1) - \sqrt{\frac{b_2^2}{a_2^2 + b_2^2}} \cdot \cos(\theta_1) \right] \\
 &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \text{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \frac{\sqrt{a_2^2} \cdot \sin(\theta_1) - \sqrt{b_2^2} \cdot \cos(\theta_1)}{\sqrt{a_2^2 + b_2^2}}
 \end{aligned}$$

To determine the value of the coordinate $c_{\frac{1}{2}}$ the steps to perform will be the following:

$$\begin{aligned}
 c_{\frac{1}{2}} &= \text{sign}(c_1) \cdot \frac{\sqrt{c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \sqrt{\frac{a_2^2 + b_2^2}{a_2^2 + b_2^2 + c_2^2}} \\
 &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \text{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2}
 \end{aligned}$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a_2, b_2, c_2 are zero (provided that $o_2(a_2, b_2, c_2)$ remains in the context of the complete numbers not belonging in the line U).

The only limitation in this regard is the need to avoid the following situation:

$$a_2^2 + b_2^2 + c_2^2 = 0$$

which confirms the impossibility to divide a complete number $o(t, \theta, \gamma)$ for zero (characterized by the values a_2, b_2, c_2 that make the above mentioned condition true).

Wanting to find relations that satisfy the division rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients a,b,c the convention $\sqrt{x^2} = x$, with the exception of c_1 for which we should adopt the convention $\sqrt{x^2} = |x|$. The reason is simple because if we adopt for c_1 the usual convention, we will have:

$$\text{sign}(c_1) \cdot \sqrt{c_1^2} = |c_1|$$

and therefore a result of the division that depends on the modulus of the coordinate c_1 . While adopting $\sqrt{x^2} = |x|$ we will have:

$$\text{sign}(c_1) \cdot \sqrt{c_1^2} = c_1$$

and therefore a result of the division that depends on the effective value of this coordinate.

The relations obtained will be the following:

$$(2.5) \quad \begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) + b_2 \cdot \sin(\theta_1)}{\sqrt{a_2^2 + b_2^2}} \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) - b_2 \cdot \cos(\theta_1)}{\sqrt{a_2^2 + b_2^2}} \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot c_1 \cdot \sqrt{a_2^2 + b_2^2} \end{aligned}$$

Since the number $o_2(a_2, b_2, c_2)$ is in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = |\sqrt{a_2^2 + b_2^2}|$$

that combined with those indicated by the formulas (2.5), proving the thesis. ■

As an example of the theorem just proved, suppose you have to divide the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30^\circ$ by a complete number in standard representation provided with coordinates: $a_2 = 1$, $b_2 = -1$, $c_2 = 1$.

Their modulus may be calculated in the following way:

$$\begin{aligned} t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1 \\ t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \end{aligned}$$

For their phases in the case of the outgoing number we have:

$$\begin{aligned} \gamma_1 &= \text{sign}(c_1) \cdot 90^\circ = 90^\circ \\ \theta_1 &= 30^\circ \end{aligned}$$

while in the case of the complete number we should refer to the formulas related to the standard representation:

$$\begin{aligned} \gamma_2 &= \arctan\left(\frac{c_2}{|\sqrt{a_2^2 + b_2^2}|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ \\ \theta_2 &= \arctan\left(\frac{b_2}{a_2}\right) = \arctan\left(\frac{-1}{1}\right) = -45^\circ \end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} = \frac{1}{\sqrt{3}} \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \simeq 54.74^\circ \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = 75^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \cos(\simeq 54.74^\circ) \cdot \cos(75^\circ) \simeq 0.09 \\ b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \cos(\simeq 54.74^\circ) \cdot \sin(75^\circ) \simeq 0.32 \\ c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \sin(\simeq 54.74^\circ) = \frac{\sqrt{2}}{3} \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) + b_2 \cdot \sin(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} = \\ &= \frac{1}{3} \cdot 1 \cdot \frac{\cos(30^\circ) - \sin(30^\circ)}{|\sqrt{2}|} \simeq 0.09 \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) - b_2 \cdot \cos(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} = \\ &= \frac{1}{3} \cdot 1 \cdot \frac{\sin(30^\circ) + \cos(30^\circ)}{|\sqrt{2}|} \simeq 0.32 \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| = \frac{1}{3} \cdot 1 \cdot |\sqrt{2}| = \frac{\sqrt{2}}{3} \end{aligned}$$

Theorem 2.53. *With only $o_1(t_1, \theta_1, \gamma_1)$ belonging to the line U and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}} \left(\frac{t_1}{t_2} \right) + i \cdot b_{\frac{1}{2}}(\theta_1 - \theta_2) + u \cdot c_{\frac{1}{2}}(\gamma_1 - \gamma_2)$$

where:

$$\begin{aligned} a_{\frac{1}{2}} &= -\frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) + b_2 \cdot \sin(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} \\ b_{\frac{1}{2}} &= -\frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) - b_2 \cdot \cos(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} \\ c_{\frac{1}{2}} &= -\frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| \end{aligned}$$

Proof. Since the number $o_2(a_2, b_2, c_2)$ is in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = -|\sqrt{a_2^2 + b_2^2}|$$

that combined with those indicated by the formulas (2.5), proving the thesis. ■

As an example of the theorem just proved, suppose you have to divide the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30^\circ$ by a complete number in complementary representation provided with coordinates: $a_2 = 1$, $b_2 = -1$, $c_2 = 1$.

Their modulus may be calculated in the following way:

$$\begin{aligned} t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1 \\ t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \end{aligned}$$

For their phases in the case of the outgoing number we have:

$$\begin{aligned} \gamma_1 &= \text{sign}(c_1) \cdot 90^\circ = 90^\circ \\ \theta_1 &= 30^\circ \end{aligned}$$

while in the case of the complete number we should refer to the formulas related to the complementary representation:

$$\begin{aligned} \gamma_2 &= \arctan\left(\frac{c_2}{-|\sqrt{a_2^2 + b_2^2}|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.74^\circ \\ \theta_2 &= \arctan\left(\frac{-b_2}{-a_2}\right) = \arctan\left(\frac{1}{-1}\right) = 135^\circ \end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} = \frac{1}{\sqrt{3}} \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \simeq -54.74^\circ \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = -105^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \cos(\simeq -54.74^\circ) \cdot \cos(-105^\circ) \simeq -0.09 \\ b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \cos(\simeq -54.74^\circ) \cdot \sin(-105^\circ) \simeq -0.32 \\ c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \sin(\simeq -54.74^\circ) = -\frac{\sqrt{2}}{3} \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{\frac{1}{2}} &= -\frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) + b_2 \cdot \sin(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} = \\ &= -\frac{1}{3} \cdot 1 \cdot \frac{\cos(30^\circ) - \sin(30^\circ)}{|\sqrt{2}|} \simeq -0.09 \\ b_{\frac{1}{2}} &= -\frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) - b_2 \cdot \cos(\theta_1)}{|\sqrt{a_2^2 + b_2^2}|} = \\ &= -\frac{1}{3} \cdot 1 \cdot \frac{\sin(30^\circ) + \cos(30^\circ)}{|\sqrt{2}|} \simeq -0.32 \\ c_{\frac{1}{2}} &= -\frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot c_1 \cdot \left| \sqrt{a_2^2 + b_2^2} \right| = -\frac{1}{3} \cdot 1 \cdot |\sqrt{2}| = -\frac{\sqrt{2}}{3} \end{aligned}$$

Theorem 2.54. *With only $o_2(t_2, \theta_2, \gamma_2)$ belonging to the line U and $o_1(t_1, \theta_1, \gamma_1)$ in standard representation, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}}\left(\frac{t_1}{t_2}\right) + i \cdot b_{\frac{1}{2}}(\theta_1 - \theta_2) + u \cdot c_{\frac{1}{2}}(\gamma_1 - \gamma_2)$$

where:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \frac{a_1 \cdot \cos(\theta_2) + b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} \\ b_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \frac{b_1 \cdot \cos(\theta_2) - a_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} \\ c_{\frac{1}{2}} &= -\frac{1}{c_2} \cdot \left| \sqrt{a_1^2 + b_1^2} \right| \end{aligned}$$

Proof. The division between two complete numbers, as we know, satisfies the following formula:

$$\begin{aligned} o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}) &= \frac{t_1}{t_2} \cdot \{ [\cos(\gamma_1 - \gamma_2) \cdot \cos(\theta_1 - \theta_2)] \\ &\quad + i \cdot [\cos(\gamma_1 - \gamma_2) \cdot \sin(\theta_1 - \theta_2)] + u \cdot [\sin(\gamma_1 - \gamma_2)] \} \end{aligned}$$

Since $o_2(t_2, \theta_2, \gamma_2)$ belongs to the line U will be provided with the following values of modulus and phases:

$$\begin{aligned} t_2 &= \sqrt{c_2^2} \\ \gamma_2 &= \text{sign}(c_2) \cdot 90^\circ \\ \theta_2 \text{ known} &\neq \arctan\left(\frac{b_2}{a_2}\right) \end{aligned}$$

unlike $o_1(t_1, \theta_1, \gamma_1)$ that will be provided with the following values:

$$\begin{aligned} t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} \\ \gamma_1 &= \arctan \left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} \right) \\ \theta_1 &= \arctan \left(\frac{b_1}{a_1} \right) \end{aligned}$$

This means that we can write the coordinates sought in the following way:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \cos \left[\arctan \left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} \right) - \text{sign}(c_2) \cdot 90^\circ \right] \cdot \\ &\quad \cdot \cos \left[\arctan \left(\frac{b_1}{a_1} \right) - \theta_2 \right] \\ b_{\frac{1}{2}} &= \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \cos \left[\arctan \left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} \right) - \text{sign}(c_2) \cdot 90^\circ \right] \cdot \\ &\quad \cdot \sin \left[\arctan \left(\frac{b_1}{a_1} \right) - \theta_2 \right] \\ c_{\frac{1}{2}} &= \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \sin \left[\arctan \left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} \right) - \text{sign}(c_2) \cdot 90^\circ \right] \end{aligned}$$

To continue with the proof, we have to use the following trigonometric relations:

$$\begin{aligned} \cos(x - y) &= \cos(x) \cdot \cos(y) + \sin(x) \cdot \sin(y) \\ \sin(x - y) &= \sin(x) \cdot \cos(y) - \cos(x) \cdot \sin(y) \\ \cos \left[\arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right) \right] &= \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + c^2}} \\ \sin \left[\arctan \left(\frac{c}{\sqrt{a^2 + b^2}} \right) \right] &= \sqrt{\frac{c^2}{a^2 + b^2 + c^2}} \\ \cos \left[\arctan \left(\frac{b}{a} \right) \right] &= \sqrt{\frac{a^2}{a^2 + b^2}} \\ \sin \left[\arctan \left(\frac{b}{a} \right) \right] &= \sqrt{\frac{b^2}{a^2 + b^2}} \\ \cos[x - \text{sign}(y) \cdot 90^\circ] &= \text{sign}(y) \cdot \sin(x) \\ \sin[x - \text{sign}(y) \cdot 90^\circ] &= -\text{sign}(y) \cdot \cos(x) \end{aligned}$$

To determine the value of the coordinate $a_{\frac{1}{2}}$ the steps to perform will be the following:

$$\begin{aligned}
 a_{\frac{1}{2}} &= \text{sign}(c_2) \cdot \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \\
 &\quad \cdot \left[\sqrt{\frac{a_1^2}{a_1^2 + b_1^2}} \cdot \cos(\theta_2) + \sqrt{\frac{b_1^2}{a_1^2 + b_1^2}} \cdot \sin(\theta_2) \right] \\
 &= \text{sign}(c_2) \cdot \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \frac{\sqrt{a_1^2} \cdot \cos(\theta_2) + \sqrt{b_1^2} \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}}
 \end{aligned}$$

To determine the value of the coordinate $b_{\frac{1}{2}}$ the steps to perform will be the following:

$$\begin{aligned}
 b_{\frac{1}{2}} &= \text{sign}(c_2) \cdot \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \\
 &\quad \cdot \left[\sqrt{\frac{b_1^2}{a_1^2 + b_1^2}} \cdot \cos(\theta_2) - \sqrt{\frac{a_1^2}{a_1^2 + b_1^2}} \cdot \sin(\theta_2) \right] \\
 &= \text{sign}(c_2) \cdot \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \frac{\sqrt{b_1^2} \cdot \cos(\theta_2) - \sqrt{a_1^2} \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}}
 \end{aligned}$$

To determine the value of the coordinate $c_{\frac{1}{2}}$ the steps to perform will be the following:

$$c_{\frac{1}{2}} = -\text{sign}(c_2) \cdot \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \sqrt{\frac{a_1^2 + b_1^2}{a_1^2 + b_1^2 + c_1^2}} = -\text{sign}(c_2) \cdot \frac{1}{\sqrt{c_2^2}} \cdot \sqrt{a_1^2 + b_1^2}$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a_1, b_1, c_1 are zero (provided that $o_1(a_1, b_1, c_1)$ remains in the context of the complete numbers not belonging in the line U).

The only limitation in this regard is the need to avoid the following situation:

$$c_2^2 = 0$$

which confirms the impossibility to divide a complete number $o(t, \theta, \gamma)$ for zero (characterized by the values c_2 that make the above mentioned condition true).

Wanting to find relations that satisfy the division rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients a,b,c the convention $\sqrt{x^2} = x$, with the exception of c_2 for which we should adopt the convention $\sqrt{x^2} = |x|$. The reason is simple because if we adopt for c_2 the usual convention, we will have:

$$\frac{\text{sign}(c_2)}{\sqrt{c_2^2}} = \frac{1}{|c_2|}$$

and therefore a result of the division that depends on the modulus of the coordinate c_2 . While adopting $\sqrt{x^2} = |x|$ we will have:

$$\frac{\text{sign}(c_2)}{\sqrt{c_2^2}} = \frac{1}{c_2}$$

and therefore a result of the division that depends on the effective value of this coordinate.

The relations obtained will be the following:

$$(2.6) \quad \begin{aligned} a_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \frac{a_1 \cdot \cos(\theta_2) + b_1 \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}} \\ b_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \frac{b_1 \cdot \cos(\theta_2) - a_1 \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}} \\ c_{\frac{1}{2}} &= -\frac{1}{c_2} \cdot \sqrt{a_1^2 + b_1^2} \end{aligned}$$

Since the number $o_1(a_1, b_1, c_1)$ is in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = |\sqrt{a_1^2 + b_1^2}|$$

that combined with those indicated by the formulas (2.6), proving the thesis. ■

As an example of the theorem just proved, suppose you have to divide the complete number in standard representation provided with coordinates: $a_1 = 1$, $b_1 = -1$, $c_1 = 1$ by an outgoing numbers of coordinate: $c_2 = 1$ and phase $\theta_2 = 30^\circ$.

Their modulus may be calculated in the following way:

$$\begin{aligned} t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \\ t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1 \end{aligned}$$

For their phases in the case of the outgoing number we have:

$$\begin{aligned} \gamma_2 &= \text{sign}(c_2) \cdot 90^\circ = 90^\circ \\ \theta_2 &= 30^\circ \end{aligned}$$

while in the case of the complete number we should refer to the formulas related to the standard representation:

$$\begin{aligned} \gamma_1 &= \arctan\left(\frac{c_1}{|\sqrt{a_1^2 + b_1^2}|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ \\ \theta_1 &= \arctan\left(\frac{b_1}{a_1}\right) = \arctan\left(\frac{-1}{1}\right) = -45^\circ \end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} = \sqrt{3} \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \simeq -54.74^\circ \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = -75^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = \sqrt{3} \cdot \cos(\simeq -54.74^\circ) \cdot \cos(-75^\circ) \simeq 0.26 \\ b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = \sqrt{3} \cdot \cos(\simeq -54.74^\circ) \cdot \sin(-75^\circ) \simeq -0.97 \\ c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = \sqrt{3} \cdot \sin(\simeq -54.74^\circ) = -\sqrt{2} \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \frac{a_1 \cdot \cos(\theta_2) + b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} = \frac{\cos(30^\circ) - \sin(30^\circ)}{|\sqrt{2}|} \simeq 0.26 \\ b_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \frac{b_1 \cdot \cos(\theta_2) - a_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} = \frac{-\cos(30^\circ) - \sin(30^\circ)}{|\sqrt{2}|} \simeq -0.97 \\ c_{\frac{1}{2}} &= -\frac{1}{c_2} \cdot |\sqrt{a_1^2 + b_1^2}| = -\sqrt{2} \end{aligned}$$

Theorem 2.55. *With only $o_2(t_2, \theta_2, \gamma_2)$ belonging to the line U and $o_1(t_1, \theta_1, \gamma_1)$ in complementary representation, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}}(\frac{t_1}{t_2}) + i \cdot b_{\frac{1}{2}}(\theta_1 - \theta_2) + u \cdot c_{\frac{1}{2}}(\gamma_1 - \gamma_2)$$

where:

$$\begin{aligned} a_{\frac{1}{2}} &= -\frac{c_1}{c_2} \cdot \frac{a_1 \cdot \cos(\theta_2) + b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} \\ b_{\frac{1}{2}} &= -\frac{c_1}{c_2} \cdot \frac{b_1 \cdot \cos(\theta_2) - a_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} \\ c_{\frac{1}{2}} &= \frac{1}{c_2} \cdot |\sqrt{a_1^2 + b_1^2}| \end{aligned}$$

Proof. Since the number $o_1(a_1, b_1, c_1)$ is in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = -|\sqrt{a_1^2 + b_1^2}|$$

that combined with those indicated by the formulas (2.6), proving the thesis. ■

As an example of the theorem just proved, suppose you have to divide a complete number in complementary representation provided with coordinates: $a_2 = 1, b_2 = -1, c_2 = 1$ by the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30^\circ$.

Their modulus may be calculated in the following way:

$$t_1 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

$$t_2 = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1$$

For their phases in the case of the outgoing number we have:

$$\gamma_2 = \text{sign}(c_2) \cdot 90^\circ = 90^\circ$$

$$\theta_2 = 30^\circ$$

while in the case of the complete number we should refer to the formulas related to the complementary representation:

$$\gamma_1 = \arctan\left(\frac{c_1}{-|\sqrt{a_1^2 + b_1^2}|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.74^\circ$$

$$\theta_1 = \arctan\left(\frac{-b_1}{-a_1}\right) = \arctan\left(\frac{1}{-1}\right) = 135^\circ$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{\frac{1}{2}} = \frac{t_1}{t_2} = \sqrt{3}$$

$$\gamma_{\frac{1}{2}} = \gamma_1 - \gamma_2 \simeq 54.74^\circ$$

$$\theta_{\frac{1}{2}} = \theta_1 - \theta_2 = 105^\circ$$

and the following coordinates:

$$a_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = \sqrt{3} \cdot \cos(\simeq 54.74^\circ) \cdot \cos(105^\circ) \simeq -0.26$$

$$b_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = \sqrt{3} \cdot \cos(\simeq 54.74^\circ) \cdot \sin(105^\circ) \simeq 0.97$$

$$c_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = \sqrt{3} \cdot \sin(\simeq 54.74^\circ) = \sqrt{2}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$a_{\frac{1}{2}} = -\frac{c_1}{c_2} \cdot \frac{a_1 \cdot \cos(\theta_2) + b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} = -\frac{\cos(30^\circ) - \sin(30^\circ)}{|\sqrt{2}|} \simeq -0.26$$

$$b_{\frac{1}{2}} = -\frac{c_1}{c_2} \cdot \frac{b_1 \cdot \cos(\theta_2) - a_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}|} = -\frac{-\cos(30^\circ) - \sin(30^\circ)}{|\sqrt{2}|} \simeq 0.97$$

$$c_{\frac{1}{2}} = \frac{1}{c_2} \cdot \left| \sqrt{a_1^2 + b_1^2} \right| = \sqrt{2}$$

Theorem 2.56. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ both belonging to the line U , their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}} \left(\frac{t_1}{t_2} \right) + i \cdot b_{\frac{1}{2}} (\theta_1 - \theta_2) + u \cdot c_{\frac{1}{2}} (\gamma_1 - \gamma_2)$$

where:

$$a_{\frac{1}{2}} = \frac{c_1}{c_2} \cdot \cos (\theta_1 - \theta_2)$$

$$b_{\frac{1}{2}} = \frac{c_1}{c_2} \cdot \sin (\theta_1 - \theta_2)$$

$$c_{\frac{1}{2}} = 0$$

Proof. The division between two complete numbers, as we know, satisfies the following formula:

$$o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}) = \frac{t_1}{t_2} \cdot \{ [\cos (\gamma_1 - \gamma_2) \cdot \cos (\theta_1 - \theta_2)] + \\ + i \cdot [\cos (\gamma_1 - \gamma_2) \cdot \sin (\theta_1 - \theta_2)] + u \cdot [\sin (\gamma_1 - \gamma_2)] \}$$

Since $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ belong to the line U will be provided with the following values of modulus and phases:

$$t_1 = \sqrt{c_1^2}$$

$$t_2 = \sqrt{c_2^2}$$

$$\gamma_1 = \text{sign} (c_1) \cdot 90^\circ$$

$$\gamma_2 = \text{sign} (c_2) \cdot 90^\circ$$

$$\theta_1 \text{ known } \neq \arctan \left(\frac{b_1}{a_1} \right)$$

$$\theta_2 \text{ known } \neq \arctan \left(\frac{b_2}{a_2} \right)$$

This means that we can write the coordinates sought in the following way:

$$a_{\frac{1}{2}} = \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \cos [\text{sign} (c_1) \cdot 90^\circ - \text{sign} (c_2) \cdot 90^\circ] \cdot \cos (\theta_1 - \theta_2)$$

$$b_{\frac{1}{2}} = \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \cos [\text{sign} (c_1) \cdot 90^\circ - \text{sign} (c_2) \cdot 90^\circ] \cdot \sin (\theta_1 - \theta_2)$$

$$c_{\frac{1}{2}} = \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \sin [\text{sign} (c_1) \cdot 90^\circ - \text{sign} (c_2) \cdot 90^\circ]$$

Considering that when c_1 and c_2 have the same sign we obtained:

$$\cos [\text{sign} (c_1) \cdot 90^\circ - \text{sign} (c_2) \cdot 90^\circ] = \cos (\pm 0^\circ) = 1 = \text{sign} (c_1) \cdot \text{sign} (c_2)$$

$$\sin [\text{sign} (c_1) \cdot 90^\circ - \text{sign} (c_2) \cdot 90^\circ] = \sin (\pm 0^\circ) = 0$$

and that when they have the opposite sign we obtained:

$$\cos [\text{sign}(c_1) \cdot 90^\circ - \text{sign}(c_2) \cdot 90^\circ] = \cos(\pm 180^\circ) = -1 = \text{sign}(c_1) \cdot \text{sign}(c_2)$$

$$\sin [\text{sign}(c_1) \cdot 90^\circ - \text{sign}(c_2) \cdot 90^\circ] = \sin(\pm 180^\circ) = 0$$

we can write:

$$a_{\frac{1}{2}} = \text{sign}(c_1) \cdot \text{sign}(c_2) \cdot \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \cos(\theta_1 - \theta_2)$$

$$b_{\frac{1}{2}} = \text{sign}(c_1) \cdot \text{sign}(c_2) \cdot \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \sin(\theta_1 - \theta_2)$$

$$c_{\frac{1}{2}} = 0$$

Wanting to find relations that satisfy the division rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients c_1, c_2 the convention $\sqrt{x^2} = |x|$. In fact in this way we obtain:

$$\begin{aligned} \text{sign}(c_1) \cdot \sqrt{c_1^2} &= c_1 \\ \frac{\text{sign}(c_2)}{\sqrt{c_2^2}} &= \frac{1}{c_2} \end{aligned}$$

and therefore a result of the division that depends on the effective values of these coordinates. The relation that we obtain following these conventions proves the thesis. ■

As an example of the theorem just proved, suppose you have to divide the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30^\circ$ by the outgoing number of coordinate: $c_2 = 1$ and phase $\theta_2 = 30^\circ$.

Their modulus may be calculated in the following way:

$$t_1 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1$$

$$t_2 = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1$$

For their phases we have:

$$\gamma_1 = \text{sign}(c_1) \cdot 90^\circ = 90^\circ$$

$$\gamma_2 = \text{sign}(c_2) \cdot 90^\circ = 90^\circ$$

$$\theta_1 = 30^\circ$$

$$\theta_2 = 30^\circ$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{\frac{1}{2}} = \frac{t_1}{t_2} = 1$$

$$\gamma_{\frac{1}{2}} = \gamma_1 - \gamma_2 = 0^\circ$$

$$\theta_{\frac{1}{2}} = \theta_1 - \theta_2 = 0^\circ$$

and the following coordinates:

$$\begin{aligned} a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = 1 \cdot \cos(0^\circ) \cdot \cos(0^\circ) = 1 \\ b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = 1 \cdot \cos(0^\circ) \cdot \sin(0^\circ) = 0 \\ c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = 1 \cdot \sin(0^\circ) = 0 \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \cos(\theta_1 - \theta_2) = 1 \cdot \cos(0^\circ) = 1 \\ b_{\frac{1}{2}} &= \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \sin(\theta_1 - \theta_2) = 1 \cdot \sin(0^\circ) = 0 \\ c_{\frac{1}{2}} &= 0 \end{aligned}$$

Theorem 2.57. *For the operation of division is defined indivisible the complete number 0, namely for:*

$$o_1(t_1, \theta_1, \gamma_1) = 0$$

we have:

$$\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} = 0$$

Proof. $t_1, \theta_1, \gamma_1, t_2, \theta_2, \gamma_2$ being real numbers, we can write:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} = \frac{0}{t_2} = 0 \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = \theta_1 - \textit{indeterminate} = \textit{indeterminate} \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 = \gamma_1 - \textit{indeterminate} = \textit{indeterminate} \end{aligned}$$

proving the thesis. ■

Theorem 2.58. *For the operation of division is defined neuter the complete number $1_{(S)}$, namely for:*

$$o_2(a_2, b_2, c_2)_{(S)} = 1_{(S)}$$

we have:

$$\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} = o_1(t_1, \theta_1, \gamma_1)$$

Proof. $t_1, \theta_1, \gamma_1, t_2, \theta_2, \gamma_2$ being real numbers, we can write:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} = \frac{t_1}{1} = t_1 \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = \theta_1 - 0 = \theta_1 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 = \gamma_1 - 0 = \gamma_1 \end{aligned}$$

proving the thesis. ■

Theorem 2.59. *For the operation of division is defined identical the same position with respect to the origin, namely for:*

$$o_2(a_2, b_2, c_2) = o_2(a_1, b_1, c_1)$$

we have:

$$\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} = 1_{(S)}$$

Proof. $t_1, \theta_1, \gamma_1, t_2, \theta_2, \gamma_2$ being real numbers, we can write:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} = \frac{t_1}{t_1} = 1 \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = \theta_1 - \theta_1 = 0 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 = \gamma_1 - \gamma_1 = 0 \end{aligned}$$

proving the thesis. ■

Theorem 2.60. *It is valid the invariantive property, namely:*

$$\begin{aligned} \frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} &= \frac{[o_1(t_1, \theta_1, \gamma_1) \cdot o_3(t_3, \theta_3, \gamma_3)]}{[o_2(t_2, \theta_2, \gamma_2) \cdot o_3(t_3, \theta_3, \gamma_3)]} \\ \frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} &= \frac{\frac{o_1(t_1, \theta_1, \gamma_1)}{o_3(t_3, \theta_3, \gamma_3)}}{\frac{o_2(t_2, \theta_2, \gamma_2)}{o_3(t_3, \theta_3, \gamma_3)}} \end{aligned}$$

Proof. $t_1, \theta_1, \gamma_1, t_2, \theta_2, \gamma_2, t_3, \theta_3, \gamma_3$ being real numbers, we can write:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} \\ \theta_{\frac{1}{2}} &= (\theta_1 - \theta_2) \\ \gamma_{\frac{1}{2}} &= (\gamma_1 - \gamma_2) \end{aligned}$$

$$\begin{aligned} t_{\frac{(1 \cdot 3)}{(2 \cdot 3)}} &= \frac{t_1 \cdot t_3}{t_2 \cdot t_3} = \frac{t_1}{t_2} \\ \theta_{\frac{(1 \cdot 3)}{(2 \cdot 3)}} &= (\theta_1 + \theta_3) - (\theta_2 + \theta_3) = \theta_1 - \theta_2 \\ \gamma_{\frac{(1 \cdot 3)}{(2 \cdot 3)}} &= (\gamma_1 + \gamma_3) - (\gamma_2 + \gamma_3) = \gamma_1 - \gamma_2 \end{aligned}$$

$$\begin{aligned} t_{\frac{(\frac{1}{3})}{(\frac{2}{3})}} &= \frac{\frac{t_1}{t_3}}{\frac{t_2}{t_3}} = \frac{t_1}{t_2} \\ \theta_{\frac{(\frac{1}{3})}{(\frac{2}{3})}} &= (\theta_1 - \theta_3) - (\theta_2 - \theta_3) = \theta_1 - \theta_2 \\ \gamma_{\frac{(\frac{1}{3})}{(\frac{2}{3})}} &= (\gamma_1 - \gamma_3) - (\gamma_2 - \gamma_3) = \gamma_1 - \gamma_2 \end{aligned}$$

proving the thesis. ■

Theorem 2.61. *It is not valid the distributive property of division over addition, namely for:*

$$o_1(t_1, \theta_1, \gamma_1) = o_3(t_3, \theta_3, \gamma_3) + o_4(t_4, \theta_4, \gamma_4)$$

we have:

$$\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} \neq \left[\frac{o_3(t_3, \theta_3, \gamma_3)}{o_2(t_2, \theta_2, \gamma_2)} \right] + \left[\frac{o_4(t_4, \theta_4, \gamma_4)}{o_2(t_2, \theta_2, \gamma_2)} \right]$$

Proof. Referring to the situation described by theorem 2.48 and considering that $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, a_4, b_4, c_4$ being real numbers, we can write:

$$\begin{aligned} c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_1 \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| - c_2 \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \right] \\ c_{\left(\frac{3}{2}\right)+\left(\frac{4}{2}\right)} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_3 \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| - c_2 \cdot \left| \sqrt{(a_3^2 + b_3^2)} \right| \right] \\ &\quad + \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_4 \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| - c_2 \cdot \left| \sqrt{(a_4^2 + b_4^2)} \right| \right] \\ &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left\{ (c_3 + c_4) \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| \right. \\ &\quad \left. - c_2 \cdot \left[\left| \sqrt{(a_3^2 + b_3^2)} \right| + \left| \sqrt{(a_4^2 + b_4^2)} \right| \right] \right\} \\ &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left\{ c_1 \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| \right. \\ &\quad \left. - c_2 \cdot \left[\left| \sqrt{(a_3^2 + b_3^2)} \right| + \left| \sqrt{(a_4^2 + b_4^2)} \right| \right] \right\} \neq c_{\frac{1}{2}} \end{aligned}$$

proving the thesis. ■

Theorem 2.62. *It is not valid the distributive property of division over subtraction, namely for:*

$$o_1(t_1, \theta_1, \gamma_1) = o_3(t_3, \theta_3, \gamma_3) - o_4(t_4, \theta_4, \gamma_4)$$

we have:

$$\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} \neq \left[\frac{o_3(t_3, \theta_3, \gamma_3)}{o_2(t_2, \theta_2, \gamma_2)} \right] - \left[\frac{o_4(t_4, \theta_4, \gamma_4)}{o_2(t_2, \theta_2, \gamma_2)} \right]$$

Proof. Referring to the situation described by theorem 2.48 and considering that $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, a_4, b_4, c_4$ being real numbers, we can write:

$$c_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_1 \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| - c_2 \cdot \left| \sqrt{(a_1^2 + b_1^2)} \right| \right]$$

$$\begin{aligned}
c_{\left(\frac{3}{2}\right)-\left(\frac{4}{2}\right)} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_3 \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| - c_2 \cdot \left| \sqrt{(a_3^2 + b_3^2)} \right| \right] \\
&\quad - \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_4 \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| - c_2 \cdot \left| \sqrt{(a_4^2 + b_4^2)} \right| \right] \\
&= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left\{ (c_3 - c_4) \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| + \right. \\
&\quad \left. - c_2 \cdot \left[\left| \sqrt{(a_3^2 + b_3^2)} \right| - \left| \sqrt{(a_4^2 + b_4^2)} \right| \right] \right\} \\
&= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left\{ c_1 \cdot \left| \sqrt{(a_2^2 + b_2^2)} \right| \right. \\
&\quad \left. - c_2 \cdot \left[\left| \sqrt{(a_3^2 + b_3^2)} \right| - \left| \sqrt{(a_4^2 + b_4^2)} \right| \right] \right\} \neq c_{\frac{1}{2}}
\end{aligned}$$

proving the thesis. ■

Theorem 2.63. *It is valid the equivalence between multiplication and division, namely:*

$$\begin{aligned}
o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) &= \frac{o_1(t_1, \theta_1, \gamma_1)}{\frac{1}{o_2(t_2, \theta_2, \gamma_2)}} \\
\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} &= o_1(t_1, \theta_1, \gamma_1) \cdot \frac{1}{o_2(t_2, \theta_2, \gamma_2)}
\end{aligned}$$

Proof. $t_1, \theta_1, \gamma_1, t_2, \theta_2, \gamma_2$ being real numbers, we can write:

$$\begin{aligned}
t_{1 \cdot 2} &= t_1 \cdot t_2 \\
\theta_{1 \cdot 2} &= \theta_1 + \theta_2 \\
\gamma_{1 \cdot 2} &= \gamma_1 + \gamma_2 \\
t_{\frac{1}{2}} &= \frac{t_1}{\frac{1}{t_2}} = t_1 \cdot t_2 \\
\theta_{\frac{1}{2}} &= \theta_1 - (-\theta_2) = \theta_1 + \theta_2 \\
\gamma_{\frac{1}{2}} &= \gamma_1 - (-\gamma_2) = \gamma_1 + \gamma_2 \\
t_{\frac{1}{2}} &= \frac{t_1}{t_2} \\
\theta_{\frac{1}{2}} &= (\theta_1 - \theta_2) \\
\gamma_{\frac{1}{2}} &= (\gamma_1 - \gamma_2) \\
t_{1 \cdot \frac{1}{2}} &= t_1 \cdot \frac{1}{t_2} = \frac{t_1}{t_2} \\
\theta_{1 \cdot \frac{1}{2}} &= \theta_1 + (-\theta_2) = \theta_1 - \theta_2 \\
\gamma_{1 \cdot \frac{1}{2}} &= \gamma_1 + (-\gamma_2) = \gamma_1 - \gamma_2
\end{aligned}$$

proving the thesis. ■

2.6. N -th power

Definition 2.64. In the space RIU we can define n -th power of the complete number $o(t, \theta, \gamma)$, with n (natural number) known as exponent and $o(t, \theta, \gamma)$ known as base, as the number $o_{\uparrow n}(t_{\uparrow n}, \theta_{\uparrow n}, \gamma_{\uparrow n})$ also represented with the symbol $o(t, \theta, \gamma)^n$ that satisfies the following conditions:

1. $o(t, \theta, \gamma)^n = o(t, \theta, \gamma) \cdot \dots \cdot o(t, \theta, \gamma)$ for $n > 0$
2. $o(t, \theta, \gamma)^n = \frac{o(t, \theta, \gamma)}{o(t, \theta, \gamma)} = 1$ for $n = 0$
3. $o(t, \theta, \gamma)^n = \frac{1}{\frac{o(t, \theta, \gamma)}{\frac{\dots}{o(t, \theta, \gamma)}}}$ for $n < 0$
4. $n > 0$ for $o(t, \theta, \gamma) = 0$

We note that the term $o(t, \theta, \gamma)$ in the first and third conditions is intended to appear $|n|$ times.

The first condition defines the repeated multiplication of the base by itself a positive number of times, the second a zero number of times, and finally the third a negative number of times. All these conditions correspond to require:

$$\begin{aligned} t_{\uparrow n} &= t^n \\ \theta_{\uparrow n} &= \theta \cdot n \\ \gamma_{\uparrow n} &= \gamma \cdot n \end{aligned}$$

The fourth condition gets its own justification by the impossibility of defining the n -th power module when to be multiplied by itself a zero number or a negative number of times is just the 0, because in this case would be present the following divisions for 0:

$$\begin{aligned} o(t, \theta, \gamma)^n &= \frac{0}{0} = 1 \text{ for } n = 0 \\ o(t, \theta, \gamma)^n &= \frac{1}{\frac{0}{\frac{\dots}{0}}} \text{ for } n < 0 \text{ with } 0 \text{ that appears } -n \text{ times} \end{aligned}$$

Theorem 2.65. *It is valid the product property of exponents, namely:*

$$(o^n)^m = o^{n \cdot m}$$

Proof. By applying to $(o^n)^m$ and $o^{n \cdot m}$ the definition of n-th power previously introduced, we really obtain the same result as we can observe by the following relations, when (m, n) are both greater than zero:

$$\begin{aligned}(o^n)^m &= (o \cdot o \cdot \dots \cdot o) \cdot (o \cdot o \cdot \dots \cdot o) \cdot \dots \cdot (o \cdot o \cdot \dots \cdot o) \\ o^{n \cdot m} &= (o \cdot o \cdot o \cdot o \cdot \dots \cdot o)\end{aligned}$$

It is easy to verify how all pairs of obtainable relations show a total of $|n \cdot m|$ terms $o(t, \theta, \gamma)$ to the numerator or to the denominator. Since this result is not depending on the particular values assumed by $o(t, \theta, \gamma)$ we can consider the property examined here as generally valid. ■

Theorem 2.66. *It is valid the sum property of exponents, namely:*

$$o^n \cdot o^m = o^{n+m}$$

Proof. By applying to $(o^n \cdot o^m)$ and o^{n+m} the definition of n-th power previously introduced, we really obtain the same result as we can observe by the following relations, when (m, n) are both greater than zero:

$$\begin{aligned}o^n \cdot o^m &= (o \cdot o \cdot o \cdot o \cdot \dots \cdot o) \cdot (o \cdot o \cdot \dots \cdot o) \\ o^{n+m} &= (o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot \dots \cdot o)\end{aligned}$$

It is easy to verify how all pairs of obtainable relations show a total of $|m + n|$ terms $o(t, \theta, \gamma)$ to the numerator or to the denominator. Since this result is not depending on the particular values assumed by $o(t, \theta, \gamma)$ we can consider the property examined here as generally valid. ■

Theorem 2.67. *It is valid the difference property of exponents, namely:*

$$\frac{o^n}{o^m} = o^{n-m}$$

Proof. By applying to $(\frac{o^n}{o^m})$ and o^{n-m} the definition of n-th power previously introduced, we really obtain the same result as we can observe by the following relations, when (m, n) are both greater than zero:

$$\begin{aligned}\frac{o^n}{o^m} &= (o \cdot o \cdot o \cdot o \cdot \dots \cdot o) && \text{if } n > m \\ o^{n-m} &= (o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot \dots \cdot o) && \text{if } n > m \\ \frac{o^n}{o^m} &= \frac{1}{(o \cdot o \cdot o \cdot o \cdot \dots \cdot o)} && \text{if } n < m \\ o^{n-m} &= \frac{1}{(o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot \dots \cdot o)} && \text{if } n < m\end{aligned}$$

It is easy to verify how all pairs of obtainable relations show a total of $|m - n|$ terms $o(t, \theta, \gamma)$ to the numerator or to the denominator. Since this result is not depending on the particular values assumed by $o(t, \theta, \gamma)$ we can consider the property examined here as generally valid. ■

Theorem 2.68. *It is valid the product property of bases, namely:*

$$o_1^n \cdot o_2^n = (o_1 \cdot o_2)^n$$

Proof. By applying to $(o_1^n \cdot o_2^n)$ and $(o_1 \cdot o_2)^n$ the definition of n-th power previously introduced, we really obtain the same result as we can observe by the following relations, when n is greater than zero:

$$\begin{aligned} o_1^n \cdot o_2^n &= (o_1 \cdot o_1 \cdot o_1 \cdot \dots \cdot o_1) \cdot (o_2 \cdot o_2 \cdot o_2 \cdot \dots \cdot o_2) \\ (o_1 \cdot o_2)^n &= (o_1 \cdot o_2) \cdot (o_1 \cdot o_2) \cdot \dots \cdot (o_1 \cdot o_2) \cdot \end{aligned}$$

It is easy to verify how all pairs of obtainable relations show a total of $|n|$ terms $o_1(t_1, \theta_1, \gamma_1)$ and $|n|$ terms $o_2(t_2, \theta_2, \gamma_2)$ to the numerator or to the denominator. Since this result is not depending on the particular values assumed by $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ we can consider the property examined here as generally valid. ■

Theorem 2.69. *It is valid the quotient property of bases, namely:*

$$\frac{o_1^n}{o_2^n} = \left(\frac{o_1}{o_2}\right)^n$$

Proof. By applying to $\frac{o_1^n}{o_2^n}$ and $\left(\frac{o_1}{o_2}\right)^n$ the definition of n-th power previously introduced, we really obtain the same result as we can observe by the following relations, when n is greater than zero:

$$\begin{aligned} \frac{o_1^n}{o_2^n} &= \frac{(o_1 \cdot o_1 \cdot o_1 \cdot \dots \cdot o_1)}{(o_2 \cdot o_2 \cdot o_2 \cdot \dots \cdot o_2)} \\ \left(\frac{o_1}{o_2}\right)^n &= \left(\frac{o_1}{o_2}\right) \cdot \left(\frac{o_1}{o_2}\right) \cdot \dots \cdot \left(\frac{o_1}{o_2}\right) \end{aligned}$$

It is easy to verify how all pairs of obtainable relations show a total of $|n|$ terms $o_1(t_1, \theta_1, \gamma_1)$ to the numerator and $|n|$ terms $o_2(t_2, \theta_2, \gamma_2)$ to the denominator or vice versa. Since this result is not depending on the particular values assumed by $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ we can consider the property examined here as generally valid. ■

2.7. N-th root

Definition 2.70. In the space RIU we can define n-th root of the complete number $o(t, \theta, \gamma)$, with n (natural number) known as degree and $o(t, \theta, \gamma)$ known as radicand, as the number $o_{\downarrow n}(t_{\downarrow n}, \theta_{\downarrow n}, \gamma_{\downarrow n})$ also represented with the symbol $\sqrt[n]{o(t, \theta, \gamma)}$ that satisfies the following conditions:

1. $\sqrt[n]{o(t, \theta, \gamma)} \cdot \dots \cdot \sqrt[n]{o(t, \theta, \gamma)} = o(t, \theta, \gamma) \quad \text{for } n > 0$
 2. $\frac{1}{\sqrt[n]{o(t, \theta, \gamma)}} = o(t, \theta, \gamma) \quad \text{for } n < 0$
- ...
- $$\frac{\dots}{\sqrt[n]{o(t, \theta, \gamma)}}$$

- 3. $\theta_{\downarrow n} = \frac{\theta}{n}, \quad \gamma_{\downarrow n} = \frac{\gamma}{n}$
- 4. $n \neq 0$ for any $o(t, \theta, \gamma)$
- 5. $n \geq 0$ for $o(t, \theta, \gamma) = 0$
- 6. $\sqrt[n]{t} > 0, \quad t > 0$

We note that the term $\sqrt[n]{o(t, \theta, \gamma)}$ in the first and second conditions is intended to appear $|n|$ times.

The first condition defines the repeated multiplication of the root by itself a positive number of times, while the second a negative number of times. Both these conditions correspond to require:

$$\begin{aligned} t_{\downarrow n} &= \sqrt[n]{t} \\ \theta_{\downarrow n} &= \frac{\theta + k \cdot 360^\circ}{n} \quad \text{for } k = \pm 1, \pm 2, \pm 3, \pm 4, \dots \\ \gamma_{\downarrow n} &= \frac{\gamma + k \cdot 360^\circ}{n} \quad \text{for } k = \pm 1, \pm 2, \pm 3, \pm 4, \dots \end{aligned}$$

The third condition gets its own justification by the necessity of defining the n -th root in an univocal way. In fact, when that condition is not valid, there are n^2 different complete numbers able to satisfy such definition: one for each distinct pair of phases $\theta_{\downarrow n}, \gamma_{\downarrow n}$ given by the relations seen above.

Also the fourth condition gets its own justification by the necessity of defining the n -th root in an univocal way. In fact when that condition is not valid, the multiplication of the root by itself a number of times equal to 0 would require the use of the following expression:

$$\frac{\sqrt[n]{o(t, \theta, \gamma)}}{\sqrt[n]{o(t, \theta, \gamma)}} = 1$$

that would be satisfied by several values of $\sqrt[n]{o(t, \theta, \gamma)}$.

The fifth condition gets its own justification by the impossibility of defining values of n -th root that multiplied by itself a negative number of times are able to give as the result just 0 value. In fact the following expression:

$$\frac{1}{\sqrt[n]{o(t, \theta, \gamma)}} = 0 \text{ for } n < 0, \sqrt[n]{o(t, \theta, \gamma)} \text{ appears } -n \text{ times}$$

$$\frac{\dots}{\sqrt[n]{o(t, \theta, \gamma)}}$$

requires the existence of a divisor of 1 that can assign to it a quotient equal to 0: a thing that we know impossible.

The sixth condition gets its own justification by the need to make acceptable the n -th root in regard the modulus t of the complete number $o(t, \theta, \gamma)$.

Theorem 2.71. *It is valid the product property of degrees, namely:*

$$\sqrt[m]{\sqrt[n]{o}} = \sqrt[m \cdot n]{o}$$

Proof. By applying the principle according to which two numbers are equal if and only if they remain as such, also once we raise them to the same power, we can raise the two member of the previous equality to the number $(m \cdot n)$, obtaining:

$$\left(\sqrt[m]{\sqrt[n]{o}} \right)^{(m \cdot n)} = \left(\sqrt[m \cdot n]{o} \right)^{(m \cdot n)}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

Considering the value $\sqrt[m]{\sqrt[n]{o}}$ of the first member as a complete number, it is possible to apply to it the theorem 2.65 concerning the product of exponents of the n -th power, obtaining:

$$\left(\sqrt[m]{\sqrt[n]{o}} \right)^{(m \cdot n)} = \left[\left(\sqrt[m]{\sqrt[n]{o}} \right)^m \right]^n$$

Then applying to this member the definition of n -th root, we obtain:

$$\left[\left(\sqrt[m]{\sqrt[n]{o}} \right)^m \right]^n = \left(\sqrt[n]{o} \right)^n = o$$

By applying the same definition to the second member we obtain an equivalent final result:

$$\left(\sqrt[m \cdot n]{o} \right)^{(m \cdot n)} = o \quad \blacksquare$$

Theorem 2.72. *It is valid the product property of radicands, namely:*

$$\sqrt[n]{o_1} \cdot \sqrt[n]{o_2} = \sqrt[n]{o_1 \cdot o_2}$$

Proof. By applying the principle according to which two numbers are equal if and only if they remain as such, also once we raise them to the same power, we can raise the two member of the previous equality to the number n , obtaining:

$$\left(\sqrt[n]{o_1} \cdot \sqrt[n]{o_2} \right)^n = \left(\sqrt[n]{o_1 \cdot o_2} \right)^n$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

Considering the values $\sqrt[n]{o_1}$ and $\sqrt[n]{o_2}$ of the first member as the complete numbers, it is possible to apply to them the theorem 2.68 concerning the product of bases of the n -th power, obtaining:

$$\left(\sqrt[n]{o_1} \cdot \sqrt[n]{o_2} \right)^n = \left(\sqrt[n]{o_1} \right)^n \cdot \left(\sqrt[n]{o_2} \right)^n$$

Then applying to two factors of this member the definition of n -th root, we obtain:

$$\left(\sqrt[n]{o_1} \right)^n \cdot \left(\sqrt[n]{o_2} \right)^n = o_1 \cdot o_2$$

By applying the same definition to the second member we obtain an equivalent final result:

$$\left(\sqrt[n]{o_1 \cdot o_2} \right)^n = o_1 \cdot o_2 \quad \blacksquare$$

Theorem 2.73. *It is valid the quotient property of radicands, namely:*

$$\frac{\sqrt[n]{o_1}}{\sqrt[n]{o_2}} = \sqrt[n]{\frac{o_1}{o_2}}$$

Proof. By applying the principle according to which two numbers are equal if and only if they remain as such, also once we raise them to the same power, we can raise the two member of the previous equality to the number n , obtaining:

$$\left(\frac{\sqrt[n]{o_1}}{\sqrt[n]{o_2}}\right)^n = \left(\sqrt[n]{\frac{o_1}{o_2}}\right)^n$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

Considering the values $\sqrt[n]{o_1}$ and $\sqrt[n]{o_2}$ of the first member as the complete numbers, it is possible to apply to them the theorem 2.69 concerning the quotient of bases of the n -th power, obtaining:

$$\left(\frac{\sqrt[n]{o_1}}{\sqrt[n]{o_2}}\right)^n = \frac{(\sqrt[n]{o_1})^n}{(\sqrt[n]{o_2})^n}$$

Then applying to two factors of this member the definition of n -th root, we obtain:

$$\frac{(\sqrt[n]{o_1})^n}{(\sqrt[n]{o_2})^n} = \frac{o_1}{o_2}$$

By applying the same definition to the second member we obtain an equivalent final result:

$$\left(\sqrt[n]{\frac{o_1}{o_2}}\right)^n = \frac{o_1}{o_2}$$

■

2.8. Power with rational exponent

Definition 2.74. In the space RIU we can define power with rational exponent $\frac{m}{n}$ (n, m both natural numbers) of the complete number $o(t, \theta, \gamma)$, with $\frac{m}{n}$ known as rational exponent and $o(t, \theta, \gamma)$ known as base, as the number $o_{\uparrow m \downarrow n}(t_{\uparrow m \downarrow n}, \theta_{\uparrow m \downarrow n}, \gamma_{\uparrow m \downarrow n})$ also represented with the symbol $o(t, \theta, \gamma)^{\frac{m}{n}}$ or $\sqrt[n]{o(t, \theta, \gamma)^m}$ that satisfies the following conditions:

1. $\left[\sqrt[n]{o(t, \theta, \gamma)^m}\right]^n = o(t, \theta, \gamma)^m$
2. $m > 0$ for $o(t, \theta, \gamma) = 0$
3. $n \neq 0$ for any $o(t, \theta, \gamma)^m$ and therefore for any $o(t, \theta, \gamma)$

4. $n \geq 0$ for $o(t, \theta, \gamma)^m = 0$ and therefore for $o(t, \theta, \gamma) = 0$
5. $\theta_{\uparrow m \downarrow n} = \frac{\theta \cdot m}{n}$, $\gamma_{\uparrow m \downarrow n} = \frac{\gamma \cdot m}{n}$
6. $\sqrt[n]{t^m} > 0$, $t^m > 0$
7. $\sqrt[n]{t} > 0$, $t > 0$

The first condition defines the power with rational exponent as a n -th root of a m -th power.

The second condition is required for the correct definition of the m -th power.

The third, the fourth, the fifth and the sixth conditions are required for the correct definition of n -th root.

The seventh condition is required to make possible the reversal of the order between root and power, namely to write:

$$\left[\sqrt[n]{o(t, \theta, \gamma)} \right]^m$$

and therefore:

$$(\sqrt[n]{t})^m$$

Theorem 2.75. *It is valid the inversion property between root and power, namely:*

$$o^{\frac{m}{n}} = (\sqrt[n]{o})^m$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\sqrt[n]{o^m} = (\sqrt[n]{o})^m$$

By applying the principle according to which two numbers are equal if and only if they remain as such, also once we raise them to the same power, we can raise the two member of the previous equality to the number n obtaining:

$$(\sqrt[n]{o^m})^n = [(\sqrt[n]{o})^m]^n$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

Considering the value $\sqrt[n]{o}$ of the second member as a complete number, it is possible to apply to it the theorem 2.65 concerning the product of exponents of the n -th power, obtaining:

$$[(\sqrt[n]{o})^m]^n = (\sqrt[n]{o})^{m \cdot n} = (\sqrt[n]{o})^{n \cdot m} = [(\sqrt[n]{o})^n]^m$$

Then applying to this member the definition of n -th root , we obtain:

$$[(\sqrt[n]{o})^n]^m = o^m$$

By applying the same definition to the first member we obtain an equivalent final result:

$$(\sqrt[n]{o^m})^n = o^m \quad \blacksquare$$

Theorem 2.76. *It is valid the equivalence property between exponent and degree, namely:*

$$o^{\frac{m}{n}} = o^{\frac{m \cdot p}{n \cdot p}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\sqrt[n]{o^m} = \sqrt[n \cdot p]{o^{m \cdot p}}$$

By applying the principle according to which two numbers are equal if and only if they remain as such, also once we raise them to the same power, we can raise the two member of the previous equality to the number $(n \cdot p)$ obtaining:

$$(\sqrt[n]{o^m})^{(n \cdot p)} = (\sqrt[n \cdot p]{o^{m \cdot p}})^{(n \cdot p)}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

By applying to the first member the theorem 2.75 concerning the inversion between root and power of the power with rational exponent, we obtain:

$$(\sqrt[n]{o^m})^{(n \cdot p)} = [(\sqrt[n]{o})^m]^{(n \cdot p)}$$

Considering the value $\sqrt[n]{o}$ of this member as a complete number, it is possible to apply to it the theorem 2.65 concerning the product of exponents of the n-th power, obtaining:

$$[(\sqrt[n]{o})^m]^{(n \cdot p)} = (\sqrt[n]{o})^{m \cdot n \cdot p} = (\sqrt[n]{o})^{n \cdot m \cdot p} = [(\sqrt[n]{o})^n]^{(m \cdot p)}$$

Then applying to this member the definition of n-th root , we obtain:

$$[(\sqrt[n]{o})^n]^{(m \cdot p)} = o^{m \cdot p}$$

By applying the same definition to the second member we obtain an equivalent final result:

$$(\sqrt[n \cdot p]{o^{m \cdot p}})^{(n \cdot p)} = o^{m \cdot p} \quad \blacksquare$$

Theorem 2.77. *It is valid the product property of rational exponents, namely:*

$$(o^{\frac{m}{n}})^{\frac{p}{q}} = o^{\frac{m}{n} \cdot \frac{p}{q}} = o^{\frac{m \cdot p}{n \cdot q}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$(\sqrt[n]{o^m})^{\frac{p}{q}} = \sqrt[n \cdot q]{o^{m \cdot p}}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

Let us start expressing the first member in the following way:

$$(\sqrt[n]{o^m})^{\frac{p}{q}} = \sqrt[q]{(\sqrt[n]{o^m})^p}$$

Considering the value o^m of this member as a complete number, it is possible to apply to it the theorem 2.75 concerning the inversion between root and power of the power with rational exponent, obtaining:

$$\sqrt[q]{(\sqrt[n]{o^m})^p} = \sqrt[q]{\sqrt[n]{(o^m)^p}}$$

Then applying to this member the theorem 2.71 concerning the product of degrees of the n -th root and the theorem 2.65 concerning the product of exponents of the n -th power, we obtain an expression coincident with the second member:

$$\sqrt[q]{\sqrt[n]{(o^m)^p}} = \sqrt[q \cdot n]{o^{m \cdot p}} \quad \blacksquare$$

Theorem 2.78. *It is valid the sum property of rational exponents, namely:*

$$(o^{\frac{m}{n}}) \cdot (o^{\frac{p}{q}}) = o^{(\frac{m}{n} + \frac{p}{q})} = o^{\frac{(m \cdot q) + (p \cdot n)}{n \cdot q}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\sqrt[n]{o^m} \cdot \sqrt[q]{o^p} = \sqrt[n \cdot q]{o^{(m \cdot q + p \cdot n)}}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

By applying to the first member the theorem 2.76 concerning the equivalence between exponent and degree of the power with rational exponent, we obtain:

$$\sqrt[n]{o^m} \cdot \sqrt[q]{o^p} = \sqrt[n \cdot q]{o^{m \cdot q}} \cdot \sqrt[q \cdot n]{o^{p \cdot n}}$$

Considering the values $o^{m \cdot q}$ and $o^{p \cdot n}$ of this member as the complete numbers, it is possible to apply to it the theorem 2.72 concerning the product of radicands of the n -th root, obtaining:

$$\sqrt[n \cdot q]{o^{m \cdot q}} \cdot \sqrt[q \cdot n]{o^{p \cdot n}} = \sqrt[n \cdot q]{o^{m \cdot q} \cdot o^{p \cdot n}}$$

Then applying to this member the theorem 2.66 concerning the sum of exponents of the n -th power, we obtain an expression coincident with the second member:

$$\sqrt[n \cdot q]{o^{m \cdot q} \cdot o^{p \cdot n}} = \sqrt[n \cdot q]{o^{(m \cdot q + p \cdot n)}} \quad \blacksquare$$

Theorem 2.79. *It is valid the difference property of rational exponents, namely:*

$$\frac{(o^{\frac{m}{n}})}{(o^{\frac{p}{q}})} = o^{(\frac{m}{n} - \frac{p}{q})} = o^{\frac{(m \cdot q) - (p \cdot n)}{n \cdot q}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\frac{\sqrt[n]{o^m}}{\sqrt[q]{o^p}} = \sqrt[n \cdot q]{o^{(m \cdot q - p \cdot n)}}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

By applying to the first member the theorem 2.76 concerning the equivalence between exponent and degree of the power with rational exponent, we obtain:

$$\frac{\sqrt[n]{o^m}}{\sqrt[q]{o^p}} = \frac{\sqrt[n \cdot q]{o^{m \cdot q}}}{\sqrt[q \cdot n]{o^{p \cdot n}}}$$

Considering the values $o^{m \cdot q}$ and $o^{p \cdot n}$ of this member as the complete numbers, it is possible to apply to it the theorem 2.73 concerning the quotient of radicands of the n-th root, obtaining:

$$\frac{\sqrt[n \cdot q]{o^{m \cdot q}}}{\sqrt[q \cdot n]{o^{p \cdot n}}} = \sqrt[n \cdot q]{\frac{o^{m \cdot q}}{o^{p \cdot n}}}$$

Then applying to this member the theorem 2.67 concerning the difference of exponents of the n-th power, we obtain an expression coincident with the second member:

$$\sqrt[n \cdot q]{\frac{o^{m \cdot q}}{o^{p \cdot n}}} = \sqrt[n \cdot q]{o^{(m \cdot q - p \cdot n)}} \quad \blacksquare$$

Theorem 2.80. *It is valid the product property of bases, namely:*

$$\left(o_1^{\frac{m}{n}}\right) \cdot \left(o_2^{\frac{m}{n}}\right) = \left(o_1 \cdot o_2\right)^{\frac{m}{n}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\sqrt[n]{o_1^m} \cdot \sqrt[n]{o_2^m} = \sqrt[n]{(o_1 \cdot o_2)^m}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

By applying to the second member the theorem 2.68 concerning the product of bases of the n-th power, we obtain:

$$\sqrt[n]{(o_1 \cdot o_2)^m} = \sqrt[n]{o_1^m \cdot o_2^m}$$

Considering the values o_1^m and o_2^m of this member as the complete numbers, it is possible to apply to it the theorem 2.72 concerning the product of radicands of the n-th root, obtaining an expression coincident with the first member:

$$\sqrt[n]{o_1^m \cdot o_2^m} = \sqrt[n]{o_1^m} \cdot \sqrt[n]{o_2^m} \quad \blacksquare$$

Theorem 2.81. *It is valid the quotient property of bases, namely:*

$$\frac{o_1^{\frac{m}{n}}}{o_2^{\frac{m}{n}}} = \left(\frac{o_1}{o_2}\right)^{\frac{m}{n}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\frac{\sqrt[n]{o_1^m}}{\sqrt[n]{o_2^m}} = \sqrt[n]{\left(\frac{o_1}{o_2}\right)^m}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

By applying to the second member the theorem 2.69 concerning the quotient of bases of the n -th power, we obtain:

$$\sqrt[n]{\left(\frac{o_1}{o_2}\right)^m} = \sqrt[n]{\frac{o_1^m}{o_2^m}}$$

Considering the values o_1^m and o_2^m of this member as the complete numbers, it is possible to apply to it the theorem 2.73 concerning the quotient of radicands of the n -th root, obtaining an expression coincident with the first member:

$$\sqrt[n]{\frac{o_1^m}{o_2^m}} = \frac{\sqrt[n]{o_1^m}}{\sqrt[n]{o_2^m}} \quad \blacksquare$$

3. Numbers in the n dimensional space

3.1. N dimensional complete numbers

To identify the n dimensional complete numbers, we will use the following notations:

1. $o(a)$ or $o(t)$ for the real numbers
2. $o(a, b)$ or $o(t, \theta)$ for the complex numbers
3. $o(a, b, c)$ or $o(t, \theta, \gamma)$ for the complete number strictly speaking
4. $o(a, b, c, d)$ or $o(t, \theta, \gamma, \varphi)$ for the four dimensional complete numbers
5. ...
6. $o(a_1, a_2, \dots, a_n)$ or $o(t, \theta_2, \theta_3, \dots, \theta_n)$ for the n dimensional complete numbers

Definition 3.1. We can define n dimensional complete number $o(t, \theta_2, \theta_3, \dots, \theta_n)$ as the position that can be reached starting from that unitary of the straight line V_1 first translating it of modulus t , then making the line R turn of the angle θ_2 in the plane V_1V_n , next making the plane V_1V_n turn of the angle θ_3 in the space $V_1V_{n-1}V_n$, after that making the space $V_1V_{n-1}V_n$ turn of the angle θ_4 in the hyperspace $V_1V_{n-2}V_{n-1}V_n$, and so on up to the rotation of angle θ_n of the n dimensional space $V_1V_2\dots V_{n-2}V_{n-1}V_n$.

In Figure 30 we can observe a complete number in the four dimensional space.

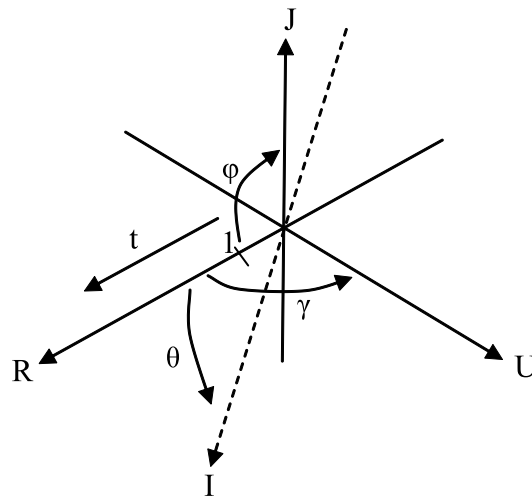


Figure 30: Cartesian representation of the four dimensional complete numbers

Theorem 3.2. *N dimensional complete numbers can be expressed in the following way:*

$$\begin{aligned}
 (3.1) \quad o(t, \theta_2, \theta_3, \dots, \theta_n) = & t \cdot \{ v_1 \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot \dots \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \cos(\theta_3) \cdot \cos(\theta_2)] \\
 & + v_2 \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot \dots \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \cos(\theta_3) \cdot \sin(\theta_2)] \\
 & + v_3 \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot \dots \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \sin(\theta_3)] \\
 & + v_4 \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot \dots \cdot \cos(\theta_5) \cdot \sin(\theta_4)] \\
 & + \dots + \\
 & + v_{n-1} \cdot [\cos(\theta_n) \cdot \sin(\theta_{n-1})] \\
 & + v_n \cdot [\sin(\theta_n)] \}
 \end{aligned}$$

with the symbols v_1, v_2, \dots, v_n that identify the versors concerning the orthogonal straight lines V_1, V_2, \dots, V_n that form the n dimensional space, the symbols $\theta_2, \theta_3, \dots, \theta_n$ the rotations used to introduce such lines (the line V_1 is introduced by the translating t), and the following symbols a_1, a_2, \dots, a_n constitute the coordinates of the complete number $o(t, \theta_2, \theta_3, \dots, \theta_n)$ in the n dimensional space:

$$\begin{aligned}
 a_1 &= t \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot \dots \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \cos(\theta_3) \cdot \cos(\theta_2)] \\
 a_2 &= t \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot \dots \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \cos(\theta_3) \cdot \sin(\theta_2)] \\
 a_3 &= t \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot \dots \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \sin(\theta_3)] \\
 a_4 &= t \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot \dots \cdot \cos(\theta_5) \cdot \sin(\theta_4)] \\
 &\dots \\
 a_n &= t \cdot [\sin(\theta_n)]
 \end{aligned}$$

Proof. By observing in Figure 31 how the addition of a new rotation allows us to express the coordinates of the complete numbers from one dimension to the next we obtain the previous relation. ■

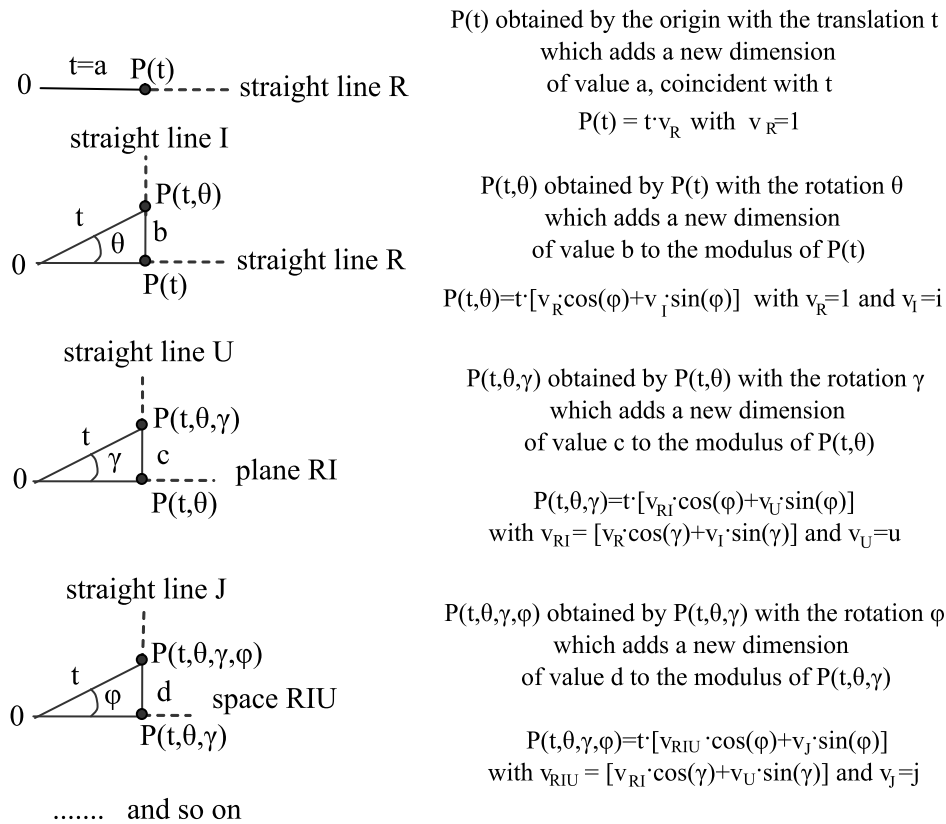


Figure 31: Construction of the n dimensional complete numbers

Theorem 3.3. *The modulus of the n dimensional complete numbers can be expressed in the following way:*

$$t = \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_n^2}$$

Proof. By applying Pythagoras' theorem to the steps leading us to the next dimensions, as shown by Figure 32 on the next page, we obtain the previous relation. ■

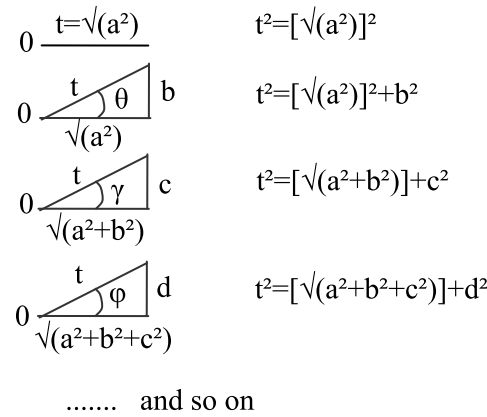


Figure 32: Representation of the modulus of the n dimensional complete numbers

Theorem 3.4. *The phases of the n dimensional complete numbers can be expressed in the following way:*

$$\theta_n = \arctan \left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \dots + a_{n-1}^2}} \right)$$

Proof. By applying the trigonometric relations of the function $\arctan()$ to the steps leading us to the next dimensions, as shown by Figure 33, we obtain the previous relation.

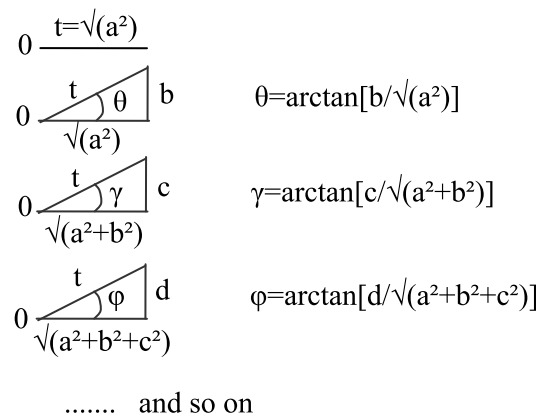


Figure 33: Representation of the phases of the n dimensional complete numbers

■

Definition 3.5. An n dimensional complete numbers with coordinates $(a_1, a_2, a_3, \dots, a_n)$ all non zero can be defined in standard representation if provided with phases $(\theta_2, \theta_3, \dots, \theta_n)$ that satisfy the conventions introduced hereunder.

For the positions $P(a_1, a_2, a_3, \dots, a_n)$ in the region $V_1^+ V_2^+ V_3^+ \dots V_n^+$, characterized by the values $a_1, a_2, a_3, \dots, a_n$ all positives, the phases chosen will lie in the first quadrant, namely:

$$0^\circ < \theta_2, \theta_3, \dots, \theta_n < 90^\circ$$

We can observe, with regard to this, Figure 34.

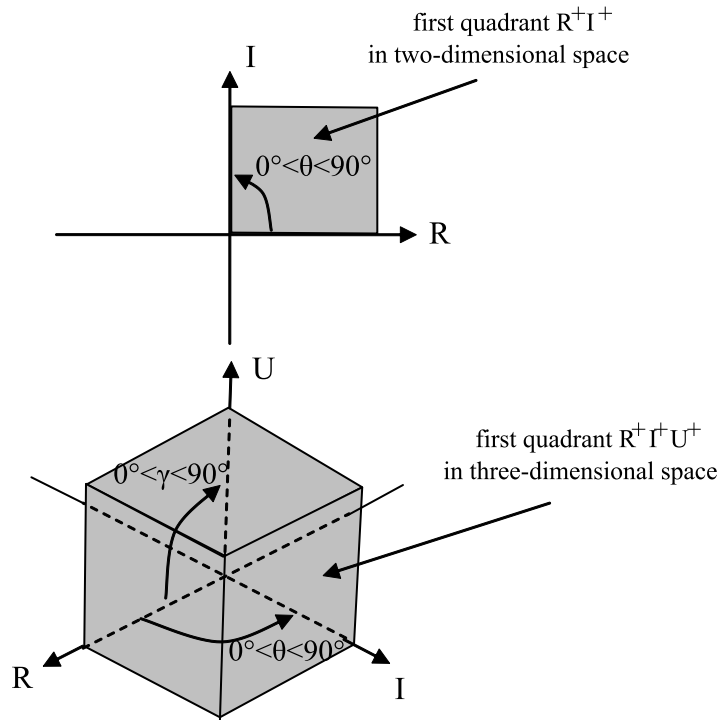


Figure 34: Standard representation of the phases θ, γ concerning the first quadrants

Since the following relations are valid:

$$\theta_2 = \arctan \left(\frac{a_2}{\sqrt{a_1^2}} \right)$$

$$\theta_3 = \arctan \left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}} \right)$$

...

$$\theta_n = \arctan \left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \dots + a_{n-1}^2}} \right)$$

to allow the phases $\theta_2, \theta_3, \dots, \theta_n$ to have a value between 0° and 90° when the coefficients a_2, a_3, \dots, a_n are all positives, also the corresponding denominators should be positives. This means that the standard representation requires that we assign the positive solutions to the following roots:

$$\begin{aligned}
\sqrt{a_1^2} &= \left| \sqrt{a_1^2} \right| \\
\sqrt{a_1^2 + a_2^2} &= \left| \sqrt{a_1^2 + a_2^2} \right| \\
&\dots \\
\sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2} &= \left| \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2} \right|
\end{aligned}$$

For the positions $P(a_1, a_2, a_3, \dots, a_n)$ in the region $V_1^- V_2^+ V_3^+ \dots V_n^+$, characterized by the values $a_2, a_3, a_4, \dots, a_n$ all positives and by the value a_1 negative, the phases chosen will be the following:

$$\begin{aligned}
90^\circ &< \theta_2 < 180^\circ \\
0^\circ &< \theta_3, \theta_4, \dots, \theta_n < 90^\circ
\end{aligned}$$

Since the following relations are valid:

$$\begin{aligned}
\sin(180^\circ - \theta_2) &= \sin(\theta_2) \\
\cos(180^\circ - \theta_2) &= -\cos(\theta_2)
\end{aligned}$$

to impose the coefficient a_1 as the only negative value in the formula (3.1), will be enough to leave unchanged all phases $\theta_3, \theta_4, \dots, \theta_n$ at the value they have in the first quadrant, and change the value of $\theta_2 = \theta_2^*$ (that is the value that this phase assumes in the first quadrant) with $\theta_2 = (180^\circ - \theta_2^*)$.

We can observe, with regard to this, Figure 35 on the facing page.

Since the following relations are valid:

$$\begin{aligned}
\theta_2 &= \arctan \left(\frac{a_2}{\sqrt{a_1^2}} \right) \\
\theta_3 &= \arctan \left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}} \right) \\
&\dots \\
\theta_n &= \arctan \left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2}} \right)
\end{aligned}$$

to allow the phases $\theta_3, \theta_4, \dots, \theta_n$ to have a value between 0° and 90° when the coefficients a_3, a_4, \dots, a_n are all positives, also the corresponding denominators should be positives. While to allow the phase θ_2 to have a value between 90° and 180° when the coefficient a_1 is negative and that a_2 is positive, we should consider the term which appears into its denominator as negative. This means that the standard representation requires that we assign the positive solutions to the following roots:

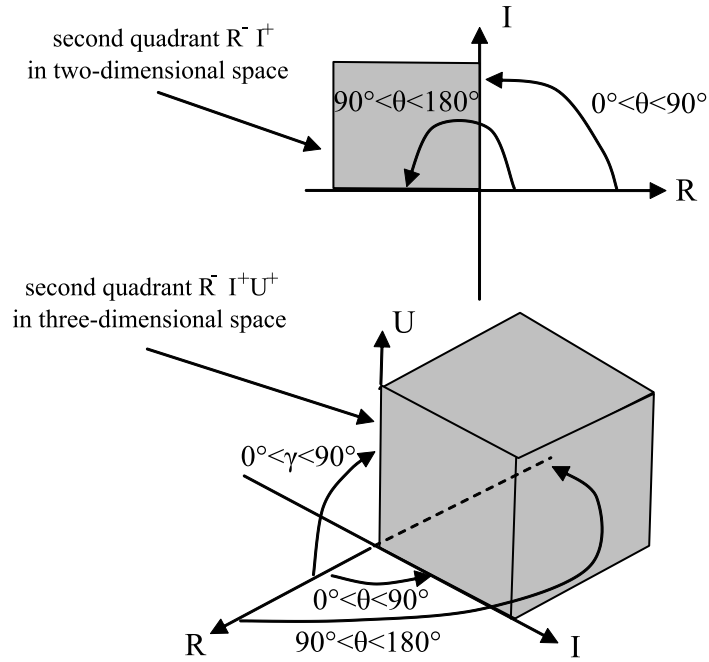


Figure 35: Standard representation of the phases θ, γ concerning the second quadrants

$$\begin{aligned} \sqrt{a_1^2 + a_2^2} &= \left| \sqrt{a_1^2 + a_2^2} \right| \\ \sqrt{a_1^2 + a_2^2 + a_3^2} &= \left| \sqrt{a_1^2 + a_2^2 + a_3^2} \right| \\ &\dots \\ \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2} &= \left| \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2} \right| \end{aligned}$$

and the negative solutions to:

$$\sqrt{a_1^2} = -\left| \sqrt{a_1^2} \right|$$

For the positions $P(a_1, a_2, a_3, \dots, a_n)$ in the region $V_1^- V_2^- V_3^+ \dots V_n^+$, characterized by the values $a_3, a_4, a_5, \dots, a_n$ all positives and by the values a_1, a_2 negative, the phases chosen will be the following:

$$\begin{aligned} 180^\circ < \theta_2 < 270^\circ \\ 0^\circ < \theta_3, \theta_4, \dots, \theta_n < 90^\circ \end{aligned}$$

Since the following relations are valid:

$$\begin{aligned} \sin(180^\circ + \theta_2) &= -\sin(\theta_2) \\ \cos(180^\circ + \theta_2) &= -\cos(\theta_2) \end{aligned}$$

to impose the coefficients a_1 and a_2 as the only negative values in the formula (3.1), will be enough to leave unchanged all phases $\theta_3, \theta_4, \dots, \theta_n$ at the value they have in the first quadrant, and change the value of $\theta_2 = \theta_2^*$ (that is the value that this phase assumes in the first quadrant) with $\theta_2 = (180^\circ + \theta_2^*)$.

We can observe, with regard to this, Figure 36.

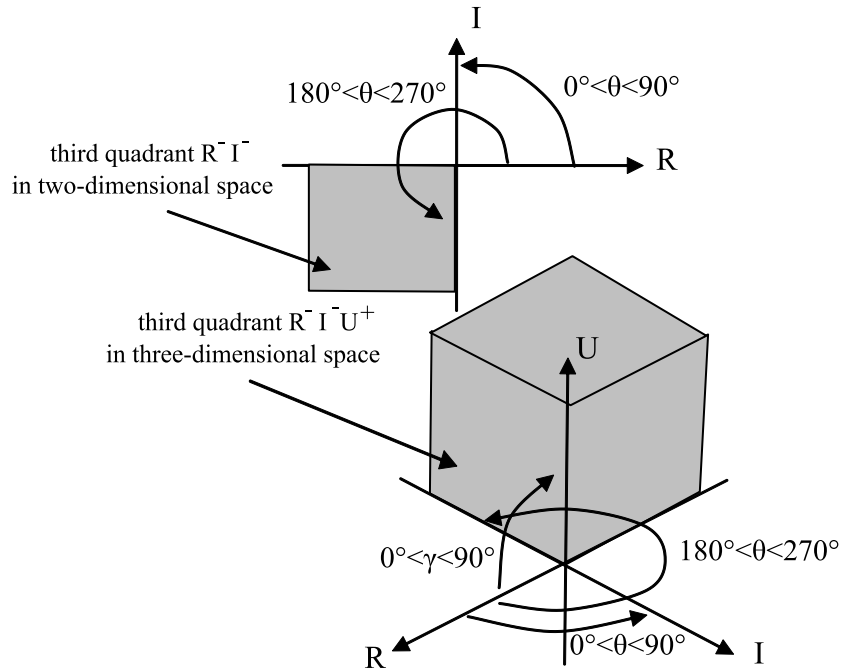


Figure 36: Standard representation of the phases θ, γ concerning the third quadrants

Since the following relations are valid:

$$\theta_2 = \arctan \left(\frac{a_2}{\sqrt{a_1^2}} \right)$$

$$\theta_3 = \arctan \left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}} \right)$$

...

$$\theta_n = \arctan \left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2}} \right)$$

to allow the phases $\theta_3, \theta_4, \dots, \theta_n$ to have a value between 0° and 90° when the coefficients a_3, a_4, \dots, a_n are all positives, also the corresponding denominators should be positives. While to allow the phase θ_2 to have a value between 180° and 270° when the coefficient a_1 and a_2 are negative, we should consider the term which appears into its denominator as negative. This means that the standard representation requires that we assign the positive solutions to the following roots:

$$\begin{aligned} \sqrt{a_1^2 + a_2^2} &= \left| \sqrt{a_1^2 + a_2^2} \right| \\ \sqrt{a_1^2 + a_2^2 + a_3^2} &= \left| \sqrt{a_1^2 + a_2^2 + a_3^2} \right| \\ &\dots \\ \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2} &= \left| \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2} \right| \end{aligned}$$

and the negative solutions to:

$$\sqrt{a_1^2} = -\left| \sqrt{a_1^2} \right|.$$

For the positions $P(a_1, a_2, a_3, \dots, a_n)$ in the region $V_1^+ V_2^- V_3^+ \dots V_n^+$, characterized by the values $a_1, a_3, a_4, \dots, a_n$ all positives and by the value a_2 negative, the phases chosen will be the following:

$$\begin{aligned} 270^\circ < \theta_2 < 360^\circ \\ 0^\circ < \theta_3, \theta_4, \dots, \theta_n < 90^\circ \end{aligned}$$

Since the following relations are valid:

$$\begin{aligned} \sin(360^\circ - \theta_2) &= -\sin(\theta_2) \\ \cos(360^\circ - \theta_2) &= \cos(\theta_2) \end{aligned}$$

to impose the coefficient a_2 as the only negative value in the formula (3.1), will be enough to leave unchanged all phases $\theta_3, \theta_4, \dots, \theta_n$ at the value they have in the first quadrant, and change the value of $\theta_2 = \theta_2^*$ (that is the value that this phase assumes in the first quadrant) with $\theta_2 = (360^\circ - \theta_2^*)$.

We can observe, with regard to this, Figure 37 on the following page.

Since the following relations are valid:

$$\begin{aligned} \theta_2 &= \arctan \left(\frac{a_2}{\sqrt{a_1^2}} \right) \\ \theta_3 &= \arctan \left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}} \right) \\ &\dots \\ \theta_n &= \arctan \left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2}} \right) \end{aligned}$$

to allow the phases $\theta_3, \theta_4, \dots, \theta_n$ to have a value between 0° and 90° when the coefficients a_3, a_4, \dots, a_n are all positives, also the corresponding denominators should be positives. While to allow the phase θ_2 to have a value between 270° and 360° when the coefficient a_2 is negative and that a_1 is positive, we should consider the

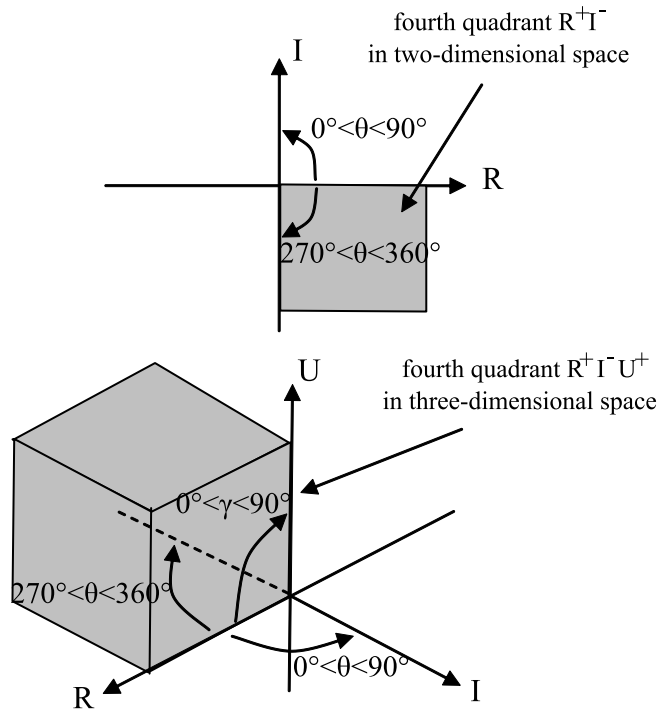


Figure 37: Standard representation of the phases θ, γ concerning the fourth quadrants

term which appears into its denominator as positive. This means that the standard representation requires that we assign the positive solutions to the following roots:

$$\begin{aligned}\sqrt{a_1^2} &= \left| \sqrt{a_1^2} \right| \\ \sqrt{a_1^2 + a_2^2} &= \left| \sqrt{a_1^2 + a_2^2} \right| \\ \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots + a_{n-1}^2} &= \left| \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots + a_{n-1}^2} \right|\end{aligned}$$

For the positions $P(a_1, a_2, a_3, \dots, a_n)$ in the region $V_1 V_2 V_3 \dots V_n$, characterized by the values $a_3, a_4, a_5, \dots, a_n$ both positives and negative, the phases chosen will be the following:

$$\begin{aligned}0^\circ < \theta_i < 90^\circ & \text{ for any } a_i > 0 \text{ with } i = 3, 4, 5, \dots, n \\ 270^\circ < \theta_i < 360^\circ & \text{ for any } a_i < 0 \text{ with } i = 3, 4, 5, \dots, n\end{aligned}$$

Since the following relations are valid:

$$\begin{aligned}\sin(360^\circ - \theta_i) &= -\sin(\theta_i) \\ \cos(360^\circ - \theta_i) &= \cos(\theta_i)\end{aligned}$$

to impose the negative sign to some of the coefficients a_3, a_4, \dots, a_n in the formula (3.1), will be enough to assign to the corresponding phases $\theta_3, \theta_4, \dots, \theta_n$ the opposite

value with respect to that they have in the first quadrant (and therefore to assign them a value between 270° and 360°) and leave all the others unchanged.

We can observe, with regard to this, Figure 38.

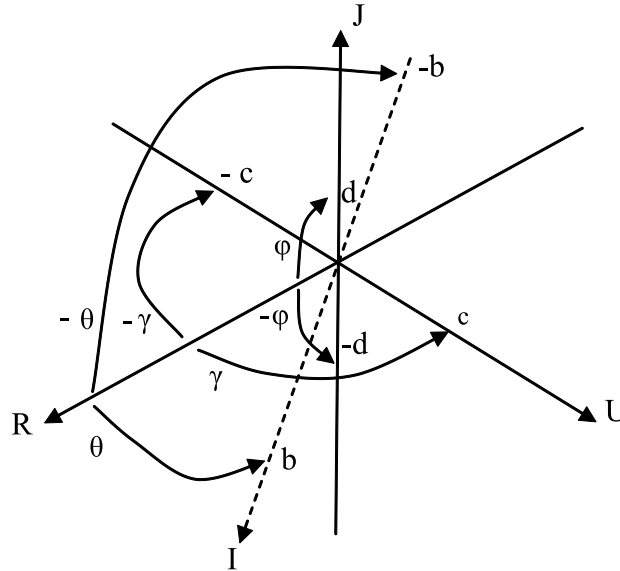


Figure 38: Variation of the sign of the phases due to the variation of sign of their corresponding coefficient

Since the following relations are valid:

$$\theta_3 = \arctan \left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}} \right)$$

$$\theta_4 = \arctan \left(\frac{a_4}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \right)$$

...

$$\theta_n = \arctan \left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2}} \right)$$

to allow the phases $\theta_3, \theta_4, \dots, \theta_n$ to have a value between 0° and 90° when the corresponding coefficients a_3, a_4, \dots, a_n are positives, and a value between 270° and 360° when the corresponding coefficients are negative, the corresponding denominators should be all positives. This means that the standard representation requires that we assign the positive solutions to the following roots:

$$\sqrt{a_1^2 + a_2^2} = \left| \sqrt{a_1^2 + a_2^2} \right|$$

$$\sqrt{a_1^2 + a_2^2 + a_3^2} = \left| \sqrt{a_1^2 + a_2^2 + a_3^2} \right|$$

$$\sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2} = \left| \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2} \right|$$

Since the management of sign of the coefficients a_3, a_4, \dots, a_n does not interfere with the angle θ_2 , we can combine it with the management of signs of a_1 and a_2 according to the manner described above.

Theorem 3.6. *The standard representation of a n dimensional complete number of coordinates $(a_1, a_2, a_3, \dots, a_n)$ all non zero requires to give the following solutions to the following algebraic roots:*

$$\begin{aligned}\sqrt{a_1^2} &= a_1 \\ \sqrt{a_1^2 + a_2^2} &= \left| \sqrt{a_1^2 + a_2^2} \right| \\ \sqrt{a_1^2 + a_2^2 + a_3^2} &= \left| \sqrt{a_1^2 + a_2^2 + a_3^2} \right| \\ &\dots \\ \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2} &= \left| \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2} \right|\end{aligned}$$

Proof. In the case of the representations previously examined the phases assume the values provided by the formulas:

$$\begin{aligned}\theta_2 &= \arctan \left(\frac{a_2}{\sqrt{a_1^2}} \right) \\ \theta_3 &= \arctan \left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}} \right) \\ &\dots \\ \theta_n &= \arctan \left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_{n-1}^2}} \right)\end{aligned}$$

when we give to the algebraic roots involved just the values considered here. And this immediately proves the thesis. ■

For example in the four dimensional space the complete number with the expression:

$$\begin{aligned}o(t, \theta, \gamma, \varphi) &= [\cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta)] + i \cdot [\cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta)] \\ &\quad + u \cdot [\cos(\varphi) \cdot \sin(\gamma)] + j \cdot [\sin(\varphi)]\end{aligned}$$

associated to the position:

$$o(a, b, c, d) = o(-1, 1, -1, 1)$$

could be expressed in standard representation through the following phases:

$$\begin{aligned}\theta &= \arctan \left(\frac{b}{a} \right) = \arctan \left(\frac{1}{-1} \right) = 135^\circ \\ \gamma &= \arctan \left(\frac{c}{|\sqrt{a^2 + b^2}|} \right) = \arctan \left(\frac{-1}{\sqrt{2}} \right) \simeq -35.26^\circ \\ \varphi &= \arctan \left(\frac{d}{|\sqrt{a^2 + b^2 + c^2}|} \right) = \arctan \left(\frac{1}{\sqrt{3}} \right) = 30^\circ\end{aligned}$$

and the following modulus:

$$t = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{4} = 2$$

To verify that the standard representation $o(\theta, \gamma, \varphi)$ thus obtained identifies just the position $o(-1, 0, 1, 0)$ it is sufficient to perform the following calculations:

$$\begin{aligned} a &= t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta) = 2 \cdot \cos(30^\circ) \cdot \cos(\simeq -35.26^\circ) \cdot \cos(135^\circ) = -1 \\ b &= t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta) = 2 \cdot \cos(30^\circ) \cdot \cos(\simeq -35.26^\circ) \cdot \sin(135^\circ) = 1 \\ c &= t \cdot \cos(\varphi) \cdot \sin(\gamma) = 2 \cdot \cos(30^\circ) \cdot \sin(\simeq -35.26^\circ) = -1 \\ d &= t \cdot \sin(\varphi) = 2 \cdot \sin(30^\circ) = 1 \end{aligned}$$

Definition 3.7. An n dimensional complete numbers with coordinates $(a_1, a_2, a_3, \dots, a_n)$ all non zero can be defined in complementary representation if provided with phases obtained by the values: $\theta_2, \theta_3, \dots, \theta_n$ of the standard representation through those substitutions which allow us to identify the same positions.

Theorem 3.8. *If we call $\theta_2, \theta_3, \dots, \theta_n$ the phases that allow to an n dimensional complete number provided with coordinates $(a_1, a_2, a_3, \dots, a_n)$ all non zero and in standard representation to identify any position of the space $V_1 V_2 V_3 \dots V_n$, the alternative sets of phases able to individuate the same position can be obtained by the following values: $\theta, (360^\circ - \theta), (180^\circ - \theta), (\theta + 180^\circ)$.*

Proof. The ability to express through the formula (3.1) the same positions of the standard representation, assigning to the phases the following values: $\theta, (360^\circ - \theta), (180^\circ - \theta), (\theta + 180^\circ)$ comes from the fact that in this way we maintain the moduli unchanged and introduce signs which can neutralize each other, as shown by the following relations:

$$\begin{aligned} \cos(\theta) &= \cos(\theta) \\ \sin(\theta) &= \sin(\theta) \\ \cos(360^\circ - \theta) &= \cos(\theta) \\ \sin(360^\circ - \theta) &= -\sin(\theta) \\ \cos(180^\circ - \theta) &= -\cos(\theta) \\ \sin(180^\circ - \theta) &= \sin(\theta) \\ \cos(\theta + 180^\circ) &= -\cos(\theta) \\ \sin(\theta + 180^\circ) &= -\sin(\theta) \end{aligned}$$

Using this process it is possible to combine, for example, the standard representation concerning the fourth dimension:

$$\begin{aligned} o(t, \theta, \gamma, \varphi) &= [\cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta)] + i \cdot [\cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta)] \\ &\quad + u \cdot [\cos(\varphi) \cdot \sin(\gamma)] + j \cdot [\sin(\varphi)] \end{aligned}$$

to the following complementary representations:

$$\begin{aligned}
o(t, \theta, \gamma + 180^\circ, 180^\circ - \varphi) &= [\cos(180^\circ - \varphi) \cdot \cos(\gamma + 180^\circ) \cdot \cos(\theta)] \\
&\quad + i \cdot [\cos(180^\circ - \varphi) \cdot \cos(\gamma + 180^\circ) \cdot \sin(\theta)] \\
&\quad + u \cdot [\cos(180^\circ - \varphi) \cdot \sin(\gamma + 180^\circ)] \\
&\quad + j \cdot [\sin(180^\circ - \varphi)]
\end{aligned}$$

$$\begin{aligned}
o(t, \theta + 180^\circ, 360^\circ - \gamma, 180^\circ - \varphi) &= [\cos(180^\circ - \varphi) \cdot \cos(360^\circ - \gamma) \cdot \cos(\theta + 180^\circ)] \\
&\quad + i \cdot [\cos(180^\circ - \varphi) \cdot \cos(360^\circ - \gamma) \cdot \sin(\theta + 180^\circ)] \\
&\quad + u \cdot [\cos(180^\circ - \varphi) \cdot \sin(360^\circ - \gamma)] \\
&\quad + j \cdot [\sin(180^\circ - \varphi)]
\end{aligned}$$

$$\begin{aligned}
o(t, \theta + 180^\circ, 180^\circ - \gamma, \varphi) &= [\cos(\varphi) \cdot \cos(180^\circ - \gamma) \cdot \cos(\theta + 180^\circ)] \\
&\quad + i \cdot [\cos(\varphi) \cdot \cos(180^\circ - \gamma) \cdot \sin(\theta + 180^\circ)] \\
&\quad + u \cdot [\cos(\varphi) \cdot \sin(180^\circ - \gamma)] \\
&\quad + j \cdot [\sin(\varphi)]
\end{aligned}$$

■

For example if you want to identify a complementary representation of the following four dimensional complete number:

$$o(a, b, c, d) = o(-1, 1, -1, 1)$$

whose standard representation is provided with the following phases:

$$\begin{aligned}
\theta^* &= 135^\circ \\
\gamma^* &\simeq -35.26^\circ \\
\varphi^* &= 30^\circ
\end{aligned}$$

and the following modulus:

$$t = 2$$

it is sufficient to perform the following calculations:

$$\begin{aligned}
\theta &= \theta^* = 135^\circ \\
\gamma &= \gamma^* + 180^\circ \simeq (\simeq -35.26^\circ) + 180^\circ \simeq 144.74^\circ \\
\varphi &= 180^\circ - \varphi^* = 180^\circ - 30^\circ = 150^\circ
\end{aligned}$$

To verify that the complementary representation $o(\theta, \gamma, \varphi)$ thus obtained identifies just the position $o(-1, 1, -1, 1)$ it is sufficient to perform the following calculations:

$$\begin{aligned}
a &= t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta) = 2 \cdot \cos(150^\circ) \cdot \cos(\simeq 144.74^\circ) \cdot \cos(135^\circ) = -1 \\
b &= t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta) = 2 \cdot \cos(150^\circ) \cdot \cos(\simeq 144.74^\circ) \cdot \sin(135^\circ) = 1 \\
c &= t \cdot \cos(\varphi) \cdot \sin(\gamma) = 2 \cdot \cos(150^\circ) \cdot \sin(\simeq 144.74^\circ) = -1 \\
d &= t \cdot \sin(\varphi) = 2 \cdot \sin(150^\circ) = 1
\end{aligned}$$

Definition 3.9. The n dimensional complete numbers with coordinates $(a_1, a_2, a_3, \dots, a_n)$ some of which are zero can be defined in standard representation if their phases besides to be consistent with those of the other standard representations (according to the definition 3.5) assume the zero value in case of indetermination.

The relations that give the values of the phases for the standard representation are the following:

$$\begin{aligned} \theta_2 &= \arctan \left(\frac{a_2}{a_1} \right) \\ \theta_3 &= \arctan \left(\frac{a_3}{|\sqrt{a_1^2 + a_2^2}|} \right) \\ \theta_3 &= \arctan \left(\frac{a_4}{|\sqrt{a_1^2 + a_2^2 + a_3^2}|} \right) \\ &\dots \\ \theta_n &= \arctan \left(\frac{a_n}{|\sqrt{a_1^2 + a_2^2 + a_3^2 + \dots + a_{n-1}^2}|} \right) \end{aligned}$$

Due to coefficients with zero value, we can have the following notable cases:

$$\begin{aligned} \theta_2 &= \arctan \left(\frac{0}{a_1} \right) = \begin{cases} 0^\circ & \text{for } a_1 > 0 \\ 180^\circ & \text{for } a_1 < 0 \end{cases} \\ \theta_i &= \arctan \left(\frac{a_i}{0} \right) = \begin{cases} 90^\circ & \text{for } a_i > 0 \\ 270^\circ & \text{for } a_i < 0 \end{cases} \\ \theta_i &= \arctan \left(\frac{0}{|x \neq 0|} \right) = 0^\circ \\ \theta_i &= \arctan \left(\frac{0}{0} \right) = 0^\circ \end{aligned}$$

For example in the four dimensional space the complete number with expression:

$$\begin{aligned} o(t, \theta, \gamma, \varphi) &= [\cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta)] + i \cdot [\cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta)] \\ &\quad + u \cdot [\cos(\varphi) \cdot \sin(\gamma)] + j \cdot [\sin(\varphi)] \end{aligned}$$

associated to the position:

$$o(a, b, c, d) = o(-1, 0, 1, 0)$$

could be expressed in standard representation through the following phases:

$$\begin{aligned} \theta &= \arctan \left(\frac{b}{a} \right) = \arctan \left(\frac{0}{-1} \right) = 180^\circ \\ \gamma &= \arctan \left(\frac{c}{|\sqrt{a^2 + b^2}|} \right) = \arctan \left(\frac{1}{1} \right) = 45^\circ \\ \varphi &= \arctan \left(\frac{d}{|\sqrt{a^2 + b^2 + c^2}|} \right) = \arctan \left(\frac{0}{\sqrt{2}} \right) = 0^\circ \end{aligned}$$

and the following modulus:

$$t = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{2}$$

To verify that the standard representation $o(\theta, \gamma, \varphi)$ thus obtained identifies just the position $o(-1, 0, 1, 0)$ it is sufficient to perform the following calculations:

$$a = t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta) = \sqrt{2} \cdot \cos(0^\circ) \cdot \cos(45^\circ) \cdot \cos(180^\circ) = -1$$

$$b = t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta) = \sqrt{2} \cdot \cos(0^\circ) \cdot \cos(45^\circ) \cdot \sin(180^\circ) = 0$$

$$c = t \cdot \cos(\varphi) \cdot \sin(\gamma) = \sqrt{2} \cdot \cos(0^\circ) \cdot \sin(45^\circ) = 1$$

$$d = t \cdot \sin(\varphi) = \sqrt{2} \cdot \sin(0^\circ) = 0$$

Definition 3.10. The n dimensional complete numbers with coordinates $(a_1, a_2, a_3, \dots, a_n)$ some of which are zero can be defined in complementary representation if their phases besides to be consistent with those of the standard representations (according to the definition 3.5) show cases of indetermination in correspondence of which they do not assume the zero value.

For example in the four dimensional space the complete number with expression:

$$o(t, \theta, \gamma, \varphi) = [\cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta)] + i \cdot [\cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta)] + \\ + u \cdot [\cos(\varphi) \cdot \sin(\gamma)] + j \cdot [\sin(\varphi)]$$

associated to the position:

$$o(a, b, c, d) = o(0, 0, 1, 0)$$

could be expressed in complementary representation through the following phases:

$$\theta = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{0}{0}\right) = 30^\circ \neq 0^\circ$$

$$\gamma = \arctan\left(\frac{c}{|\sqrt{a^2 + b^2}|}\right) = \arctan\left(\frac{1}{0}\right) = 90^\circ$$

$$\varphi = \arctan\left(\frac{d}{|\sqrt{a^2 + b^2 + c^2}|}\right) = \arctan\left(\frac{0}{1}\right) = 0^\circ$$

and the following modulus:

$$t = \sqrt{a^2 + b^2 + c^2 + d^2} = 1$$

To verify that the complementary representation $o(\theta, \gamma, \varphi)$ thus obtained identifies just the position $o(0, 0, 1, 0)$ it is sufficient to perform the following calculations:

$$a = t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta) = 1 \cdot \cos(0^\circ) \cdot \cos(90^\circ) \cdot \cos(30^\circ) = 0$$

$$b = t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta) = 1 \cdot \cos(0^\circ) \cdot \cos(90^\circ) \cdot \sin(30^\circ) = 0$$

$$c = t \cdot \cos(\varphi) \cdot \sin(\gamma) = 1 \cdot \cos(0^\circ) \cdot \sin(90^\circ) = 1$$

$$d = t \cdot \sin(\varphi) = 1 \cdot \sin(0^\circ) = 0$$

Since the non zero values associated to the indeterminate phases are unlimited, unlimited will also be the complementary representation defined here.

Theorem 3.11. N dimensional complete numbers (with $n > 2$) provided with coordinates $(a_1, a_2, a_3, \dots, a_n)$ cannot be expressed in the following way:

$$o(a_1, a_2, \dots, a_n) = v_1 \cdot a_1 + v_2 \cdot a_2 + \dots + v_n \cdot a_n$$

namely:

$$o(t, \theta_2, \dots, \theta_n) \neq o(a_1, a_2, \dots, a_n) = v_1 \cdot a_1 + v_2 \cdot a_2 + \dots + v_n \cdot a_n$$

Proof. The proof comes from the absence of bijection between translation and rotation operations of values $(t, \theta_2, \dots, \theta_n)$ and the positions (a_1, a_2, \dots, a_n) of the n dimensional space, since always exists (for every dimension higher than the second) the complementary representation with the following phases:

$$o(t, \theta_2, \dots, \theta_{(n-2)}, \theta_{(n-1)} + 180^\circ, 180^\circ - \theta_n)$$

In fact, if $(t, \theta_2, \dots, \theta_n)$ are the values that make true the formula (3.1) of the n dimensional complete numbers, this same expression will also be satisfied by values:

$$(t, \theta_2, \dots, \theta_{(n-2)}, \theta_{(n-1)} + 180^\circ, 180^\circ - \theta_n)$$

as shown by the following trigonometric relations:

$$\begin{aligned} \cos(180^\circ - \theta_n) \cdot \cos(\theta_{(n-1)} + 180^\circ) &= \cos(\theta_n) \cdot \cos(\theta_{(n-1)}) \\ \cos(180^\circ - \theta_n) \cdot \sin(\theta_{(n-1)} + 180^\circ) &= \cos(\theta_n) \cdot \sin(\theta_{(n-1)}) \\ \sin(180^\circ - \theta_n) &= \sin(\theta_n) \end{aligned}$$

■

Since it is impossible to associate the complete numbers to the individual positions of the n dimensional space, we can always express them in terms of their coordinates (a_1, a_2, \dots, a_n) , provided that we make explicit the phases involved as well.

In other words we should use the following notation:

$$o(a_1, a_2, \dots, a_n)_{(t, \theta_2, \dots, \theta_n)} = v_1 \cdot a_1(t) + v_2 \cdot a_2(\theta_2) + v_3 \cdot a_3(\theta_3) + \dots + v_n \cdot a_n(\theta_n)$$

where the values of $t, \theta_2, \dots, \theta_n$, if not yet given, should be reported to those which characterize the standard representation.

While any other notation of the following type:

$$o(a_1, a_2, \dots, a_n) = v_1 \cdot a_1 + v_2 \cdot a_2 + \dots + v_n \cdot a_n$$

that is devoid of sufficient information to trace the values of the phases $\theta_2, \dots, \theta_n$ will be able to represent the positions of the n dimensional space, but not the complete numbers.

3.2. N dimensional operations

Definition 3.12. In the space $V_1V_2V_3\dots V_n$ we can define addition between two positions $o(a_{11}, a_{21}, \dots, a_{n1})$ and $o(a_{12}, a_{22}, \dots, a_{n2})$ as the position $o(a_{1(1+2)}, a_{2(1+2)}, \dots, a_{n(1+2)})$ represented also with the symbol $o(a_{11}, a_{21}, \dots, a_{n1}) + o(a_{12}, a_{22}, \dots, a_{n2})$ that satisfies the following condition:

$$o_{1+2}(a_{1(1+2)}, a_{2(1+2)}, \dots, a_{n(1+2)}) = o_{1+2}(a_{11} + a_{12}, a_{21} + a_{22}, \dots, a_{n1} + a_{n2})$$

This condition is equivalent to take the position of the space $V_1V_2V_3\dots V_n$ provided with the following coordinates:

$$\begin{aligned} a_{1(1+2)} &= a_{11} + a_{12} \\ a_{2(1+2)} &= a_{21} + a_{22} \\ &\dots \\ a_{n(1+2)} &= a_{n1} + a_{n2} \end{aligned}$$

For example in the fourth dimension we have:

$$o_{1+2}(a_{1+2}, b_{1+2}, c_{1+2}, d_{1+2}) = o_{1+2}(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

with:

$$\begin{aligned} a_{1+2} &= a_1 + a_2 \\ b_{1+2} &= b_1 + b_2 \\ c_{1+2} &= c_1 + c_2 \\ d_{1+2} &= d_1 + d_2 \end{aligned}$$

It should be emphasized that the addition must be considered an operation that works on the positions and not on the complete numbers, at least for every dimension higher than the second, for which there is no bijection between the positions and the complete numbers.

To integrate the operation of addition, working on the positions, with the others, working on the complete numbers, will be enough making reference to the complete number that we can obtain assigning to the sum the phases of the standard representation.

Theorem 3.13. *The properties embodied by Theorems 2.21, 2.22, 2.23, 2.24 for the third dimension remain valid for the next dimensions as well.*

Proof. In practise, the proofs of these theorems can be merely extended to a number of dimensions at will, since each coordinate is treated independently of the others, and has the same properties. ■

Definition 3.14. In the space $V_1V_2V_3\dots V_n$ we can define subtraction between two positions $o(a_{11}, a_{21}, \dots, a_{n1})$ and $o(a_{12}, a_{22}, \dots, a_{n2})$ as the position $o(a_{1(1-2)}, a_{2(1-2)}, \dots, a_{n(1-2)})$ represented also with the symbol $o(a_{11}, a_{21}, \dots, a_{n1}) - o(a_{12}, a_{22}, \dots, a_{n2})$ that satisfies the following condition:

$$o_{1-2}(a_{1(1-2)}, a_{2(1-2)}, \dots, a_{n(1-2)}) + o(a_{12}, a_{22}, \dots, a_{n2}) = o(a_{11}, a_{21}, \dots, a_{n1})$$

This condition defines the subtraction as the inverse operation of addition, and it is equivalent to require:

$$\begin{aligned} a_{1(1-2)} &= a_{11} - a_{12} \\ a_{2(1-2)} &= a_{21} - a_{22} \\ &\dots \\ a_{n(1-2)} &= a_{n1} - a_{n2} \end{aligned}$$

For example in the fourth dimension we have:

$$o_{1-2}(a_{1-2}, b_{1-2}, c_{1-2}, d_{1-2}) = o_{1-2}(a_1 - a_2, b_1 - b_2, c_1 - c_2, d_1 - d_2)$$

with:

$$\begin{aligned} a_{1-2} &= a_1 - a_2 \\ b_{1-2} &= b_1 - b_2 \\ c_{1-2} &= c_1 - c_2 \\ d_{1-2} &= d_1 - d_2 \end{aligned}$$

It should be emphasized that the subtraction must be considered an operation that works on the positions and not on the complete numbers, at least for every dimension higher than the second, for which there is no bijection between the positions and the complete numbers.

To integrate the operation of subtraction, working on the positions, with the others, working on the complete numbers, will be enough making reference to the complete number that we can obtain assigning to the difference the phases of the standard representation.

Theorem 3.15. *The properties embodied by Theorems 2.26, 2.27, 2.28, 2.29 for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be merely extended to a number of dimensions at will, since each coordinate is treated independently of the others, and has the same properties. ■

Definition 3.16. In the space $V_1V_2\dots V_n$ we can define multiplication between two complete numbers $o_1(t_1, \theta_{21}, \dots, \theta_{n1})$ and $o_2(t_2, \theta_{22}, \dots, \theta_{n2})$ as the number $o_{1.2}(t_{1.2}, \theta_{2(1.2)}, \dots, \theta_{n(1.2)})$ represented also with the symbol $o_1(t_1, \theta_{21}, \dots, \theta_{n1}) \cdot o_2(t_2, \theta_{22}, \dots, \theta_{n2})$ that satisfies the following condition:

$$o_{1.2}(t_{1.2}, \theta_{2(1.2)}, \dots, \theta_{n(1.2)}) = o_{1.2}(t_1 \cdot t_2, \theta_{21} + \theta_{22}, \dots, \theta_{n1} + \theta_{n2})$$

This condition defines the multiplication and it is equivalent to require:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 \\ \theta_{2(1.2)} &= \theta_{21} + \theta_{22} \\ &\dots \\ \theta_{n(1.2)} &= \theta_{n1} + \theta_{n2} \end{aligned}$$

For example in the fourth dimension we have:

$$o_{1.2}(t_{1.2}, \theta_{1.2}, \gamma_{1.2}, \varphi_{1.2}) = o_{1.2}(t_1 \cdot t_2, \theta_1 + \theta_2, \gamma_1 + \gamma_2, \varphi_1 + \varphi_2)$$

with:

$$\begin{aligned} t_{1.2} &= t_1 \cdot t_2 \\ \theta_{1.2} &= \theta_1 + \theta_2 \\ \gamma_{1.2} &= \gamma_1 + \gamma_2 \\ \varphi_{1.2} &= \varphi_1 + \varphi_2 \end{aligned}$$

Theorem 3.17. *The properties embodied by Theorems 2.40, 2.41, 2.42, 2.43, 2.44, 2.45, 2.46 for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be merely extended to a number of dimensions at will, since each phases is treated independently of the others, and has the same properties.

Special reference also needs to be made to the distributive properties over addition and subtraction for which we must consider that the dimensions higher than the third, of fact extend them. This means that if these properties had been valid for the dimensions higher than the third, they had been such even in the third, as a sub-case, but we know that this does not happen. ■

Definition 3.18. In the space $V_1 V_2 \dots V_n$ we can define division between two complete numbers $o_1(t_1, \theta_{21}, \dots, \theta_{n1})$ and $o_2(t_2, \theta_{22}, \dots, \theta_{n2})$ as the number $o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{2(\frac{1}{2})}, \dots, \theta_{n(\frac{1}{2})})$ represented also with the symbol $\frac{o_1(t_1, \theta_{21}, \dots, \theta_{n1})}{o_2(t_2, \theta_{22}, \dots, \theta_{n2})}$ that satisfies the following conditions:

1. $o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{2(\frac{1}{2})}, \dots, \theta_{n(\frac{1}{2})}) \cdot o_2(t_2, \theta_{22}, \dots, \theta_{n2}) = o_1(t_1, \theta_{21}, \dots, \theta_{n1})$
2. $o_2(t_2, \theta_{22}, \dots, \theta_{n2}) \neq 0$

The first condition defines the division as the inverse operation of multiplication, and it is equivalent to require that:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} \\ \theta_{2(\frac{1}{2})} &= \theta_{21} - \theta_{22} \\ &\dots \\ \theta_{n(\frac{1}{2})} &= \theta_{n1} - \theta_{n2} \end{aligned}$$

For example in the fourth dimension we have:

$$o_{1.2}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}, \varphi_{\frac{1}{2}}) = o_{1.2}(t_1 \cdot t_2, \theta_1 - \theta_2, \gamma_1 - \gamma_2, \varphi_1 - \varphi_2)$$

with:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \\ \varphi_{\frac{1}{2}} &= \varphi_1 - \varphi_2 \end{aligned}$$

The second condition gets its own justification by the necessity of defining the divisions in an univocal way. In fact when that condition is not valid, the expression:

$$o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{2(\frac{1}{2})}, \dots, \theta_{n(\frac{1}{2})}) \cdot 0 = 0$$

besides to require a zero dividend $o_1(t_1, \theta_{21}, \dots, \theta_{n1})$ as well, would be satisfied by more values of $o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{2(\frac{1}{2})}, \dots, \theta_{n(\frac{1}{2})})$.

Theorem 3.19. *The properties embodied by Theorems 2.57,2.58,2.59, 2.60, 2.61, 2.62, 2.63 for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be merely extended to a number of dimensions at will, since each phases is treated independently of the others, and has the same properties.

Special reference also needs to be made to the distributive properties over addition and subtraction for which we must consider that the dimensions higher than the third, of fact extend them. This means that if these properties had been valid for the dimensions higher than the third, they had been such even in the third, as a sub-case, but we know that this does not happen. ■

Definition 3.20. In the space $V_1V_2\dots V_n$ we can define n-th power of the complete number $o(t, \theta_2, \dots, \theta_i)$ with n (natural number) known as exponent and $o(t, \theta_2, \dots, \theta_i)$ known as base, as the number $o_{\uparrow n}(t_{\uparrow n}, \theta_{2(\uparrow n)}, \dots, \theta_{i(\uparrow n)})$ also represented with the symbol $o(t, \theta_2, \dots, \theta_i)^n$ that satisfies the following conditions:

1. $o(t, \theta_2, \dots, \theta_i)^n = o(t, \theta_2, \dots, \theta_i) \cdot \dots \cdot o(t, \theta_2, \dots, \theta_i)$ for $n > 0$
2. $o(t, \theta_2, \dots, \theta_i)^n = \frac{o(t, \theta_2, \dots, \theta_i)}{o(t, \theta_2, \dots, \theta_i)} = 1$ for $n = 0$
3. $o(t, \theta_2, \dots, \theta_i)^n = \frac{1}{\frac{o(t, \theta_2, \dots, \theta_i)}{\dots \frac{1}{o(t, \theta_2, \dots, \theta_i)}}}$ for $n < 0$
4. $n > 0$ for $o(t, \theta_2, \dots, \theta_i) = 0$

We note that the term $o(t, \theta_2, \dots, \theta_i)$ in the first and third conditions is intended to appear $|n|$ times.

The first condition defines the repeated multiplication of the base by itself a positive number of times, the second a zero number of times, and finally the third a negative number of times. All these conditions correspond to require:

$$\begin{aligned} t_{\uparrow n} &= t^n \\ \theta_{2(\uparrow n)} &= \theta_2 \cdot n \\ \dots \\ \theta_{i(\uparrow n)} &= \theta_i \cdot n \end{aligned}$$

For example in the fourth dimension we have:

$$o_{\uparrow n}(t, \theta, \gamma, \varphi)^n = o_{\uparrow n}(t_{\uparrow n}, \theta_{\uparrow n}, \gamma_{\uparrow n}, \varphi_{\uparrow n})$$

with:

$$\begin{aligned} t_{\uparrow n} &= t^n \\ \theta_{\uparrow n} &= \theta \cdot n \\ \gamma_{\uparrow n} &= \gamma \cdot n \\ \varphi_{\uparrow n} &= \varphi \cdot n \end{aligned}$$

The fourth condition gets its own justification by the impossibility of defining the n-th power module when to be multiplied by itself a zero number or a negative number of times is just the 0, because in this case would be present the following divisions for 0:

$$\begin{aligned} o(t, \theta_2, \dots, \theta_i)^n &= \frac{0}{0} = 1 \text{ for } n = 0 \\ o(t, \theta_2, \dots, \theta_i)^n &= \frac{1}{0} \text{ for } n < 0 \text{ with } 0 \text{ that appears } -n \text{ times} \\ &\frac{\dots}{0} \end{aligned}$$

Theorem 3.21. *The properties embodied by Theorems 2.65, 2.66, 2.67, 2.68, 2.69 for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be repeated unchanged for dimensions higher than the third, since they do not depend on the number of dimensions considered but on the structure of the n-th power. ■

Definition 3.22. In the space $V_1V_2\dots V_n$ we can define n-th root of the complete number $o(t, \theta_2, \dots, \theta_i)$ with n (natural number) known as degree and $o(t, \theta_2, \dots, \theta_i)$ known as radicand, as the number $o_{\downarrow n}(t_{\downarrow n}, \theta_{2(\downarrow n)}, \dots, \theta_{i(\downarrow n)})$ also represented with the symbol $\sqrt[n]{o(t, \theta_2, \dots, \theta_i)}$ that satisfies the following conditions:

1. $\sqrt[n]{o(t, \theta_2, \dots, \theta_i)} \cdot \dots \cdot \sqrt[n]{o(t, \theta_2, \dots, \theta_i)} = o(t, \theta_2, \dots, \theta_i)$ for $n > 0$
2. $\frac{1}{\sqrt[n]{o(t, \theta_2, \dots, \theta_i)}} = o(t, \theta_2, \dots, \theta_i)$ for $n < 0$
 \dots
 $\frac{\dots}{\sqrt[n]{o(t, \theta_2, \dots, \theta_i)}}$
3. $\theta_{2(\downarrow n)} = \frac{\theta_2}{n}, \quad \theta_{3(\downarrow n)} = \frac{\theta_3}{n}, \quad \dots, \quad \theta_{i(\downarrow n)} = \frac{\theta_i}{n}$
4. $n \neq 0$ for any $o(t, \theta_2, \dots, \theta_i)$
5. $n \geq 0$ for $o(t, \theta_2, \dots, \theta_i) = 0$
6. $\sqrt[n]{t} > 0, \quad t > 0$

We note that the term $\sqrt[n]{o(t, \theta_2, \dots, \theta_i)}$ in the first and second conditions is intended to appear $|n|$ times.

The first condition defines the repeated multiplication of the root by itself a positive number of times, while the second a negative number of times. Both these conditions correspond to require:

$$\begin{aligned}
 t_{\downarrow n} &= \sqrt[n]{t} \\
 \theta_{2(\downarrow n)} &= \frac{\theta_2 + k \cdot 360^\circ}{n} \quad \text{for } k = \pm 1, \pm 2, \pm 3, \pm 4, \dots \\
 \dots & \\
 \theta_{i(\downarrow n)} &= \frac{\theta_i + k \cdot 360^\circ}{n} \quad \text{for } k = \pm 1, \pm 2, \pm 3, \pm 4, \dots
 \end{aligned}$$

The third condition gets its own justification by the necessity of defining the n -th root in an univocal way. In fact, when that condition is not valid, there are $n^{(i-1)}$ different complete numbers able to satisfy this definition: one for each distinct set of phases $\theta_{2(\downarrow n)}, \theta_{3(\downarrow n)}, \dots, \theta_{i(\downarrow n)}$ given by the relations seen above.

For example in the fourth dimension we have:

$$\sqrt[n]{o(t, \theta, \gamma, \varphi)} = o_{\downarrow n}(t_{\downarrow n}, \theta_{\downarrow n}, \gamma_{\downarrow n}, \varphi_{\downarrow n})$$

with:

$$\begin{aligned}
 t_{\downarrow n} &= \sqrt[n]{t} \\
 \theta_{\downarrow n} &= \frac{\theta}{n} \\
 \gamma_{\downarrow n} &= \frac{\gamma}{n} \\
 \varphi_{\downarrow n} &= \frac{\varphi}{n}
 \end{aligned}$$

Also the fourth condition gets its own justification by the necessity of defining the n-th root in an univocal way. In fact when that condition is not valid, the multiplication of the root by itself a number of times equal to 0 would require the use of the following expression:

$$\frac{\sqrt[n]{o(t, \theta_2, \dots, \theta_i)}}{\sqrt[n]{o(t, \theta_2, \dots, \theta_i)}} = 1$$

that would be satisfied by several values of $\sqrt[n]{[o(t, \theta_2, \dots, \theta_i)]}$.

The fifth condition gets its own justification by the impossibility of defining values of n-th root that multiplied by itself a negative number of times are able to give as the result just 0 value. In fact the following expression:

$$\frac{1}{\sqrt[n]{o(t, \theta_2, \dots, \theta_i)}} = 0 \text{ for } n < 0, \sqrt[n]{o(t, \theta_2, \dots, \theta_i)} \text{ appears } -n \text{ times}$$

$$\frac{\dots}{\sqrt[n]{o(t, \theta_2, \dots, \theta_i)}}$$

requires the existence of a divisor of 1 that can assign to it a quotient equal to 0: a thing that we know impossible.

The sixth condition gets its own justification by the need to make acceptable the n-th root in regard the modulus t of the complete number $o(t, \theta_2, \dots, \theta_i)$.

Theorem 3.23. *The properties embodied by Theorems 2.71, 2.72, 2.73 for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be repeated unchanged for dimensions higher than the third, since they do not depend on the number of dimensions considered but on the structure of the n-th root. ■

Definition 3.24. In the space $V_1 V_2 \dots V_n$ we can define power with rational exponent $\frac{m}{n}$ (n, m both natural numbers) of the complete number $o(t, \theta_2, \dots, \theta_i)$ with $\frac{m}{n}$ known as the rational exponent and $o(t, \theta_2, \dots, \theta_i)$ known as base, as the number $o_{\uparrow m \downarrow n}(t_{\uparrow m \downarrow n}, \theta_{2(\uparrow m \downarrow n)}, \dots, \theta_{i(\uparrow m \downarrow n)})$ also represented with the symbol $o(t, \theta_2, \dots, \theta_i)^{\frac{m}{n}}$ or $\sqrt[n]{o(t, \theta_2, \dots, \theta_i)^m}$ that satisfies the following conditions:

1. $\left[\sqrt[n]{o(t, \theta_2, \dots, \theta_i)^m} \right]^n = o(t, \theta_2, \dots, \theta_i)^m$
2. $m > 0$ for $o(t, \theta_2, \dots, \theta_i) = 0$
3. $n \neq 0$ for any $o(t, \theta_2, \dots, \theta_i)^m$ and therefore for any $o(t, \theta_2, \dots, \theta_i)$
4. $n \geq 0$ for $o(t, \theta_2, \dots, \theta_i)^m = 0$ and therefore for $o(t, \theta_2, \dots, \theta_i) = 0$

5. $\theta_{2(\uparrow m \downarrow n)} = \frac{\theta_2 \cdot m}{n}, \quad \theta_{3(\uparrow m \downarrow n)} = \frac{\theta_3 \cdot m}{n}, \quad \dots, \quad \theta_{i(\uparrow m \downarrow n)} = \frac{\theta_i \cdot m}{n}$
6. $\sqrt[n]{t^m} > 0, \quad t^m > 0$
7. $\sqrt[n]{t} > 0, \quad t > 0$

The first condition defines the power with rational exponent as a n -th root of a m -th power.

The second condition is required for the correct definition of the m -th power.

The third, the fourth, the fifth and the sixth conditions are required for the correct definition of n -th root.

For example in the fourth dimension we have:

$$\sqrt[n]{o(t, \theta, \gamma, \varphi)^m} = o_{\uparrow m \downarrow n}(t_{\uparrow m \downarrow n}, \theta_{\uparrow m \downarrow n}, \gamma_{\uparrow m \downarrow n}, \varphi_{\uparrow m \downarrow n})$$

with:

$$\begin{aligned} t_{\downarrow n} &= t^{\frac{m}{n}} \\ \theta_{\downarrow n} &= \frac{m}{n} \cdot \theta \\ \gamma_{\downarrow n} &= \frac{m}{n} \cdot \gamma \\ \varphi_{\downarrow n} &= \frac{m}{n} \cdot \varphi \end{aligned}$$

The seventh condition is required to make possible the reversal of the order between root and power, namely to write:

$$\left[\sqrt[n]{o(t, \theta_2, \dots, \theta_i)} \right]^m$$

and therefore:

$$(\sqrt[n]{t})^m$$

Theorem 3.25. *The properties embodied by Theorems 2.75, 2.76, 2.77, 2.78, 2.79, 2.80, 2.81 for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be repeated unchanged for dimensions higher than the third, since they do not depend on the number of dimensions considered but on the structure of the power with rational exponent. ■

References

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Accepted: 28.05.2011