FUZZY HYPERVECTOR SPACES (REDEFINED)

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Abstract. In this paper, we introduce and analyze a new type of fuzzy hypervector spaces, as a generalization of fuzzy vector spaces. In this regards, we investigate the basic properties of fuzzy hyper vector spaces and obtain some related results.

Keywords: fuzzy hypervector space, hypervector space, fuzzy vector space, subfuzzy hypervector space.

1. Introduction

The notion of a hypergroup was introduced by F. Marty in 1934 [19]. Since then many researchers have worked on hyperalgebraic structures and developed this theory (for more details see [9], [10]). In 1990, M.S. Tallini introduced the notion of hypervector spaces ([24], [25]) and studied basic properties of them.

As it is well-known the concept of a fuzzy subset of a nonempty set was introduced by Zadeh in 1965 [27] as a function from a nonempty set X into the unit real interval $I = [0, 1]$. Rosenfeld [21] applied this to the group theory and then many researchers developed it in all branches of algebra. The concepts of fuzzy field and fuzzy linear space over a fuzzy field were introduced and discussed by Nanda [20]. In 1977, Katsaras and Liu [15] formulated and studied the notion of fuzzy vector subspaces over the field of real or complex numbers.

Fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. (for example see [1]-[6], [11], [13], [14]). The study of fuzzy hyperstructure is divided into three groups. Crisp hyperoperations defined through fuzzy sets have been initiated by Corsini [8]. Fuzzy hyperalgebras which is a direct extension of the concept of fuzzy algebras. This idea has been extended to fuzzy hypergroups by Zahedi [28]. A completely different approach is an idea defining a fuzzy hypersemigroup considering a fuzzy hyperoperation and a nonempty set that assigns to every pair of elements a fuzzy set. This idea was studied by Corsini and Tofan [12] and then studied by Kehagias, Konstantinidou and Serafimidis [23]. This idea was continued by Sen, Ameri and Chowdhury in [22], where fuzzy semihypergroups are introduced and analyzed. In 2009, Leoreanu and Davvaz [17] introduced the notion of a fuzzy hyperring and then fuzzy hepermodule based on the fuzzy semihypergroup in [22] and made connections.

In [1], Ameri introduced and studied fuzzy hypervector spaces. Now in this paper we introduce and study a new type of a fuzzy hypervector spaces (which is different from that) and obtain some results. We will proceed by giving a connection between fuzzy hypervector spaces and hypervector spaces.

2. Preliminaries

In this section, we present some definitions and simple properties of hypervector spaces and fuzzy subsets, that we need for developing our paper.

A mapping $\circ : H \times H \longrightarrow P^*(H)$ is called a *hyperoperation* (or a join operation), where $P^*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A \text{ and } b \in B \}$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$, respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Definition 2.1 Let K be a field and $(V, +)$ be an abelian group. We define a *hypervector* space over K to be the quadrupled $(V, +, \circ, K)$, where " $\circ$ " is a mapping

$$\circ : K \times V \quad \longrightarrow \quad P^*(V),$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

(H_1) $a \circ (x + y) \subseteq a \circ x + a \circ y$;

(H_2) $(a + b) \circ x \subseteq a \circ x + b \circ x$;

(H_3) $a \circ (b \circ x) = (ab) \circ x$;

(H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$;

(H_5) $x \in 1 \circ x$.

Remark.

(i) In the right hand side of the right distributivity law (H_1) the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly we have that for left distributivity law (H_2).

(ii) We say $(V, +, \circ, K)$ is *anti-left distributive* if

$$\forall a, b \in K,\ \forall x \in V,\ (a + b) \circ x \supseteq a \circ x + b \circ x,$$

and *strongly left distributive*, if

$$\forall a, b \in K,\ \forall x \in V,\ (a + b) \circ x = a \circ x + b \circ x$$

In a similar way, we define the *anti-right distributive and* strongly right distributive hypervector spaces, respectively. V is called strongly distributive if it is both strongly left and strongly right distributive. (For more details see [25]).

(iii) The left hand side of associativity law (H_3) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$, i.e.,

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) Let $\Omega_V = 0 \circ \underline{0}$, where $\underline{0}$ is the zero of $(V, +)$. It has been shown if V is either strongly right or left distributive, then Ω_V is a subgroup of $(V, +)$. (For more details see [24]).

Example 2.2 [24] In $(\mathbb{R}^2, +)$ we define the product times a scalar in $\mathbb{R}$ by setting:

$$\forall a \in \mathbb{R},\ \forall x \in \mathbb{R}^2 : a \circ x = \begin{cases} \text{line } ox & \text{if } x \neq \underline{0}, \\ \{\underline{0}\} & \text{if } x = \underline{0}, \end{cases}$$

where $\underline{0} = (0, 0)$. Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space.

Definition 2.3 [3] A nonempty subset W of V is a *subhyperspace* if W is itself a hypervector space with the hyperoperation on V, i.e.,

$$\begin{cases} W \neq \varnothing, \\ \forall x, y \in W \Longrightarrow x - y \in W, \\ \forall a \in K,\ \forall x \in W \Longrightarrow a \circ x \subseteq W. \end{cases}$$

In this case, we write $W \leqslant V$.

Definition 2.4

(i) (**Extension principle**) Let $f : X \longrightarrow Y$ be a mapping and $\mu \in FS(X)$ and $\nu \in FS(Y)$. Then we define $f(\mu) \in FS(Y)$ and $f^{-1}(\nu) \in FS(X)$ respectively as follows:

$$f(\mu)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

(ii) $f^{-1}(\nu)(x) = \nu(f(x)), \quad \forall x \in X.$

Definition 2.5 [22] Let S be a nonempty set. $F^*(S)$ denotes the set of all fuzzy subsets of S. A fuzzy hyperoperation on S is a mapping $\circ : S \times S \longmapsto F^*(S)$ written as $(a,b) \longmapsto a \circ b$. In other words the fuzzy hyperoperation "$\circ$", assigns to every pair (a,b) in H^2 , a nonempty fuzzy subset of H. S together with a fuzzy hyperoperation $\circ$ is called a *fuzzy hypergroupoid.*

Definition 2.6 [22] A fuzzy hypergroupoid $(S,\circ)$ is called a *fuzzy hypersemigroup* if

$$\forall a,b,c \in S,\ (a \circ b) \circ c = a \circ (b \circ c),$$

where for any fuzzy subset μ of S and for all $r \in S$:

(1) $(a \circ \mu)(r) = \bigvee_{t \in S} ((a \circ t)(r) \wedge \mu(t)),\ (\mu \circ a)(r) = \bigvee_{t \in S} ((t \circ a)(r) \wedge \mu(t)),$

(2) If A is a nonempty subset of S and $x \in S$, then for all $t \in S$ we have

$$(x \circ A)(t) = \bigvee_{a \in A} (x \circ a)(t) \text{ and } (A \circ x)(t) = \bigvee_{a \in A} (a \circ x)(t),$$

(3) Let μ, ν be two fuzzy subsets of a fuzzy hypergroupoid $(S,\circ)$ then

$$(\mu \circ \nu)(t) = \bigvee_{p,q \in S} (\mu(p) \wedge (p \circ q)(t) \wedge \nu(q)),\ \text{for all } t \in S.$$

3. Fuzzy hypervector space

In this section, we introduce a new type of fuzzy hyper vector spaces dealing with the new definition of fuzzy hyperstrucures [22], and obtain some basic properties of such spaces.

Definition 3.1 Let K be a field and $(V,+)$ an abelian group. A *fuzzy hypervector space* over K is a quadruple $(V,+,\odot,K)$, where "$\odot$" is a fuzzy hyper operation

$$\begin{array}{rcl} \odot : K \times V & \longrightarrow & F^*(V) \\ (a,v) & \longmapsto & a \odot v \end{array}$$

such that for all $\alpha, \beta \in K$ and $a, b \in V$ the followings hold:

$(FH1)$ $\alpha \odot (a+b) \subseteq (\alpha \odot a) + (\alpha \odot b);$
$(FH2)$ $(\alpha + \beta) \odot a \subseteq (\alpha \odot a) + (\beta \odot a);$
$(FH3)$ $\alpha \odot (\beta \odot x) = (\alpha\beta) \odot x;$
$(FH4)$ $a \odot (-x) = (-a) \odot x = -(a \odot x);$
$(FH5)$ $\chi_x \subseteq 1 \odot x.$

Remark.

(i) In the right hand side of the right distributivity law $(FH1)$ the sum is meant in the sense of fuzzy sum, that is for fuzzy subsets μ and ν of V

$$(\mu+\nu)(z)=\bigvee_{z=x+y}(\mu(x)\bigwedge\nu(y)).$$

Similarly, we have for left distributivity law (H_2).

(ii) We say that $(V,+,\odot,K)$ is *anti-left distributive* if

$$\forall a,b\in K,\ \forall x\in V,\ (a+b)\odot x\supseteq a\odot x+b\odot x,$$

and strongly left distributive, if

$$\forall a,b\in K,\ \forall x\in V,\ (a+b)\odot x=a\odot x+b\odot x,$$

(iii) Let $\Omega_V=0\odot\underline{0}$, where $\underline{0}$ is the zero of $(V,+)$. It can be easily shown that if V is either strongly right or left distributive, then Ω_V is a subgroup of $(V,+)$.

Here, we present examples of fuzzy hypervector spaces.

Example 3.2 Let $(V,+)$ be an arbitrary abelian group and K be a field. Define fuzzy hyperoperation: $\odot:K\times V\longrightarrow F^*(V)$ by

$$\forall a\in V, r\in K,\qquad r\odot a=\chi_{\{ra\}}$$

where $\chi_{\{ra\}}$ is the characteristic function. It is easy to verify that $(V,+,\odot,K)$ is a fuzzy hypervector space over the field K. ∎

This example shows that every fuzzy hypervector space is a generalization of a classic hypervector space.

Example 3.3 Let $(V,+)$ be a an abelian group and K be a field. Define following fuzzy hyperoperation "$\odot$" by

$$\forall a\in V, r\in K,\qquad (r\odot a)(t)=\frac{1}{2}\qquad \text{if } t\in r\circ a$$

and 0 otherwise. Then $(V,+,\odot,K)$ is a fuzzy hypervector space over the field K. ∎

Example 3.4 Let $(V,+)$ be an abelian group and μ be a nonzero fuzzy semigroup of V, then for $a,b\in V$, we define the fuzzy hyperoperation

$$(a\odot b)(t)=\begin{cases}\mu(a)\wedge\mu(b) & \text{if } t=ab,\\ 0 & \text{otherwise,}\end{cases}$$

then $(V,+,\odot)$ is a fuzzy hypervector space over field K. ∎

Definition 3.5 A nonempty subset W of V is a *subfuzzy hypervector space* if W is itself a fuzzy hypervector space with the fuzzy hyper operation on V, that is,

$$\begin{cases} W \neq \varnothing, \\ \forall x, y \in W \Longrightarrow x - y \in W, \\ (\forall a \in K,\ \forall x \in W,\ \ (a \odot x)(v) > 0) \Longrightarrow v \in W. \end{cases}$$

Lemma 3.6 *A nonempty subset W of V is a subfuzzy hypervector space if and only if, $\forall a, b \in K, \forall u, v \in W$, we have*

$$(a \odot u + b \odot v)(t) > 0 \Longrightarrow t \in W.$$

Proof. Let W be a subfuzzy hypervector space of V. Suppose that for $a, b \in K$ and $u, v \in W$, we have

$$(a \odot u + b \odot v)(t) > 0.$$

On the other hand,

$$(a \odot u + b \odot v)(t) = \bigvee_{t = t_1 + t_2} ((a \odot u)(t_1) \wedge (b \odot v)(t_2))(t) > 0.$$

Then, there exists $u_1, u_2 \in V$ such that $t = u_1 + u_2$ and $(a \odot u)(u_1) > 0$, $(b \odot v)(u_2) > 0$, by Definition 3.5 we obtain $u_1 \in W, u_2 \in W$ and hence $t \in W$.

Conversely, for $u, v \in W$ then by Definition 3.5 we have $\chi_u \subseteq 1 \odot u$ and $\chi_v \subseteq 1 \odot v$, so $(\chi_u + \chi_v)(u + v) \subseteq (1 \odot u + 1 \odot v)(u + v) > 0$ then $u + v \in W$.

Also, if $(a \odot x)(t) > 0$, then $(a \odot x + \chi_0)(t) > 0$, which means $(a \odot x + 1 \odot x)(t) > 0$ and implies that $t \in W$. ∎

Definition 3.7 Let V, W be two fuzzy hypervector spaces over a field K. Then, the mapping $T : V \longrightarrow W$ is called

(i) *weak linear transformation* if
$T(x + y) = T(x) + T(y)$ and $T(a \odot x) \cap a \odot T(x) \neq \phi$.

(ii) *linear transformation* if
$T(x + y) = T(x) + T(y)$ and $T(a \odot x) \subseteq a \odot T(x)$.

(iii) *good linear transformation* if
$T(x + y) = T(x) + T(y)$ and $T(a \odot x) = a \odot T(x)$.

Theorem 3.8 *Let $(V, +, \odot, K)$ be a fuzzy hypervector space over a field K and S be a vector space over the field K. If we consider the mapping $T : V \to S$ which is onto, then $(T(V), +, \overline{\odot}, K)$ is a fuzzy hypervector space where $a \overline{\odot} \nu = T(a \odot \nu)$, $a \in K$, $v \in V$.*

Proof. For $\alpha \in K, a, b \in V$ we have

$$\begin{array}{rcl}(\alpha\overline{\odot}(a+b))(t) & = & T(\alpha \odot (a+b))(t) \\ & = & \bigvee_{T(x)=t} (\alpha \odot (a+b))(x) \\ & \subseteq & \bigvee_{T(x)=t} ((\alpha \odot a) + (\alpha \odot b))(x)) \\ & = & \bigvee_{T(x)=t} (\bigvee_{x=u+v} ((\alpha \odot a)(u) \wedge (\alpha \odot b)(v)) \\ & = & \bigvee_{t=T(u+v)=T(u)+T(v)} ((\alpha \odot a)(u) \wedge (\alpha \odot b)(v))\end{array}$$

On the other hand we have:

$$\begin{array}{rcl}((\alpha\overline{\odot}a) + (\alpha\overline{\odot}b)(t) & = & \bigvee_{t=r+s} ((\alpha\overline{\odot}a)(r) \wedge (\alpha\overline{\odot}b)(s)) \\ & = & \bigvee_{t=r+s} (T(\alpha \odot a)(r) \wedge T(\alpha \odot b)(s)) \\ & = & \bigvee_{t=r+s} (\bigvee_{T(u)=r} (\alpha \odot a)(u)) \wedge (\sup_{T(v)=s} (\alpha \odot b)(v)) \\ & = & \bigvee_{t=T(u+v)=T(u)+T(v)} ((\alpha \odot a)(u) \wedge (\alpha \odot b)(v)).\end{array}$$

Similarly, we can prove conditions (FH2), (FH3), (FH4) and (FH5). ■

Let $(V, +, \odot, K)$ be a fuzzy hypervector space (resp. strong left distributive) and W be a subfuzzy hypervector space of V . Let $\pi : V \longrightarrow V/W$ be the projection map. Define the fuzzy hyperoperation "$*$ " on the abelian group $(V/W, +)$ by

$$\begin{array}{rcl} * : K \times V/W & \longrightarrow & F^*(V/W) \\ (a, v + W) & \longmapsto & \overline{a \odot v}\end{array}$$

in which $(\overline{a \odot v}) = \pi(a \odot v)$. Note that by Theorem 3.8, $(V/W, +, *, K)$ is a fuzzy hypervector space (resp. strong left distributive).

The next result immediately follows:

Corollary 3.9 *Let $(V, +, \odot)$ be a fuzzy hypervector space over a field K and W be a subfuzzy hypervector space of V. Then $(V/W, +, *, K)$ is a fuzzy hypervector space.*

Definition 3.10 If μ is a nonempty subset of V, then the smallest sub-fuzzy hypervector space of V containing μ is called *fuzzy linear space generated by μ* and is denoted by $\langle\mu\rangle$. In other words, $\langle\mu\rangle = \bigcap_{\mu \subseteq \nu \leq V} \nu$.

Lemma 3.11 *If μ is a nonempty subset of V then*

$$\langle\mu\rangle = \left\{ t \in V : \chi_t \subseteq \sum_{i=1}^{n}(a_i \odot s_i), a_i \in K, s_i \in V, \mu(s_i) > 0, n \in N \right\}.$$

Proof. Let $A = \left\{ t \in V | \chi_t \subseteq \sum_{i=1}^{n}(a_i \odot s_i), a_i \in K, s_i \in V, \mu(s_i) > 0, n \in N \right\}$.

We will show that A is the smallest subfuzzy hypervector space of V containing S. First, we show that A is a subfuzzy hypervector space of V containing S. Let $t_1, t_2 \in A$; then there exists $a_i, \acute{a}_i \in K$, $s_i, \acute{s}_i \in V$ such that

$$\chi_{t_1} \subseteq \bigcup_{i=1}^{n} a_i \odot s_i, \quad \chi_{t_2} \subseteq \bigcup_{i=1}^{m} \acute{a}_i \odot \acute{s}_i.$$

Then,

$$\chi_{t_1 - t_2} = \chi_{t_1} - \chi_{t_2} \subseteq \sum_{i=1}^{n} a_i \odot s_i - \sum_{j=1}^{m} \acute{a}_j \odot \acute{s}_j = \sum_{k=1}^{m+n} b_k \odot l_k,$$

where $b_k = a_k$, $b_{k+j} = a_{\acute{j}}$, $l_k = s_k$ and $l_{k+j} = s_{\acute{j}}$, for $1 \leq k \leq n$, and $1 \leq j \leq m$. Thus, $t_1 - t_2 \in A$.

Also, let us suppose that, for $t \in A$, $k \in K$, we have $(k \odot t)(x) > 0$. We will show that $x \in A$. For this, we have

$$(k \odot \chi_t)(x) = \sup_{s \in V}((k \odot s)(x) \wedge \chi_t(s)) = (k \odot t)(x) > 0.$$

On the other hand, we have

$$\begin{aligned}
0 < (k \odot t)(x) &= (k \odot \chi_t)(x) \subseteq k \odot \left(\sum_{i=1}^{n} a_i \odot s_i \right)(x) \\
&= \sum_{i=1}^{n}((ka_i) \odot s_i)(x) = \sum_{i=1}^{m}(b \odot s_i)(x) > 0 \\
\Longrightarrow &\bigvee_{x=\sum_{i=1}^{m} x_i} ((b \odot s_i) \wedge ... \wedge (b \odot s_m))(x_m) > 0 \\
&\Longrightarrow \exists x_1, ..., x_m \in W; x = \sum_{i=1}^{n} x_i \text{ and } (b \odot s_i)(x_i) > 0 \text{ for } 1 \leq i \leq m \\
&\Longrightarrow x_i \in A \Longrightarrow x \in A.
\end{aligned}$$

Thus, A is a subfuzzy hypervector space of V.

Now, let θ be a subfuzzy hypervector space of V containing μ and $t \in A$. Then, $\chi_t \subseteq \sum_{i=1}^{n} a_i \odot s_i$, for $a_i \in K, \mu(s_i) > 0, n \in N$. Since θ is a subfuzzy hypervector space containing μ, so for $s_i \in V$, $\theta(s_i) > 0$ we have $\sum_{i=1}^{n} a_i \odot s_i \subseteq \theta$. Thus, $A \leq \theta$. Hence, A is the smallest and for all $s \in V$ such that $\mu(s) > 0$, we have $\chi_s \subseteq 1_k \odot a$ then $s \in A$ and so $\mu \leq A$. ∎

Definition 3.12 Let V, W be two fuzzy hypervector space over a field K, and $T : V \longrightarrow W$ be a linear transformation. Then the kernel of T is denoted by $kerT$ and defined by

$$KerT = \{x \in V \mid \chi_{T(x)} \subseteq \Omega_W\}$$

where $\Omega_W = 0_K \odot 0_W$.

Theorem 3.13 *Let U, V be two fuzzy hypervector spaces (resp.strongly left) over K and $T : V \longrightarrow U$ be a linear transformation. Then, $KerT$ is a subfuzzy hypervector space of V.*

Proof. $T(\Omega_V) = T(0 \odot \underline{0}_V) \subseteq 0 \odot T(\underline{0}_V) = 0 \odot \underline{0}_U = \Omega_U$. Therefore, $KerT \neq \phi$. Also, for all $a, b \in K, x, y \in KerT$, we have $\chi_{T(x)} \in \Omega_U$ and $\chi_{T(y)} \in \Omega_U$ so

$$\begin{aligned} \chi_{T(a\odot x + a\odot y)} &= \chi_{T(a\odot x)} + \chi_{T(b\odot y)} \subseteq \chi_{a\odot T(x)} + \chi_{b\odot T(y)} \\ &\subseteq a \odot \chi_{T(x)} + b \odot \chi_{T(x)} \subseteq a \odot \Omega_U + b \odot \Omega_U = \Omega_U. \end{aligned}$$

Now, by Lemma 3.6 since $(a \odot x + b \odot y)(v) > 0$, we have $\chi_{T(v)} \subseteq \Omega_U$. Hence, $v \in KerT$ and so $KerT$ is subfuzzy hypervector space of U. ∎

It is easy to see that, if W is a subfuzzy hypervector space of V over a field K, then

$$\begin{aligned} \Pi : V &\longrightarrow V/W \\ x &\longmapsto x + W \end{aligned}$$

is a good linear transformation, such that $\Omega_V \subseteq KerT$ and it is called projection or canonical transformation.

Theorem 3.14 *Let V, U be two fuzzy hypervector spaces and $T : V \longrightarrow U$ be a good linear transformation:*

(i) *if W is a subfuzzy hypervector space of V, then $T(W)$ is a subfuzzy hypervector space of U.*

(ii) *if L is a subfuzzy hypervector space of U, then $T^{-1}(L)$ is a subfuzzy hypervector space of V containing $kerT$.*

Proof. (i) Let $a \in K$ and $x', y' \in T(W)$, such that $x' = T(x), y' = T(y)$ for $x, y \in W$. Then $x + y \in W$ and if $(a \odot x)(t) > 0 \Longrightarrow t \in W$. So, $x' - y' = T(x) - T(y) = T(x - y) \in T(W)$.

Now, let $(a \odot x')(t) > 0$. Then, $(a \odot T(x))(t) > 0$, and hence $T(a \odot x)(t) > 0$. Thus, by extension principle, we have $\sup_{T(z)=t} (a \odot x)(z) > 0$ so, there exists y such that $(a \odot x)(y) > 0, T(y) = t$. Then, $y \in W$, and so $T(y) \in T(W)$, thus $t \in T(W)$, and hence $T(W) \leq U$.

(ii) The first part can be proved in a similar way as in (i). Now, if $x \in KerT$, then $T(x) \in 0_U \subseteq 0 \odot L \subseteq L$. Therefore, $x \in T^{-1}(L)$ and so $KerT \subseteq T^{-1}(L)$. ■

Theorem 3.15 *Let V and U be two left distributive fuzzy hypervector spaces and $T : V \longrightarrow U$ be a good linear transformation. Then there is an one-to-one correspondence between subfuzzy hypervector spaces of V containing $KerT$ and subfuzzy hypervector spaces of U.*

Proof. Let $\mathbf{A} = \{W | W \leq V, W \supseteq T\}$ and $\mathbf{B} = \{L | L \leq U\}$. We will show that the following map is one-to-one and onto:

$$\begin{array}{rcl} \varphi : \mathbf{A} & \longrightarrow & \mathbf{B} \\ W & \longmapsto & T(W) \end{array}$$

Then, $T(W)$ is an element of $\mathbf{B}$ for all $W \in \mathbf{A}$. Let W_1, W_2 be two elements of $\mathbf{A}$, such that $W_1 \neq W_2$ then there exists $w_1 \in W_1 - W_2$ or $w_2 \in W_2 - W_1$. If $w_1 \in W_1 - W_2$ then $T(w_1) \in T(W_1) - T(W_2)$, and so $T(W_1) \neq T(W_2)$, and if $w_2 \in W_2 - W_1$, similarly $T(W_1) \neq T(W_2)$. Also, for an arbitrary $L \in \mathbf{B}$, suppose $W = T^{-1}(L)$. Then, by Theorem 3.10, $W \in \mathbf{A}$ and $T(W) \in \mathbf{B}$. Hence φ is one-to-one and onto. ■

The next result follows immediately from Theorem 3.15:

Corollary 3.16 *If V is a left distributive fuzzy hypervector space, then every subfuzzy hypervector space of V/W, is of the form L/W, in which L is a subfuzzy hypervector space V containing W.*

4. Connections between fuzzy hypervector spaces and hypervector spaces

Connections between fuzzy hyperoperations and hyperoperations on fuzzy hypersemigroups, fuzzy hyperrings and fuzzy hypermodules have been studied in [22],[17].

Now, in the next theorem, we establish a similar result for hypervector spaces.

Theorem 4.1 *If $(V, +, \odot)$ is a fuzzy hypervector space over a field K, then $(V, +, \circ)$ is a hypervector space over the field K.*

Proof. For all $x \in V, \alpha \in K$ define a hyperoperation "$\circ$" on V as $\alpha \circ x = \{z \in V \mid (\alpha \odot x)(z) > 0\}$. We have to check the conditions of Definition 2.1. First, for all $x, y \in V$, $\alpha \in K$, we have:

$$t \in \alpha(x + y) \Longleftrightarrow (\alpha \odot (x + y))(t) > 0.$$

This means that

$$(\alpha\odot(x+y))(t) \subseteq ((\alpha\odot x)+(\alpha\odot y))(t) = \bigvee_{t=u+v} ((\alpha\odot x)(u) \wedge (\alpha\odot y)(v)) > 0.$$

Hence, there exists $u, v \in V$ such that $((\alpha\odot x))(u) > 0$, and so $u \in \alpha \circ x$ and $((\alpha\odot y))(v) > 0$. Thus $v \in \alpha \circ y$, and so $t = u + v \in \alpha \circ x + \alpha \circ y$.

Similarly, we can obtain other conditions of Definition 2.1. Therefore, $(V, +, \circ)$ is a hypervector space over field K, as desired. ∎

Hence, the exists a map $\psi : FHV \to HV$ with $\psi((V, +, \odot)) = (V, +, \circ)$, where HV denotes the class of all hypervector spaces and FHV the class of all fuzzy hypervector spaces.

Now, we will obtain a fuzzy hypervector space from a hypervector space $(V, +, \circ)$.

Theorem 4.2 *If $(V, +, \circ)$ is a hypervector space over a field K, then $(V, +, \odot)$ is a fuzzy hypervector space over the field K.*

Proof. We will show that for all $x, y, t \in V$, $\alpha \in K$ we have $\alpha\odot(x+y) \subseteq (\alpha\odot x)+(\alpha\odot y)$. Let $(V, +, \circ)$ is a hypervector space over a field K, then $\forall x \in V, \forall \alpha \in R$ we define the fuzzy hyperoperation: $\alpha\odot x = \chi_{\alpha\circ x}$. Now,

$$\begin{aligned}(\alpha\odot(x+y))(t) &= \chi_{\alpha\circ(x+y)}(t) \subseteq \chi_{\alpha\circ x+\alpha\circ y}(t) \\ &= \begin{cases} 1 & \text{if } t = \alpha\circ x + \alpha\circ y, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

On the other hand,

$$\begin{aligned}((\alpha\odot x)+(\alpha\odot y))(t) &= \bigvee_{t=u+v} ((\alpha\odot x)(u) \wedge (\alpha\odot y)(v)) \\ &= \bigvee_{t=u+v} (\chi_{\alpha\circ x}(u) \wedge \chi_{\alpha\circ y}(v)) \\ &= \begin{cases} 1 & \text{if } t = u+v = \alpha\circ x + \alpha\circ y, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

Similarly, we obtain other conditions of Definition 3.1. ∎

Therefore, there exists a map $\varphi : HV \to FHV$ such that

$$\varphi((V, +, \circ)) = (V, +, \odot).$$

Recall that if V, W are two fuzzy hypervector spaces, the map $f : V \to W$ is called a homomorphism if $T : V \to W$ is a linear transformation and if T is an one to one correspondence then it is called an isomorphism.

The next two theorems will make connections between homomorphisms of fuzzy hypervector spaces and homomorphism of hypervector spaces.

Theorem 4.3 *Let $(V_1,+,\odot_1)$ and $(V_2,+,\odot_2)$ be fuzzy hypervector spaces over a field K and $(V_1,+,\circ_1)=\psi(V_1,+,\odot_1)$, $(V_2,+,\circ_2)=\psi(V_2,+,\odot_2)$ be the associated hypervector spaces over the field K. If $f:V_1\to V_2$ is a homomorphism of fuzzy hypervector spaces, then f is a homomorphism of hypervector spaces, too.*

Proof. For all $x,y\in V,\alpha\in K$ we have $f(\alpha\odot_1 x)\leq\alpha\odot_2 f(x)$. If $u\in\alpha\circ_1 x$, then $(\alpha\odot_1 x)(u)>0$. Denote $v=f(u)$. We have

$$(f(\alpha\odot_1 x))(v)=\bigvee_{f(s)=v}(\alpha\odot_1 x)(s)\geq(\alpha\odot_1 x)(u)>0.$$

Hence, $(\alpha\odot_2 f(x))(v)>0$ and so $v\in\alpha\circ_2 f(x)$, which means that $f(\alpha\circ_1 x)\subseteq\alpha\circ_2 f(x)$. And obviously, $f(x+y)=f(x)+f(y)$. ■

Theorem 4.4 *Let $(V_1,+,\circ_1)$ and $(V_2,+,\circ_2)$ be two hypervector spaces over field K and $(V_1,+,\odot_1)=\psi(V_1,+,\circ_1)$, $(V_2,+,\odot_2)=\psi(V_2,+,\circ_2)$ be the associated hypervector spaces over field K. The map $f:V_1\to V_2$ is a homomorphism of fuzzy hypervector spaces if and only if it is a homomorphism of hypervector spaces.*

Proof. Suppose that f is a homomorphism of hypervector spaces. Let $x\in V$, $\alpha\in K$. For all $t\in Imf$ we have

$$\begin{aligned}(f(\alpha\odot_1 x))(t) &= \bigvee_{f(r)=t}(\alpha\odot_1 x)(r)=\bigvee_{f(r)=t}\chi_{\alpha\circ_1 x}(r)\\ &= \begin{cases}1 & \text{if } t\in f(\alpha\circ x_1),\\ 0 & \text{otherwise,}\end{cases}\\ &= \chi_{f(\alpha\circ_1 x)(t)}\leq\chi_{\alpha\circ_2 f(x)}(t)=(\alpha\odot_2 f(x))(t).\end{aligned}$$

Obviously, $f(x+y)=f(x)+f(y)$.

Conversely, let $x,y\in V_1,\ \alpha\in K$. We have $f(\alpha\odot_1 x)\leq\alpha\odot_2 f(x)$, whence $\chi_f(\alpha\circ_1 x)\leq\chi_{\alpha\circ_2 f(x)}$. This means $f(\alpha\circ_1 x)\subseteq\alpha\circ_2 f(x)$. ■

The next theorem establishes a connection between subfuzzy hypervector spaces of a fuzzy hypervector spaces and subhypervector spaces of the corresponding hypervector space.

Theorem 4.5 (i) *If $(V',+,\odot)$ is a subfuzzy hypervector space of $(V,+,\odot)$ over a field K, then $(V',+,\circ)=\psi(V',+,\odot)$ is a subhypervector space of $(V,+,\circ)=\psi(V,+,\odot)$ over the field K.*

(ii) *$(V',+,\circ)$ is a subhypervector space of $(V,+,\circ)$ over a field K if and only if $(V',+,\odot)=\varphi(V',+,\circ)$ is a subfuzzy hypervector space of $(V,+,\odot)=\psi(V,+,\circ)$.*

Proof. (i) For all $x \in V', \alpha \in K$ we will show that $\alpha \circ x \subseteq V'$. since $(V', +, \odot)$ is a subfuzzy hypervector space of $(V, +, \odot)$ so if for all $x \in V'$, $\alpha \in K$, $(\alpha \odot x)(t) > 0 \Rightarrow t \in V'$. This means that $t \in \alpha \circ x \Rightarrow t \in V'$. Hence, $\alpha \circ x \subseteq V'$.

(ii) It can be shown by a similar way as in (i). ∎

The above theorem is a connection between subfuzzy hypervector spaces of a fuzzy hypervector spaces and subhypervector spaces of the corresponding hypervector space.

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