C-ESSENTIALNESS AND WELL-BEHAVEDNESS OF C-INJECTIVITY IN Act-S

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Abstract. An important notion related to injectivity with respect to monomorphisms or any other class $\mathcal{M}$ of morphisms in a category $\mathcal{A}$ is essentialness. In this paper, taking $\mathcal{A}$ to be the category of right acts over a semigroup S, C to be an arbitrary closure operator in the category **Act-S**, and $\mathcal{M}_d$ to be the class of C-dense monomorphisms resulting from a closure operator C, we study the properties of $\mathcal{M}_d$-essential monomorphisms and we show the existence of a maximal $\mathcal{M}_d$-essential extension for any given act. Finally, the behavior of $\mathcal{M}_d$-injectivity in the sense that the three so called Well-behavedness propositions hold is studied. We show that the idempotency and weak hereditariness of a closure operator C are sufficient, but not necessary, conditions for the well-behavedness of $\mathcal{M}_d$-injectivity. The class of sequentially dense monomorphisms resulting from a special closure operator (sequential closure operator) and injectivity with respect to this class of monomorphisms have been studied by Giuli, Ebrahimi, Mahmoudi, Moghaddasi, and the author. Some of these results generalize some of the results about the class of sequentially dense monomorphisms.

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1. Introduction and preliminaries

An important notion related to injectivity with respect to monomorphisms or any other class $\mathcal{M}$ of morphisms in a category $\mathcal{A}$ is essentialness. In fact, injectivity is characterized and injective hulls are defined using essentialness (see, for example, [1], [18], and [6]). Recall that for a subclass $\mathcal{M}$ of the class $\mathcal{M}ono$ of monomorphisms of a category $\mathcal{A}$ and $M \xrightarrow{m} X \in \mathcal{M}$, one usually uses one of the following definitions to say that m is *essential*:

(1) $M \overset{m}{\rightarrowtail} X \xrightarrow{f} Y \in \mathcal{M} \Rightarrow f \in \mathcal{M}$.

(2) $M \overset{m}{\rightarrowtail} X \xrightarrow{f} Y \in \mathcal{M}ono \Rightarrow f \in \mathcal{M}ono$.

(3) $M \overset{m}{\rightarrowtail} X \xrightarrow{f} Y \in \mathcal{M} \Rightarrow f \in \mathcal{M}ono$.

Clearly, condition (3) is weaker than the other two and if $\mathcal{M}$ is taken to be the class $\mathcal{M}ono$ of all monomorphisms (in which case m is said to be an *essential monomorphism*), all the above three conditions are equivalent, but not necessarily otherwise (see, for example, [2], [3], [19]). Definition (1) is usually used for an arbitrary class $\mathcal{M}$ of morphisms of an arbitrary category $\mathcal{A}$ (see [1], [6], and [18]). The second is the one which is used in Universal Algebra, and the third one has been used when $\mathcal{M}$ is an special class of monomorphisms, in particular pure monomorphisms in an equational class of algebras. Further, Banaschewski [1] defines and studies conditions on a category $\mathcal{A}$ and a subclass $\mathcal{M}$ of monomorphisms in $\mathcal{A}$ under which $\mathcal{M}$-injectivity *behaves well* in the sense that the following three propositions hold (the definition of the terms will be given in the sequel):

Proposition 1.1 (First Theorem of Well-Behavedness) *For every $A \in \mathcal{A}$, the following conditions are equivalent:*

(I1) *A is $\mathcal{M}$-injective.*

(I2) *A is an $\mathcal{M}$-absolute retract.*

(I3) *A has no proper $\mathcal{M}$-essential extensions.*

Proposition 1.2 (Second Theorem of Well-Behavedness) *Every $A \in \mathcal{A}$ has an $\mathcal{M}$-injective hull which is unique up to isomorphism.*

Proposition 1.3 (Third Theorem of Well-Behavedness) *For an extension B of A, the following conditions are equivalent:*

(H1) *B is an $\mathcal{M}$-injective hull of A.*

(H2) *B is a maximal $\mathcal{M}$-essential extension of A.*

(H3) *B is a minimal $\mathcal{M}$-injective extension of A.*

Banaschewski [1] gives the following sufficient conditions on the pair $\mathcal{M}$ and $\mathcal{A}$ which ensure the well-behavedness of $\mathcal{M}$-injectivity in $\mathcal{A}$.

Proposition 1.4 *$\mathcal{M}$-injectivity behaves well in $\mathcal{A}$ if the following conditions hold:*

(E1) *$\mathcal{M}$ is transitive (closed under composition).*

(E2) *$\mathcal{M}$ is isomorphism closed.*

(E3) *$\mathcal{A}$ fulfills Banaschewski's $\mathcal{M}$-condition.*

(E4) *$\mathcal{A}$ satisfies the $\mathcal{M}$-transferability property.*

(E5) *$\mathcal{A}$ has $\mathcal{M}$-direct limits.*

(E6) *$\mathcal{A}$ is $\mathcal{M}$-essentially bounded.*

In this paper, we take $\mathcal{A}$ to be the category **Act-S** of acts over a semigroup S, C to be an arbitrary closure operator in the category **Act-S**, and $\mathcal{M}_d$ to be the class of C-dense monomorphisms and study the above notions of essentiality with respect to this class. We will see that the above notions of essentiality are equivalent for this subclass $\mathcal{M}_d$ of $\mathcal{M}ono$, too, and investigate some of the properties of $\mathcal{M}_d$-essential monomorphisms normally needed in the study of $\mathcal{M}_d$-injectivity. Among other things, the existence of a maximal such essential extension for any given act is shown. Finally, the behavior of $\mathcal{M}_d$-injectivity in the sense that the above so called well-behavedness propositions hold is studied. We show that the idempotency and weakly hereditariness of a closure operator C are sufficient, but not necessary, conditions for the well-behavedness of $\mathcal{M}_d$-injectivity. Some of these results generalize some of the results in [8], [11], [12], [14], [15], and [16].

In the following we first recall from [10] and [7] some facts about the category **Act-S** needed in this paper.

Let S be a semigroup, A be a set, and

$$\begin{array}{rcl} \mu : A \times S & \longrightarrow & A \\ (a, s) & \longmapsto & as := \mu(a, s), \end{array}$$

be a map. The set A is called a (*right*) S-*act* or a (*right*) *act* over S, if the map μ satisfies $a(st) = (as)t$ for $a \in A$ and $s, t \in S$. In this case, μ is called the action of S on A.

If S is a monoid with 1 as its identity, we usually also require that $a1 = a$ for $a \in A$.

A subset A' of an S-act A is said to be a *subact* of A if $a's \in A'$ for all $s \in S$ and $a' \in A'$; and in this case we write $A' \leq A$.

A *homomorphism* (also called an *equivariant* map or an S-*map*) from an S-act A to an S-act B is a function from A to B such that for each $a \in A$, $s \in S$, $f(as) = f(a)s$.

Since id_A and the composition of two S-maps are S-maps, we have the category **Act-S** of all right S-acts and S-maps between them.

Note that the class of S-acts is an equational class, and so the category **Act-S** is complete and cocomplete (has all products, equalizers, pullbacks, coproducts, coequalizers, and pushouts). In fact, limits and colimits in this category are computed as in the category **Set** of sets and equipped with a natural action. Also, monomorphisms (epimorphisms) in **Act-S** are exactly one-one (onto) S-maps. Therefore, we do not distinguish between monomorphisms of acts and inclusions, and call an S-act B containing (an isomorphic copy of) an S-act A an *extension* of A.

For an S-act A and $a \in A$ we denote the S-map $f : S \to A$, given by $f(s) = as$ for all $s \in S$, by λ_a.

Recall that an element a of an S-act A is called a *fixed* or a *zero* element if $as = a$ for all $s \in S$.

Also, recall that for a family $\{A_i : i \in I\}$ of S-acts with a unique fixed element 0, the *direct sum* $\bigoplus_{i \in I} A_i$ is defined to be the subact of the product

$\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number.

Denoting the lattice of all subacts of an S-act B by $SubB$, following [5] for the general definition of closure operators on a category, we get:

Definition 1.5 A family $C = (C_B)_{B\in\mathbf{Act-S}}$, with $C_B : SubB \to SubB$, taking $A \leq B$ to $C_B(A)$, is called a *closure operator* on **Act-S** if it satisfies the following laws:

(c_1) (*Extension*) $A \leq C_B(A)$,

(c_2) (*Monotonicity*) $A_1 \leq A_2$ implies $C_B(A_1) \leq C_B(A_2)$,

(c_3) (*Continuity*) $f(C_B(A)) \leq C_D(f(A))$, for all morphisms $f : B \to D$.

Now, one has the usual two classes of monomorphisms related to the notion of a closure operator as follows:

Definition 1.6 Let $A \leq B$ be in **Act-S**. We say that A is C-*closed* in B if $C_B(A) = A$, and it is C-*dense* in B if $C_B(A) = B$. Also, an S-map $f : A \to B$ is said to be C-dense (C-closed) if $f(A)$ is a C-dense (C-closed) subact of B.

We denote the class of all C-dense monomorphisms by $\mathcal{M}_d$ and recall some of the properties of this class from [17].

Definition 1.7 A closure operator C is said to be:

(a) *Weakly hereditary* if for every S-act B and every $A \leq B$, A is C-dense in $C_B(A)$.

(b) *Idempotent* if $C_B(C_B(A)) = C_B(A)$ for all S-acts B and $A \leq B$.

Remark 1.8 Notice that all isomorphisms are C-dense and the composition of an isomorphism with a C-dense monomorphism is C-dense. Also, the composition of a C-dense monomorphism with a surjective morphism is a C-dense morphism.

As the following result of [17] shows, the class of C-dense monomorphisms is not always closed under composition.

Theorem 1.9 *For a semigroup S and a closure operator C, the following are equivalent:*

(i) *The closure operator C is idempotent and weakly hereditary.*

(ii) *The class $\mathcal{M}_d$ is closed under composition and the closure operator C is weakly hereditary.*

(iii) *Each S-map $f : A \to B$ has a (C-dense, C-closed) factorization.*

We recall the following lemma from [9]:

Lemma 1.10 *Pushouts transfer monomorphisms in* **Act-**S.

We recall the following from [17] which is a counterpart of (E4) in [1].

Proposition 1.11 *In* **Act-S***, pushouts transfer C-dense monomorphisms.*

We recall the following from [17] which is a counterpart of (E5) in [1].

Proposition 1.12 **Act-S** *has $\mathcal{M}_d$-directed colimits.*

Definition 1.13 We call an S-act A, *C-dense injective* or *C-injective* if it is injective with respect to C-dense monomorphisms; that is, for every C-dense monomorphism $h : B \to D$ and every S-map $f : B \to A$ there exists an S-map $g : D \to A$ such that $gh = f$.

We recall the following theorem from [17] which is desirable in the study of any type of injectivity.

Theorem 1.14 *Let S be a semigroup. Then, an S-act A is C-injective if and only if it is C-absolute retract (retract of any of its C-dense extensions).*

2. C-dense essential monomorphisms

Now that we have introduced the class $\mathcal{M}_d$ of C-dense monomorphisms, we begin the study of essentiality with respect to this class. Recall the three different notions of essentiality with respect to a subclass $\mathcal{M}$ of monomorphisms given in the introduction. We also mentioned there that for some classes $\mathcal{M}$, specially for the class $\mathcal{M}ono$, these three notions of essentiality are in fact equivalent. In the following theorem we prove that this is also the case for the class $\mathcal{M}_d$. We then investigate some properties of essentiality, usually needed in the study of injectivity with respect to the class $\mathcal{M}_d$.

Theorem 2.15 *For a C-dense monomorphism $f : A \to B$, the following are equivalent:*

(i) *Any S-map $g : B \to D$ for which gf is a C-dense monomorphism is itself a C-dense monomorphism.*

(ii) *Any S-map $g : B \to D$ for which gf is a C-dense monomorphism is a monomorphism.*

(iii) *Any S-map $g : B \to D$ for which gf is a monomorphism is itself a monomorphism.*

(iv) *For every congruence ρ on B with $\rho \neq \Delta_B$ one has $\rho\mid_A = \rho \cap (A \times A) \neq \Delta_A$.*

Proof. (i)$\Rightarrow$(ii) Let $g : B \to D$ be such that $gf \in \mathcal{M}_d$, then by the assumption $g \in \mathcal{M}_d$. Thus g is a monomorphism.

(ii)$\Rightarrow$(iii) Let $g : B \to D$ be an S-map such that gf is a monomorphism. Then since $gf : A \to g(B)$ is a C-dense monomorphism, and by (ii), we get that $g : B \to g(B)$ is a monomorphism and hence g is a monomorphism.

(iii)$\Leftrightarrow$(iv) It is obtained using Lemma III.1.15 of [10].

(iv)$\Rightarrow$(i) Let $g : B \to D$ be such that $gf \in \mathcal{M}_d$, by (iii)$\Leftrightarrow$(iv), we get that g is a monomorphism. Since the class $\mathcal{M}_d$ is right cancellable, g is C-dense. Thus $g \in \mathcal{M}_d$. ∎

Definition 2.16 We call a C-dense monomorphism satisfying one of the equivalent conditions of the above theorem an *$\mathcal{M}_d$-essential* or *C-dense essential* monomorphism.

It follows by the above theorem that:

Corollary 2.17 *A monomorphism f is $\mathcal{M}_d$-essential if and only if it is essential as well as C-dense.*

Remark 2.18

(a) Since the composition of two essential monomorphisms is clearly essential, if the closure operator C is idempotent and weakly hereditary, we get from Corollary 2.17 that the composition of $\mathcal{M}_d$-essential monomorphisms is an $\mathcal{M}_d$-essential monomorphism.

(b) Let the closure operator C be idempotent and weakly hereditary and $A \subseteq A' \subseteq B$. Then A is $\mathcal{M}_d$-essential in B if and only if A is $\mathcal{M}_d$-essential in A' and A' is $\mathcal{M}_d$-essential in B.

(c) If gf is $\mathcal{M}_d$-essential and g is a monomorphism then g is $\mathcal{M}_d$-essential.

(d) Any directed colimit of $\mathcal{M}_d$-essential monomorphisms is an $\mathcal{M}_d$-essential monomorphism.

Definition 2.19 A category $\mathcal{A}$ is called $\mathcal{M}$-*essentially bounded*, for a subclass $\mathcal{M}$ of its monomorphisms, if every $A \in \mathcal{A}$ has only a set of $\mathcal{M}$-essential extensions.

The following is a counterpart of (E6) in [1].

Proposition 2.20 *The category* **Act-S** *is $\mathcal{M}_d$-essentially bounded.*

Proof. By using the fact that each S-act admits only a set of essential extensions and Corollary 2.17, we get that each S-act has only a set of $\mathcal{M}_d$-essential extensions. ∎

Definition 2.21 For a category $\mathcal{A}$, a class $\mathcal{M}$ of monomorphisms is said to satisfy *Banaschewski's $\mathcal{M}$-condition* if for every $\mathcal{M}$-morphism $f : A \to B$ there exists a homomorphism $g : B \to D$ such that gf is an $\mathcal{M}$-essential morphism.

The following is a counterpart of (E3) in [1].

Proposition 2.22 **Act-S** *fulfills Banaschewski's $\mathcal{M}_d$-condition.*

Proof. Let $A \xrightarrow{f} B \in \mathcal{M}_d$. Consider the poset

$$\mathcal{P} = \{\theta \in Con(B) : A \xrightarrow{f} B \xrightarrow{\gamma_\theta} B/\theta \text{ is a C} - \text{dense monomorphism}\}$$

under the usual ordering of relations. Let

$$... \leq \rho_i \leq ...$$

$i \in I$, be a chain in $\mathcal{P}$. Then $\rho = \bigcup_{i \in I} \rho_i$ is also a congruence which is an upper bound of this chain which belongs to $\mathcal{P}$. Indeed, let $x, y \in A$ with $x\rho y$. Then $x\rho_j y$ for some $j \in I$. Since $\gamma_{\rho_j} f$ is a monomorphism we have $x = y$. This means that $\rho \in \mathcal{P}$. Applying Zorn's Lemma, there exists a maximal such a congruence, say θ. Let $g : B \to B/\theta$. Then maximality of θ implies that $g \circ f : A \to B/\theta$ is an essential monomorphism. Indeed, suppose $h : B/\theta \to D$ is a homomorphism whose restriction on A is monomorphism. Define a relation σ on B by

$$x\sigma y \Leftrightarrow [x]_\theta (ker f)[y]_\theta$$

for any $x, y \in B$. Then σ is a congruence on B such that $\theta \leq \sigma$ and $\gamma_\sigma f$ is a monomorphism. Hence $\sigma = \theta$ which means that h is a monomorphism. Since g is surjective, it is C-dense and so, by Corollary 2.17, it is $\mathcal{M}_d$-essential. ■

Lemma 2.23 *Let A be a C-dense subact of B. If A is a proper retract of B ($A \ncong B$) then A is not $\mathcal{M}_d$-essential in B.*

Definition 2.24 Let A be an S-act. Then by a *maximal $\mathcal{M}_d$-essential extension of A* we mean an $\mathcal{M}_d$-essential extension B of A such that every homomorphism $h : B \to D$ from B to an $\mathcal{M}_d$-essential extension D of A for which $h \mid_A$ is the inclusion map, is an isomorphism.

Lemma 2.25 *If B is an $\mathcal{M}_d$-essential extension of A and A is embedded into some $(C-)$ injective act Q, then B can also be embedded into Q.*

Proof. Suppose A is $\mathcal{M}_d$-essential in B and consider the diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "i"'] & B \arrow[dl, "\bar{i}"] \\ Q & \end{tikzcd}$$

where Q is $(C-)$ injective and i is a monomorphism. Since Q is $(C-)$ injective, there exists an S-map $\bar{i}$ such that $\bar{i} \mid_A = i$. Since A is $\mathcal{M}_d$-essential in B, $\bar{i}$ is a monomorphism. ■

Proposition 2.26 *Every right S-act has a maximal $\mathcal{M}_d$-essential extension.*

Proof. Let A be an arbitrary act and Q be an injective act into which A can be embedded which exists by [4]. By the above Lemma A and all its $\mathcal{M}_d$-essential extensions are subacts of Q. Let $\mathcal{P}$ be the set of all $\mathcal{M}_d$-essential extensions of A. Consider $\mathcal{P}$ as a partially ordered set under inclusion. By Zorn's Lemma, $\mathcal{P}$ has a maximal element, say E. Then E is clearly a maximal $\mathcal{M}_d$-essential extension of A. ∎

3. Well-behavedness of C-dense injectivity

Banaschewski defines and gives some sufficient, but not necessary, conditions on a category $\mathcal{A}$ and a subclass $\mathcal{M}$ of its monomorphisms under which $\mathcal{M}$-injectivity *is well behaved* with respect to the notions such as $\mathcal{M}$-absolute retract and $\mathcal{M}$-essentialness. Recall the three well-behavedness theorems given in the introduction. In this section we study these so called well-behavedness theorems of injectivity for C-injectivity. We show that the idempotency and weakly hereditariness of the closure operator C are sufficient, but not necessary (take C as the sequential closure operator and see [14]), conditions for C-injectivity to be well behaved.

First, applying Proposition 1.4, and the results of former sections about (E1)-(E6) for the class $\mathcal{M}_d$ of C-dense monomorphisms in the category **Act-S**, we get:

Theorem 3.27 *If C is an idempotent and weakly hereditary closure operator then $\mathcal{M}_d$-injectivity behaves well in the category* **Act-S**.

But, we see that the mentioned condition on C is not necessary for the First Theorem of Well-Behavedness.

Theorem 3.28 (First Theorem of Well-Behavedness) *For a semigroup S, a closure operator C, and any S-act A, the following are equivalent:*

(i) *A is C-injective.*

(ii) *A is C-absolute retract.*

(iii) *A has no proper C-essential extension.*

Proof. (i)$\Longleftrightarrow$(ii) is clear by Theorem 1.14.

(ii)$\Longleftrightarrow$(iii) Let A be C-absolute retract and B be a proper C-dense extension of A. By hypothesis, A is a retract of B. Then, by Lemma 2.23, B is not an $\mathcal{M}_d$-essential extension of A. For the converse, let B be a C-dense extension of A. Then, by Proposition 2.22, there is an S-map $g : B \to D$ such that gi is $\mathcal{M}_d$-essential, where $i : A \to B$ is the inclusion map. Then, by hypothesis, gi has to be an isomorphism. Now, $\pi = (gi)^{-1}g : B \to A$ is an epimorphism and $\pi(a) = a$ for all $a \in A$. ∎

Now, giving a definition, we state the Second Theorem of Well-Behavedness of C-injectivity.

Definition 3.29 By a *C-dense injective hull* or *C-injective hull* of an S-act A we mean a C-essential extension of A which is C-injective.

For an S-act A, C-injective hull is unique up to isomorphism (if it exists).

The Second Theorem of Well-Behavedness of C-injectivity is about the existence of C-injective hull, which is proved in the following theorem for S-acts, for an idempotent and weakly hereditary closure operator C.

Theorem 3.30 (Second Theorem of Well-Behavedness) *If C is an idempotent and weakly hereditary closure operator then for each S-act A the C-injective hull of A exists.*

Proof. Take a maximal C-essential extension E of an S-act A which exists by Proposition 2.26. We claim that E is C-injective. To prove this, let $g : B \to D$ be any C-dense monomorphism and $h : B \to E$ be any homomorphism. Form the following pushout

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "h"'] & D \arrow[d, "v"] \\ E \arrow[r, "u"] & P = (E \sqcup D)/\theta \end{tikzcd}$$

by Proposition 1.11, u is a C-dense monomorphism and hence retractable by Theorem 3.28 and Remark 2.18(b). This proves that E is C-injective. ∎

Finally, we give the Third Theorem of Well-Behavedness of C-injectivity, which is about the relation between C-injective hull and C-essential extension.

Definition 3.31 Let A be an S-act. Then, by a *minimal C-injective C-dense extension* of A we mean a C-dense extension B of A such that B is C-injective, and every (C-dense) monomorphism $k : D \to B$ from a C-injective C-dense extension D of A which maps A identically is an isomorphism.

Theorem 3.32 (Third Theorem of Well-Behavedness) *If C is an idempotent and weakly hereditary closure operator then for every extension B of an S-act A, the following are equivalent:*

(i) *B is the C-injective hull of A.*

(ii) *B is a maximal C-essential extension of A.*

(iii) *B is a minimal C-injective C-dense extension of A.*

Proof. (i)⇒(ii) Let D be an extension of B which is a C-essential extension of A. Then applying Remark 2.18 (b), D is a C-essential extension of B. But, by Theorem 3.28, B being C-injective has no proper C-essential extension and so $D = B$.

(ii)⇒(i) If B is a maximal C-essential extension of A then, using Lemma 2.18, it has no proper C-essential extension. So, by Theorem 3.28, B is C-injective and hence the C-injective hull of A.

(i)⇒(iii) Similar to the first part of the proof, if $D \leq B$ is a C-injective extension of A, since A is C-essential in B it is concluded that the same is true for D and then since D is C-injective, applying Theorem 3.28, we get $B = D$.

(iii)⇒(i) Let $E(A)$ be the C-injective hull of A, which exists by Theorem 3.30. Since B is C-injective, there is an S-map $f : E(A) \to B$ such that $f|_A = A \hookrightarrow B$. Since A is essential in $E(A)$, f has to be a monomorphism. So, by (iii), $B \cong E(A)$. ■

Two other conditions can be added to the equivalent conditions given in the preceding theorem. To give them we need the following definition:

Definition 3.33

(a) By a *smallest C-injective C-dense extension* of an act A we mean a C-dense C-injective extension B of A such that for each C-injective extension D of A there exists a monomorphism $g : B \to D$ such that $g\,|_A$ is the inclusion map.

(b) By a *largest $\mathcal{M}_d$-essential extension* of an act A we mean an $\mathcal{M}_d$-essential extension B of A such that for each $\mathcal{M}_d$-essential extension D of A there exists an S-map $h : D \to B$ such that $h\,|_A$ is the inclusion map.

Theorem 3.34 *The following conditions are equivalent to the conditions of Theorem* 3.32*:*

(iv) *B is a largest C-essential extension of A.*

(v) *B is a smallest C-injective C-dense extension of A.*

Proof. Using the notations of Theorem 3.32, we have:

(iii)⇒(iv) Let $f : A \to B$ be a minimal C-injective extension of A. Consider $h : A \to B'$ as the C-injective hull of A which exists by Theorem 3.30. Then, by maximality of f, we get that the S-map $g : B' \to B$ which exists, since B is C-injective, and is a monomorphism, (since h is C-essential), is an isomorphism. So f is C-essential and evidently is a largest C-essential extension of A.

(iv)⇒(v) Take $E(A)$ to be the C-injective hull of A which exists by Theorem 3.30. Since $E(A)$ is a C-essential extension of A and B is a largest C-essential extension of A, we obtain an S-map $h : E(A) \to B$ such that $h\,|_A$ is the inclusion map. Now, since A is C-essential in $E(A)$, h is a monomorphism and so, since B is a C-essential extension of A, Remark 2.18 (b), implies that h is C-essential. But, $E(A)$ is C-injective, and so, by Theorem 3.28, has no proper C-essential extension. Hence, h is an isomorphism. Therefore, B is C-injective. So, B is evidently a smallest C-injective C-dense extension of A.

(v)⇒(i) Suppose $E(A)$ is the C-injective hull of A which exists by Theorem 3.30. Then, since $E(A)$ is C-injective and B is a smallest C-injective C-dense extension of A, there exists an S-map $g : B \to E(A)$ such that $g\mid_A$ is the inclusion map. Also since A is C-essential in $E(A)$ we get that g is C-essential by Remark 2.18 (b). But, B is C-injective and so has no proper C-essential extension. Thus, g is an isomorphism. Hence, B is a C-essential extension and so it is a C-injective hull of A. ∎

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