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# ON KÖTHE-TOEPLITZ DUALS OF SOME NEW AND GENERALIZED DIFFERENCE SEQUENCE SPACES

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**Abstract.** In this paper we define the sequence spaces $\triangle^m_{v,r}(l_\infty)$, $\triangle^m_{v,r}(c)$ and $\triangle^m_{v,r}(c_0)$, $(m \in N,\ r \in R)$, and have studied some of their topological properties and have computed their Köthe-Toeplitz duals.

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## 1. Introduction

Let $l_\infty$, $c$ and $c_0$ be the linear spaces of bounded convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in N = \{1, 2, 3, ...\}$, the set of positive integers.

In 1981, Kizmaz [8] introduced the concept of difference sequences and have defined $\triangle-$bounded, $\triangle-$ convergent and $\triangle-$ null sequence spaces. Using the concept of difference sequences, Et [4] has defined $\triangle^2-$bounded, $\triangle^2-$ convergent and $\triangle^2-$null sequence spaces. Further, this notion was generalized by Et and Colak [6] and have defined $\triangle^m-$bounded, $\triangle^m-$convergent and $\triangle^m-$null sequence spaces. Later on, Et and Esi [5] have defined $\triangle^m_v-$bounded, $\triangle^m_v-$convergent and $\triangle^m_v-$null sequence spaces where $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Recently, Bektas and Colak [1] have defined the sequence spaces

$$l_\infty(\triangle_r^m) = \{x = (x_k) : (k^r \triangle^m x_k) \in l_\infty\},$$
$$c(\triangle_r^m) = \{x = (x_k) : (k^r \triangle^m x_k) \in c\},$$
$$c_0(\triangle_r^m) = \{x = (x_k) : (k^r \triangle^m x_k) \in c_0\}.$$

where $m \in N$, $r \in R$, $\triangle_r^m x = (\triangle_r^m x_k) = (k^r \triangle^m x_k) = (k^r(\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1}))$ and

$$\triangle^m x_k = \sum_{j=1}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} x_{k+j}.$$

These are Banach spaces with norm

$$||x||_{\triangle_r} = \sum_{i=1}^{m} |x_i| + \sup_k k^r |\triangle^m x_k|.$$

It is trivial that $c_0(\triangle_r^m) \subset c_0(\triangle_r^{m+1})$, $c(\triangle_r^m) \subset c(\triangle_r^{m+1})$, $l_\infty(\triangle_r^m) \subset l_\infty(\triangle_r^{m+1})$ and $c_0(\triangle_r^m) \subset c(\triangle_r^m) \subset l_\infty(\triangle_r^m)$ are satisfied and are strict [1].

For convenience, we denote these spaces $\triangle_r^m(l_\infty) = l_\infty(\triangle_r^m)$, $\triangle_r^m(c) = c(\triangle_r^m)$ and $\triangle_r^m(c_0) = c_0(\triangle_r^m)$.

Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Now, we define

$$\triangle_{v,r}^m(l_\infty) = \{x = (x_k) \ : \ (k^r \triangle_v^m x_k) \in l_\infty\},$$
$$\triangle_{v,r}^m(c) = \{x = (x_k) \ : \ (k^r \triangle_v^m x_k) \in c\},$$
$$\triangle_{v,r}^m(c_0) = \{x = (x_k) \ : \ (k^r \triangle_v^m x_k) \in c_0\}.$$

where $m \in N$, $r \in R$, $\triangle_{v,r}^m(x) = (k^r \triangle_v^m x_k) = (k^r(\triangle_v^{m-1} x_k - \triangle_v^{m-1} x_{k+1}))$ and

$$\triangle_v^m x_k = \sum_{j=1}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} v_{k+j}\, x_{k+j}.$$

Throughout the paper, we write $X$ for $l_\infty$ or $c$ or $c_0$. $\triangle_{v,r}^m(X)$ is the generalization of several known sequence spaces, for instance, the following classes arise from $\triangle_{v,r}^m(X)$ as the special cases.

(i) If we take $v = (v_k) = (1, 1, ...)$, then $\triangle_{v,r}^m(X) = \triangle_r^m(X)$ [1].

(ii) If we take $r = 0$, then $\triangle_{v,r}^m(X) = \triangle_v^m(X)$ [5].

(iii) If we take $v = (v_k) = (1, 1, ...)$ and $r = 0$, then $\triangle_{v,r}^m(X) = \triangle^m(X)$ [6].

(iv) If we take $v = (v_k) = (1, 1, ...)$ , $r = 0$ and $m = 2$, then $\triangle_{v,r}^m(X) = \triangle^2(X)$ [4].

(v) If we take $v = (v_k) = (1, 1, ...)$ , $r = 0$ and $m = 1$, then $\triangle_{v,r}^m(X) = \triangle(X)$ [8].

(vi) If we take $r = 0$ and $m = 1$, then $\triangle_{v,r}^m(X) = \triangle_v(X)$ [2].

(vii) If we take $v_k = 1$ for all $k \in N$, $r < 1$ and $m = 1$, then $\triangle^m_{v,r}(X) = \triangle_r(X)$ [10].

## 2. Main results

**Theorem 2.1.** *The sequence spaces $\triangle^m_{v,r}(l_\infty)$, $\triangle^m_{v,r}(c)$ and $\triangle^m_{v,r}(c_0)$ are Banach spaces normed by*

$$||x||_{\triangle_{v,r}} = \sum_{i=1}^{m} |v_i x_i| + \sup_k k^r |\triangle^m_v x_k| \tag{2.1}$$

Let us define the operator

$$D : \triangle^m_{v,r}(X) \to \triangle^m_{v,r}(X)$$

by

$$Dx = (0, 0, 0, ....x_{m+1}, x_{m+2}, ...),$$

where $x = (x_1, x_2, x_3, ....)$. It is trivial that $D$ is bounded linear operator on $\triangle^m_{v,r}(X)$. Furthermore, the set

$$D[\triangle^m_{v,r}(X)] = \{x = (x_k) : x \in \triangle^m_{v,r}(X),\ x_1 = x_2 = .... = x_m = 0\}$$

is a subspace of $\triangle^m_{v,r}(X)$ and $\|x\|_{\triangle_{v,r}} = \|\triangle^m_{v,r}(X)\|_\infty$ in $D[\triangle^m_{v,r}(X)]$. $D[\triangle^m_{v,r}(X)]$ and $X$ are equivalent as topological space. Hence

$$\triangle^m_{v,r} : D[\triangle^m_{v,r}(X)] \to X,$$

defined by

$$\triangle^m_{v,r} x = y = (\triangle^m_{v,r} x_k) = (k^r(\triangle^m_v x_k)) \tag{2.2}$$

is a linear homeomorphism [9].

## 3. Dual spaces

In this section, we give Köthe-Toeplitz duals of $\triangle^m_{v,r}(l_\infty)$, $\triangle^m_{v,r}(c)$ and $\triangle^m_{v,r}(c_0)$. Also, we show that these spaces are not perfect spaces. Further, we show that $\triangle^m_{v,r}(l_\infty)$, and $\triangle^m_{v,r}(c)$ are not normal and not monotone spaces.

**Lemma 3.1.** $\sup_k k^r |\triangle^m_v x_k| < \infty$ *if and only if*

(i) $\sup_k k^{r-1} |\triangle^{m-1}_v x_k| < \infty$

(ii) $\sup_k k^r |\triangle^{m-1}_v x_k - k(k+1)^{-1} \triangle^{m-1}_v x_{k+1}| < \infty.$

**Proof.** Let $\sup_k k^r|\triangle_v^m x_k| < \infty$. Then

$$\begin{aligned} |\triangle_v^{m-1}x_1 - \triangle_v^{m-1}x_{k+1}| &= \left|\sum_{j=1}^{k}(\triangle_v^{m-1}x_j - \triangle_v^{m-1}x_{j+1})\right| \\ &\leq \sum_{j=1}^{k}|\triangle_v^m x_j| = O(k^{1-r}). \end{aligned}$$

This implies $\sup_k k^{r-1}|\triangle_v^{m-1}x_k| < \infty$,

$$|\triangle_v^{m-1}x_k - k(k+1)^{-1}\triangle_v^{m-1}x_{k+1}| = |k(k+1)^{-1}\triangle_v^m x_k + (k+1)^{-1}\triangle_v^{m-1}x_k| = O(k^{-r})$$

We have (ii).

Now, suppose (i) and (ii) hold. Then

$$k^r|\triangle_v^{m-1}x_k - k(k+1)^{-1}\triangle_v^{m-1}x_{k+1}| \geq k^{r+1}(k+1)^{-1}|\triangle_v^m x_k| - k^r(k+1)^{-1}|\triangle_v^{m-1}x_k.|$$

This implies $\sup_k k^r|\triangle_v^m x_k| < \infty$.

**Lemma 3.2.** *$\sup_k k^{r-n}|\triangle_v x_k| < \infty$ implies $\sup_k k^{r-(n+1)}|v_k x_k| < \infty$ for all $n \in N$ with $r < (n+1)$.*

**Proof.** Let $\sup_k k^{r-n}|\triangle_v x_k| < \infty$. Then

$$\begin{aligned} |v_1x_1 - v_{k+1}x_{k+1}| &\leq \sum_{i=1}^{k}|v_ix_i - v_{i+1}x_{i+1}| \\ &\leq \sum_{i=1}^{k}|\triangle_v x_i| = O(k^{(n+1)-r}) \end{aligned}$$

This implies $\sup_k k^{r-(n+1)}|v_k x_k| < \infty$.

**Lemma 3.3.** *$\sup_k k^{r-n}|\triangle_v^{m-n}x_k| < \infty$ implies $\sup_k k^{r-(n+1)}|\triangle_v^{m-(n+1)}x_k| < \infty$ for all $n, m \in N$ and $r \leq n < m$.*

**Proof.** If we put $\triangle_v^{m-n}x_k$ instead of $\triangle_v x_k$ in Lemma 2.2, the result is immediate.

**Corollary 3.4.** *$\sup_k k^{r-1}|\triangle_v^{m-1}x_k| < \infty$ implies $\sup_k k^{r-m}|v_k\, x_k| < \infty$.*

**Corollary 3.5.** *$x \in \triangle_{v,r}^m(l_\infty)$ implies $\sup_k k^{r-m}|v_k\, x_k| < \infty$.*

**Lemma 3.6.** ([8]) *Let $(P_n)$ be sequence of positive real numbers increasing monotonically to infinity, then*

(i) *If* $\sup_n \Big|\sum_{i=1}^{n} P_i a_i\Big| < \infty$, *then* $\sup_n \Big|P_n \sum_{k=n+1}^{\infty} a_k\Big| < \infty$,

(ii) *If* $\sum_{k=1}^{\infty} P_k a_k$ *is convergent, then* $\lim_{n\to\infty} P_n \sum_{k=n+1}^{\infty} a_k = 0$.

**Definition 3.7.** ([7]) Let $X$ be a sequence space and define

$$\begin{aligned} X^{\alpha} &= \Big\{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for all } x \in X\Big\}, \\ X^{\beta} &= \Big\{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for all } x \in X\Big\}, \\ X^{\gamma} &= \Big\{a = (a_k) : \sup_n |\sum_{k=1}^{n} a_k x_k| < \infty, \text{ for all } x \in X\Big\}. \end{aligned}$$

**Definition 3.8.** ([7]) Let $X$ be a sequence space. Then $X$ is called

(i) *perfect* if $X = X^{\alpha\alpha}$,

(ii) *normal* if $y \in X$ whenever $|y_k| \le |x_k|$, $k \ge 1$, for some $x \in X$,

(iii) *monotone* provided $X$ contains the canonical preimages of all its stepspaces.

**Lemma 3.9.** ([7]) *Let $X$ be a sequence space. Then we have*

(i) *$X$ is perfect $\Rightarrow$ $X$ is normal $\Rightarrow$ $X$ is monotone,*

(ii) *$X$ is normal $\Rightarrow X^{\alpha} = X^{\gamma}$,*

(iii) *$X$ is monotone $\Rightarrow X^{\alpha} = X^{\beta}$.*

**Theorem 3.10.** *Let $m$ be a positive integer and $r \in R$,*

(a) *We put*

$$M_{\alpha}(v,r) = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^{m-r} |a_k v_k^{-1}| < \infty\Big\}.$$

*Then*

$$[\triangle_{v,r}^{m}(l_{\infty})]^{\alpha} = [\triangle_{v,r}^{m}(c)]^{\alpha} = [\triangle_{v,r}^{m}(c_0)]^{\alpha} = M_{\alpha}(v,r) \tag{3.1}$$

(b) *We put*

$$M_{\alpha\alpha}(v,r)=\Big\{a=(a_k):\sup_k k^{r-m}|a_kv_k|<\infty\Big\}.$$

*Then*

$$[\triangle^m_{v,r}(l_\infty)]^{\alpha\alpha}=[\triangle^m_{v,r}(c)]^{\alpha\alpha}=[\triangle^m_{v,r}(c_0)]^{\alpha\alpha}=M_{\alpha\alpha}(v,r) \tag{3.2}$$

**Proof.** (a) First, we assume that $a\in M_\alpha(v,r)$. Then

$$\sum_{k=1}^{\infty}|a_kx_k|=\sum_{k=1}^{\infty}k^{m-r}|a_kv_k^{-1}|k^{r-m}|x_kv_k|<\infty,$$

for each $x\in\triangle^m_{v,r}(l_\infty)$, by Corollary 3.5.

Thus, we have shown

$$M_\alpha(v,r)\subset[\triangle^m_{v,r}(l_\infty)]^{\alpha} \tag{3.3}$$

Conversely, let $a\notin M_\alpha(v,r)$. Then, for some $k$, we have

$$\sum_{k=1}^{\infty}k^{m-r}|a_kv_k^{-1}|=\infty.$$

So, there is a strictly increasing sequence $(n_i)$ of positive integers $n_i$ such that

$$\sum_{k=n_i+1}^{n_{i+1}}k^{m-r}|a_kv_k^{-1}|>i.$$

We define a sequence $x=(x_k)$ by

$$x_k=\begin{cases}0 & (1\le k\le n_i)\\ \dfrac{v_k^{-1}k^{m-r}}{i} & (n_i+1<k\le n_{i+1};\ i=1,2,...)\end{cases}$$

Then, we see that

$$k^r|\triangle^m v_kx_k|=\frac{m!}{i}\ \ (n_i+1<k\le n_{i+1};\ i=1,2,...).$$

Hence, $x\in\triangle^m_{v,r}(c_0)$ and $\sum_{k=1}^{\infty}|a_kx_k|>\sum_{i=1}^{\infty}1=\infty$.

Thus, $a\notin[\triangle^m_{v,r}(c_0)]^{\alpha}$, and hence we have shown

$$[\triangle^m_{v,r}(c_0)]^{\alpha}\subset M_\alpha(v,r) \tag{3.4}$$

Since $\triangle_{v,r}^m(c_0) \subset \triangle_{v,r}^m(c) \subset \triangle_{v,r}^m(l_\infty)$ implies $[\triangle_{v,r}^m(l_\infty)]^\alpha \subset [\triangle_{v,r}^m(c)]^\alpha \subset [\triangle_{v,r}^m(c_0)]^\alpha$, (3.1) follows from (3.3) and (3.4).

(b) First, we assume that $a \in M_{\alpha\alpha}(v,r)$. Then

$$\sum_{k=1}^{\infty} |a_k x_k| \leq \sup_k k^{r-m} |a_k v_k| \sum_{k=1}^{\infty} k^{m-r} |x_k v_k^{-1}| < \infty,$$

for each $x \in [\triangle_{v,r}^m(c_0)]^\alpha = M_\alpha(v,r)$, by part (a).

Thus, we have shown

$$M_{\alpha\alpha}(v,r) \subset [\triangle_{v,r}^m(c_0)]^{\alpha\alpha} \tag{3.5}$$

Conversely, let $a \notin M_{\alpha\alpha}(v,r)$. Then, we have

$$\sup_k k^{r-m} |a_k v_k| = \infty.$$

Hence, there is a strictly increasing sequence of $(k(i))$ of positive integers $k(i)$ such that

$$[k(i)]^{r-m} \, |a_{k(i)} v_{k(i)}| > i^m.$$

We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} |a_{k(i)}|^{-1}, & k = k(i) \\ 0, & k \neq k(i). \end{cases}$$

Then, we see that

$$\sum_{k=1}^{\infty} k^{m-r} |x_k v_k^{-1}| = \sum_{i=1}^{\infty} [k(i)]^{m-r} |a_{k(i)} v_{k(i)}|^{-1} \leq \sum_{i=1}^{\infty} i^{-m} < \infty.$$

Hence, $x \in [\triangle_{v,r}^m(l_\infty)]^\alpha$ and $\sum_{k=1}^{\infty} |a_k x_k| = \sum_{i=1}^{\infty} 1 = \infty$.

Thus, $a \notin [\triangle_{v,r}^m(l_\infty)]^{\alpha\alpha}$, and hence, we have shown

$$[\triangle_{v,r}^m(l_\infty)]^{\alpha\alpha} \subset M_{\alpha\alpha}(v,r). \tag{3.6}$$

Since $[\triangle_{v,r}^m(l_\infty)]^\alpha \subset [\triangle_{v,r}^m(c)]^\alpha \subset [\triangle_{v,r}^m(c_0)]^\alpha$ implies $[\triangle_{v,r}^m(c_0)]^{\alpha\alpha} \subset [\triangle_{v,r}^m(c)]^{\alpha\alpha} \subset [\triangle_{v,r}^m(l_\infty)]^{\alpha\alpha}$, (3.2) follows from (3.5) and (3.6).

From Theorem 3.10, we have the following corollaries:

**Corollary 3.11.** *If we take $v_k = (1, 1, ...)$, then we obtain*

(i) $[\triangle_r^m(l_\infty)]^\alpha = [\triangle_r^m(c)]^\alpha = [\triangle_r^m(c_0)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k^{m-r}|a_k| < \infty\}$

(ii) $[\triangle_r^m(l_\infty)]^{\alpha\alpha} = [\triangle_r^m(c)]^{\alpha\alpha} = [\triangle_r^m(c_0)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{r-m}|a_k| < \infty\}$.

**Corollary 3.12.** *If we take $r = 0$, then we obtain*

(i) $[\triangle_v^m(l_\infty)]^\alpha = [\triangle_v^m(c)]^\alpha = [\triangle_v^m(c_0)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k^m|a_k v_k^{-1}| < \infty\}$ [5]

(ii) $[\triangle_v^m(l_\infty)]^{\alpha\alpha}=[\triangle_v^m(c)]^{\alpha\alpha}=[\triangle_v^m(c_0)]^{\alpha\alpha}=\{a = (a_k) : \sup_k k^{-m}|a_k v_k| < \infty\}$. [5]

**Corollary 3.13.** *If we take $v_k = k^m$ and $r = 0$, then we obtain*

(i) $[\triangle_v^m(l_\infty)]^\alpha = [\triangle_v^m(c)]^\alpha = [\triangle_v^m(c_0)]^\alpha = l_1$ [5]

(ii) $[\triangle_v^m(l_\infty)]^{\alpha\alpha} = [\triangle_v^m(c)]^{\alpha\alpha} = [\triangle_v^m(c_0)]^{\alpha\alpha} = l_\infty$. [5]

**Corollary 3.14.** *If we take $v_k = (1, 1, ...)$ and $r = 0$, then we obtain*

(i) $[\triangle^m(l_\infty)]^\alpha = [\triangle^m(c)]^\alpha = [\triangle^m(c_0)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k^m|a_k| < \infty\}$ [6]

(ii) $[\triangle^m(l_\infty)]^{\alpha\alpha} = [\triangle^m(c)]^{\alpha\alpha} = [\triangle^m(c_0)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{-m}|a_k| < \infty\}$. [6]

**Corollary 3.15.** *If we take $v_k = (1, 1, ...)$, $r = 0$ and $m = 2$, then we obtain*

(i) $[\triangle^2(l_\infty)]^\alpha = [\triangle^2(c)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k^2|a_k| < \infty\}$ [4]

(ii) $[\triangle^2(l_\infty)]^{\alpha\alpha} = [\triangle^2(c)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{-2}|a_k| < \infty\}$. [4]

**Corollary 3.16.** *If we take $v_k = (1, 1, ...)$, $r = 0$ and $m = 1$, then we obtain*

(i) $[\triangle(l_\infty)]^\alpha = [\triangle(c)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k|a_k| < \infty\}$ [8]

(ii) $[\triangle(l_\infty)]^{\alpha\alpha} = [\triangle(c)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{-1}|a_k| < \infty\}$. [5]

**Corollary 3.17.** *If we take $v_k = k^m$, then we obtain*

(i) $[\triangle^m_{v,r}(l_\infty)]^\alpha = [\triangle^m_{v,r}(c)]^\alpha = [\triangle^m_{v,r}(c_0)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k^{-r}|a_k| < \infty\}$.

(ii) $[\triangle^m_{v,r}(l_\infty)]^{\alpha\alpha} = [\triangle^m_{v,r}(c)]^{\alpha\alpha} = [\triangle^m_{v,r}(c_0)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^r|a_k| < \infty\}$.

By Lemma 3.8, we also have

**Corollary 3.18.** *The sequence spaces $\triangle^m_{v,r}(l_\infty)$, $\triangle^m_{v,r}(c)$ and $\triangle^m_{v,r}(c_0)$ are not perfect.*

**Lemma 3.19.** *Let $m$ be any positive integers and let $r$ be any real number.*

a) *We put*

$$M_\beta(v,r) = \{a = (a_k) : \sum_{k=1}^{\infty} k^{m-r} a_k v_k^{-1} \text{ is convergent, } \sum_{k=1}^{\infty} k^{m-(r+1)}|R_k| < \infty\},$$

*where $R_k = \sum_{j=k+1}^{\infty} a_j v_j^{-1}$. Then*

$$(D[\triangle^m_{v,r}(l_\infty)])^\beta = M_\beta(v,r).$$

b) *We put*

$$M_\gamma(v,r) = \{a = (a_k) : \sup_n |\sum_{k=1}^{n} k^{m-r} a_k v_k^{-1}| < \infty, \ \sum_{k=1}^{\infty} k^{m-(r+1)}|R_k| < \infty\},$$

*where $R_k = \sum_{j=k+1}^{\infty} a_j v_j^{-1}$. Then*

$$(D[\triangle^m_{v,r}(l_\infty)])^\gamma = M_\gamma(v,r).$$

**Proof.** (a) If $x \in D[\triangle^m_{v,r}(l_\infty)]$, then there exist one and only one $y = (y_k) \in l_\infty$ such that

$$\begin{aligned} x_k &= v_k^{-1} \sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} j^{-r} y_j \\ &= v_k^{-1} \sum_{j=1}^{k} (-1)^m \binom{k+m-j-1}{m-1} (j-m)^{-r} y_{j-m} \\ y_{1-m} &= y_{2-m} = ....... = y_0 = 0, \end{aligned}$$

for sufficiently large $k$, for instance $k > m$ by (2.2). Let $a \in M_\beta(v,r)$ and suppose that $\binom{-1}{-1} = 1$ (in some literature it is assumed that $\binom{r}{k} = 0$ for $k < 0$). Then, we may write

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k \Big( v_k^{-1} \sum_{j=1}^{k-m} (-1)^m \begin{pmatrix} k-j-1 \\ m-1 \end{pmatrix} j^{-r} y_j \Big)$$

$$= (-1)^m \sum_{k=1}^{n-m} R_{k+m-1} (k+m-1)^{m-(r+1)} \frac{1}{(k+m-1)^{m-(r+1)}}$$

$$\sum_{j=1}^{k} \begin{pmatrix} k+m-j-2 \\ m-2 \end{pmatrix} j^{-r} y_j - n^{m-r} R_n n^{r-m} x_n v_n \tag{3.7}$$

Since $\sum\limits_{k=1}^{\infty} k^{m-(r+1)} |R_k| < \infty$, the series $\sum\limits_{k=1}^{\infty} (k+m-1)^{m-(r+1)} R_{k+m-1} z_k$ is absolutely convergent, where

$$z = (z_k) = \left( \frac{1}{(k+m-1)^{m-(r+1)}} \sum_{j=1}^{k} \begin{pmatrix} k+m-j-2 \\ m-2 \end{pmatrix} j^{-r} y_j \right).$$

Moreover, we have $n^{m-r} R_n \to 0$ as $n \to \infty$ (Lemma 3.6), $\sup\limits_k n^{r-m} |x_n v_n| < \infty$ (Corollary 3.5), hence $\sum\limits_{k=1}^{\infty} a_k x_k$ is convergent for each $x \in D[\triangle_{v,r}^m (l_\infty)]$, so $a \in (D[\triangle_{v,r}^m (l_\infty)])^\beta$.

Let $a \in (D[\triangle_{v,r}^m (l_\infty)])^\beta$. Then, $\sum\limits_{k=1}^{\infty} a_k x_k$ is convergent for each $x \in D[\triangle_{v,r}^m (l_\infty)]$. For the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 0, & k \le m, \\ v_k^{-1} k^{m-r}, & k > m, \end{cases}$$

we may write

$$\sum_{k=1}^{\infty} k^{m-r} a_k v_k^{-1} = \sum_{k=1}^{m} k^{m-r} a_k v_k^{-1} + \sum_{k=m+1}^{\infty} a_k x_k < \infty.$$

Thus, the series $\sum\limits_{k=1}^{\infty} k^{m-r} a_k v_k^{-1}$ is convergent. This implies $n^{m-r} R_n \to 0$ as $n \to \infty$ by Lemma 3.6.

Now, let $a \in D[\triangle_{v.r}^m (l_\infty)]^\beta - M_\beta(v, r)$. Then $\sum\limits_{k=1}^{\infty} k^{m-(r+1)} |R_k|$ is divergent, that is $\sum\limits_{k=1}^{\infty} k^{m-(r+1)} |R_k| = \infty$.

We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 0, & k \le m \\ v_k^{-1} \sum\limits_{j=1}^{k-1} j^{m-(r+1)} \operatorname{sgn} R_j, & k > m, \end{cases}$$

where $a_k > 0$ for all $k$ or $a_k < 0$ for all $k$. Since $k^r|\triangle_v^m x_k| = (m-1)!$ for $k > m$, it is trivial that $x = (x_k) \in D[\triangle_{v,r}^m(l_\infty)]$. Then, we may write for $n > m$

$$\sum_{k=1}^{n} a_k x_k = -\sum_{k=1}^{m} R_{k-1}\triangle_v x_{k-1} - \sum_{k=1}^{n-1} R_{k+m-1}\triangle_v x_{k+m-1} - n^{m-r}R_n n^{r-m} x_n v_n$$

Now, letting $n \to \infty$, we get

$$\begin{aligned}\sum_{k=1}^{\infty} a_k x_k &= -\sum_{k=1}^{\infty} R_{k+m-1}\triangle_v x_{k+m-1} \\ &= \sum_{k=1}^{\infty}(k+m-1)^{m-(r+1)}|R_{k+m-1}| = \infty.\end{aligned}$$

This contradicts to $a \in (D[\triangle_{v,r}^m(l_\infty)])^\beta$. Hence, $a \in M_\beta(v,r)$.

(b) can be proved by the same way as above, using Lemma 3.6.

**Lemma 3.20.** $(D[\triangle_{v,r}^m(l_\infty)])^\eta = (D[\triangle_{v,r}^m(c)])^\eta$ *for* $\eta = \beta$ *or* $\gamma$.

**Lemma 3.21.**

(i) $[\triangle_{v,r}^m(l_\infty)]^\eta = (D[\triangle_{v,r}^m(l_\infty)])^\eta$

(ii) $[\triangle_{v,r}^m(c)]^\eta = (D[\triangle_{v,r}^m(l_\infty)])^\eta$

*for* $\eta = \beta$ *or* $\gamma$.

**Proof.** (i) We give the proof for $\eta = \beta$ only. It can be proved in a similar way for $\eta = \gamma$. Since $D[\triangle_{v,r}^m(l_\infty)] \subset \triangle_{v,r}^m(l_\infty)$, then $[\triangle_{v,r}^m(l_\infty)]^\beta \subset (D[\triangle_{v,r}^m(l_\infty)])^\beta$. Let $a \in (D[\triangle_{v,r}^m(l_\infty)])^\beta$. If $x = (x_k) \in \triangle_{v,r}^m(l_\infty)$ defined by

$$x_k = \begin{cases} x_k, & k \le m \\ x'_k, & k > m, \end{cases}$$

where $x' = (x'_k) \in D[\triangle_{v,r}^m(l_\infty)]$, then we may write for $n > m$

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{m} a_k x_k + \sum_{k=m+1}^{n} a_k x'_k.$$

Now, letting $n \to \infty$, we get the series, in the same way as the proof of Lemma 3.19,

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{m} a_k x_k + (-1)^m \sum_{k=1}^{\infty}(k+m-1)^{m-(r+1)} R_{k+m-1} z_k$$

is convergent. This implies that $a \in [\triangle_{v,r}^m(l_\infty)]^\beta$.

(ii) can be proved by the same way as above.

**Theorem 3.22.** *Let $X$ stands for $l_\infty$ or $c$. Then*

a) $$[\triangle^m_{v,r}(X)]^\beta = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^{m-r} a_k v_k^{-1} \text{ is convergent}, \sum_{k=1}^{\infty} k^{m-(r+1)} |R_k| < \infty\Big\},$$

b) $$[\triangle^m_{v,r}(X)]^\gamma = \Big\{a = (a_k) : \sup_n \Big|\sum_{k=1}^{n} k^{m-r} a_k v_k^{-1}\Big| < \infty, \sum_{k=1}^{\infty} k^{m-(r+1)} |R_k| < \infty\Big\},$$

*where* $R_k = \sum\limits_{j=k+1}^{\infty} a_j v_j^{-1}$.

**Proof.** The proof follows from Lemma 3.19, Lemma 3.20 and Lemma 3.21.

From Theorem 3.22, we have the following corollaries:

**Corollary 3.23.** *If we take $v_k = (1, 1, ...)$, then we obtain*

(i) $$[\triangle^m_{r}(X)]^\beta = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^{m-r} a_k \text{ is convergent}, \sum_{k=1}^{\infty} k^{m-(r+1)} |R_k| < \infty\Big\},$$

(ii) $$[\triangle^m_{r}(X)]^\gamma = \Big\{a = (a_k) : \sup_n |\sum_{k=1}^{n} k^{m-r} a_k| < \infty, \sum_{k=1}^{\infty} k^{m-(r+1)} |R_k| < \infty\Big\},$$

*where* $R_k = \sum\limits_{j=k+1}^{\infty} a_j$.

**Corollary 3.24.** *If we take $r = 0$, then we obtain*

(i) $$[\triangle^m_{v}(X)]^\beta = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^{m} a_k v_k^{-1} \text{ is convergent}, \sum_{k=1}^{\infty} k^{m-1} |R_k| < \infty\Big\},$$

(ii) $[\triangle_v^m(X)]^\gamma = \Big\{a = (a_k) : z\sup_n \Big| \sum_{k=1}^{n} k^m a_k v_k^{-1} \Big| < \infty,$

$$\sum_{k=1}^{\infty} k^{m-1} |R_k| < \infty \Big\},$$

*where* $R_k = \sum_{j=k+1}^{\infty} a_j v_j^{-1}$.

**Corollary 3.25.** *If we take* $v_k = (1, 1, ...)$ *and* $r = 0$, *then we obtain*

(i) $[\triangle^m(X)]^\beta = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^m a_k$ *is convergent,* $\sum_{k=1}^{\infty} k^{m-1} |R_k| < \infty \Big\}$, [3]

(ii) $[\triangle^m(X)]^\gamma = \Big\{a = (a_k) : \sup_n \Big| \sum_{k=1}^{n} k^m a_k \Big| < \infty, \sum_{k=1}^{\infty} k^{m-1} |R_k| < \infty \Big\}$, [3]

*where* $R_k = \sum_{j=k+1}^{\infty} a_j$.

**Corollary 3.26.** *If we take* $v_k = (1, 1, ...)$, $r = 0$ *and* $m = 1$, *then we obtain*

(i) $[\triangle(X)]^\beta = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k a_k$ *is convergent,* $\sum_{k=1}^{\infty} |R_k| < \infty \Big\}$, [8]

(ii) $[\triangle(X)]^\beta = \Big\{a = (a_k) : \sup_n \Big| \sum_{k=1}^{n} k a_k \Big| < \infty, \sum_{k=1}^{\infty} |R_k| < \infty \Big\}$, [8]

*where* $R_k = \sum_{j=k+1}^{\infty} a_j$.

By Lemma 3.9, Theorem 3.10 and Corollary 3.18, we also have

**Corollary 3.27.**

(i) $\triangle_{v,r}^m(l_\infty), \triangle_{v,r}^m(c)$ *are not normal.*

(ii) $\triangle_{v,r}^m(l_\infty), \triangle_{v,r}^m(c)$ *are not monotone.*

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