COMMON FIXED POINT THEOREMS FOR FINITE NUMBER OF MAPPINGS WITHOUT CONTINUITY AND COMPATIBILITY ON UNIFORMLY CONVEX BANACH SPACE

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Abstract. The purpose of this paper is to prove some common fixed point theorems for finite number of discontinuous, noncompatible mappings on noncomplete uniformly convex Banach space. Our results extend, generalize several known results of fixed point theory in different spaces. We give an example and also give formulas for total number of commutativity conditions for finite number of mappings.

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1. Introduction

Husain and Sehgal [2] proved common fixed point theorems for a family of mappings. Khan and Imdad [8] extended result of Husain and sehgal [2] and proved fixed point theorems for a class of mappings. Imdad, Khan and Sessa [3] extended above results and proved common fixed points for three mappings defined on a closed subset of a uniformly convex Banach space.

Rashwan [9] extended result of Imdad, khan and Sessa [3] by employing four compatible mappings of type (A) instead of weakly commuting mappings and by using one continuous mapping as opposed to two.

Sharma and Bamboria [11] improved results of Rashwan [9] by removing the condition of continuity and replacing the compatibility of mappings of type (A) by weak compatibility.

Sharma and Tilwankar [12] proved a common fixed point theorem for four mappings under the condition of weak compatible mappings by using the new property (E.A). For the study of discontinuous and noncompatible mappings in fixed point theory we refer to Sharma and Deshpande [13] and Sharma, Deshpande and Tiwari [14].

Several observations motivated us to prove common fixed point theorem for ten noncompatible, discontinuous mappings in noncomplete uniformly convex Banach space. We also extend our results for finite number of mappings. Our main theorems extend, improve, generalize some known results in uniformly convex Banach space. We give an example to validate our result.

To prove existence of common fixed point for finite number of mappings some commutativity conditions are required. How many commutativity conditions are necessary? We give answer of this question by giving formulas.

Throughout the paper X stands for a Banach space. Let R^+ denote the set of all non-negative real numbers and F be the family of mappings f from $(R^+)^5$ into R^+ such that each f is upper-semicontinuous, non-decreasing in each coordinate variable.

The modulus of convexity of X is a function δ from $(0,2]$ into $(0,1]$ defined by

$$\delta(\varepsilon)=\inf\left\{1-\frac{1}{2}\|x-y\|,\ x,y\in X,\ \|x\|=\|y\|=1,\ \|x-y\|\geq\varepsilon\right\}.$$

Moreover, if X is uniformly convex, then δ is strictly increasing, $\delta(\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0,\ \delta(2)=1,\ \eta(t)<2$ when $t<1$ and η is the inverse of δ.

For our theorem we need the following lemma:

Lemma 1.1. ([1]) *Let X be uniformly convex Banach space and B_r, the closed ball in X centered at the origin with radius $r>0$. If $x_1,x_2,x_3\in B_r$ satisfy*

$$\|x_1-x_2\|\geq\|x_2-x_3\|\geq d>0 \text{ and if } \|x_2\|\geq\left(1-\frac{1}{2}\delta\left(\frac{d}{\ell}\right)\right)\ell,$$

then

$$\|x_1-x_3\|\leq\eta\left(1-\frac{1}{2}\delta\left(\frac{d}{\ell}\right)\right)\|x_1-x_2\|.$$

Now, we begin with some known definitions:

Definition 1.1. ([10]) Let S and T be self-mappings on X. Then $\{S,T\}$ is called a *weakly commuting pair* on X if

$$\|STx-TSx\|\leq\|Sx-Tx\| \text{ for all } x\in X.$$

Definition 1.2. ([4]) Let $S,T:X\longrightarrow X$ be mappings. S and T are said to be compatible if

$$\lim_{n\longrightarrow\infty}\|STx_n-TSx_n\|=0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\longrightarrow\infty}Sx_n=\lim_{n\longrightarrow\infty}Tx_n=t \text{ for some } t\in X.$$

Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible. On the other hand, examples are given by Jungck [4], [5], [6] and Sessa [10] to show neither of the above implications are reversible.

Definition 1.3. [7] Two self mappings S and T are said to be weakly compatible if they commute at their coincidence points; i.e., if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

2. Common fixed point theorems

In a paper, Imdad, Khan and Seesa [3] proved the following theorem:

Theorem A. *Let X be uniformly convex and K a non-empty closed subset of X. Let A, S and T be three self-mappings of K satisfying the following conditions:*

(1) *S and T are continuous, $AK \subset SK \cap TK$,*

(2) *$\{A, S\}$ and $\{A, T\}$ are weakly commuting pairs on K,*

(3) *there exists a function $f \in F$ such that for every $x, y \in K$:*

$$\|Ax-Ay\| \leq f(\|Sx-Ty\|,\ \|Sx-Ax\|,\ \|Sx-Ay\|,\ \|Ty-Ax\|,\ \|Ty-Ay\|),$$

where f has the additional requirements:

(a) *for $t > 0, f(t, t, 0, \alpha t, t) \leq \beta t$ and $f(t, t, \alpha t, 0, t) \leq \beta t$ being $\beta < 1$ for $\alpha < 2$ and $\beta = 1$ for $\alpha = 2$, $\alpha, \beta \in R^+$,*

(b) *$f(t, 0, t, t, 0) < t$ for $t > 0$.*

Then, there exists a point u in K such that

(c) *u is the unique common fixed point of A, S and T.*

(d) *For any $x_0 \in K$, the sequence $\{Ax_n\}$ defined by*

$$Tx_{2n} = Ax_{2n-1},\ Sx_{2n+1} = Ax_{2n},\ \text{for } n = 0, 1, 2...,$$

converges strongly to u.

Rashwan [9] extended Theorem A for compatible mappings of type (A) and proved the following:

Theorem B. *Let X and K be as in Theorem* A. *Let A, B, S and T be mappings on K satisfying the following conditions:*

(1) *one of A, B, S and T is continuous and $AK \subset TK$, $BK \subset SK$,*

(2) *$\{A, S\}$ and $\{B, T\}$ are compatible of type (A),*

(3) *there exists a function $f \in F$ such that for every $x, y \in K$:*

$$\|Ax-By\| \le f(\|Sx-Ty\|,\ \|Sx-Ax\|,\ \|Sx-By\|,\ \|Ty-Ax\|,\ \|Ty-By\|),$$

where f satisfies the conditions (a) *and* (b) *as in Theorem Arm.*

Then, there exists a point u in K such that

(a) *u is the unique common fixed point of A, B, S and T,*

(b) *for any $x_0 \in K$, the sequence $\{y_n\}$ defined by*

$$y_{2n} = Sx_{2n} = Bx_{2n-1},\ y_{2n+1} = Tx_{2n+1} = Ax_{2n},\ n = 1, 2, 3, ...$$

converges strongly to u.

Sharma and Bamboria [11] proved the following.

Theorem C. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S and T be mappings on K satisfying the following conditions:*

(1) *$AK \subset TK$ and $BK \subset SK$,*

(2) *there exists a function $f \in F$ such that for every $x, y \in K$:*

$$\|Ax-By\| \le f(\|Sx-Ty\|,\ \|Sx-Ax\|,\ \|Sx-By\|,\ \|Ty-Ax\|,\ \|Ty-By\|),$$

where f satisfies the conditions (a) *and* (b) *as in Theorem* A,

(3) *one of AK, BK, SK or TK is complete subspace of X, then*

  (a) *A and S have a coincidence point,*
  (b) *B and T have a coincidence point.*

Further if

(4) *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then A, B, S and T have a common fixed point z in K.*

Further, z is the unique common fixed point of A and S and of B and T.

Sharma and Tilwankar [12] proved the following by using (E.A) property.

Theorem D. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S and T be mappings on K satisfying the following conditions:*

(1) *$AK \subset TK$ and $BK \subset SK$,*

(2) *$\{A, S\}$ or $\{B, T\}$ satisfies the property* (E.A),

(3) *for every* $x, y \in K$:

$$\|Ax - By\| \leq \max(\|Sx - Ty\|, \|Sx - By\|, \|Ty - By\|),$$

(4) *one of AK, BK, SK or TK is closed subset of X, then*

(a) *A and S have a coincidence point,*

(b) *B and T have a coincidence point.*

Further if

(5) *the pairs* $\{A, S\}$ *and* $\{B, T\}$ *are weakly compatible, then*

(c) *A, B, S and T have a common fixed point z in K.*

Further z is the unique common fixed point of A and S and of B and T.

3. Main results

Theorem 3.1. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S, T, I, J, L, U, P and Q be mappings on K satisfying the following conditions:*

(3.1) $P(K) \subset STJU(K)$ *and* $Q(K) \subset ABIL(K)$,

(3.2) *there exists a function* $f \in F$ *such that for every* $x, y \in K$:

$$\|Px - Qy\| \leq f(\|ABILy - STJUx\|, \|Px - STJUx\|, \|Qy - STJUx\|, \|Px - ABILy\|, \|Qy - ABILy\|),$$

(3.3) *if one of* $P(K), ABIL(K), STJU(K)$ *or* $Q(K)$ *is complete subspace of X, then*

(i) *P and STJU have a coincidence point,*

(ii) *Q and ABIL have a coincidence point,*

(3.4) $AB = BA,\ AI = IA,\ AL = LA,\ BI = IB,\ BL = LB,\ IL = LI,$
$QL = LQ,\ QI = IQ,\ QB = BQ,\ ST = TS,\ SJ = JS,\ SU = US,$
$TJ = JT,\ TU = UT,\ JU = UJ,\ PU = UP,\ PJ = JP,\ PT = TP.$

Further if

(3.5) *the pairs* $\{P, STJU\}$ *and* $\{Q, ABIL\}$ *are weakly compatible, then A, B, S, T, I, J, L, U, P and Q have a common fixed point z in X.*

Here f satisfy the following two conditions.

(a) *for* $t > 0$, $f(t,t,0,\alpha t,t) \leq \beta t$ *and* $f(t,t,\alpha t,0,t) \leq \beta t$ *being* $\beta < 1$ *for* $\alpha < 2$ *and* $\beta = 1$ *for* $\alpha = 2$, $\alpha, \beta \in R^+$,

(b) $f(t,0,t,t,0) < t$ *or* $f(0,t,0,t,0) < t$ *for* $t > 0$.

Proof. Let $x_0 \in K$, since $P(K) \subset STJU(K)$ and $Q(K) \subset ABIL(K)$, we can always define a sequence $\{y_n\}$ such that

$$\begin{aligned} y_{2n} &= Qx_{2n-1} = ABILx_{2n}, \\ y_{2n+1} &= Px_{2n} = STJUx_{2n+1}, \quad n = 1,2,3,.... \end{aligned}$$

Let $d_n = \|y_n - y_{n+1}\|, \; n = 0,1,2,...$

$$d = \lim_{n \longrightarrow \infty} d_n.$$

Now, for an even n, we have

$$\begin{aligned} d_n &= \|y_n - y_{n+1}\| = \|Px_n - Qx_{n-1}\| \\ &\leq f(\|ABILx_{n-1} - STJUx_n\|, \|Px_n - STJUx_n\|, \\ &\|Qx_{n-1} - STJUx_n\|, \|Px_n - ABILx_{n-1}\|, \|Qx_{n-1} - ABILx_{n-1}\|) \\ &= f(\|y_{n-1} - y_n\|, \|y_{n+1} - y_n\|, \|y_n - y_n\|, \|y_{n+1} - y_{n-1}\|, \|y_n - y_{n-1}\|) \\ &= f(\|y_{n-1} - y_n\|, \|y_{n+1} - y_n\|, 0, \|y_{n+1} - y_{n-1}\|, \|y_n - y_{n-1}\|) \\ &\leq f(\|y_{n-1} - y_n\|, \|y_{n+1} - y_n\|, 0, \|y_{n+1} - y_n\| + \|y_n - y_{n-1}\|, \|y_n - y_{n-1}\|) \end{aligned} \tag{3.6}$$

which implies

$$d_n = f(d_{n-1}, d_n, 0, d_n + d_{n-1}, d_{n-1}).$$

Similarly. for an odd n, we obtain

$$\begin{aligned} d_n &= \|y_n - y_{n+1}\| = \|Px_{n-1} - Qx_n\| \\ &\leq f(\|ABILx_n - STJUx_{n-1}\|, \|Px_{n-1} - STJUx_{n-1}\|, \\ &\|Qx_n - STJUx_{n-1}\|, \|Px_{n-1} - ABILx_n\|, \|Qx_n - ABILx_n\|) \\ &= f(\|y_n - y_{n-1}\|, \|y_n - y_{n-1}\|, \|y_{n+1} - y_{n-1}\|, \|y_n - y_n\|, \|y_{n+1} - y_n\|) \\ &= f(\|y_n - y_{n-1}\|, \|y_n - y_{n-1}\|, \|y_{n+1} - y_{n-1}\|, 0, \|y_{n+1} - y_n\|) \\ &\leq f(\|y_n - y_{n-1}\|, \|y_n - y_{n-1}\|, \|y_{n+1} - y_n\| + \|y_n - y_{n-1}\|, 0, \|y_{n+1} - y_n\|) \\ d_n &= f(d_{n-1}, d_{n-1}, d_n + d_{n-1}, 0, d_n) \end{aligned} \tag{3.7}$$

If $d_n > d_{n-1}$, for some $n \geq 1$, then $d_{n-1} + d_n = \alpha d_n$ with $\alpha < 2$, $\alpha \in R$. Since f is nondecreasing in each coordinate variable

$$d_n \leq \begin{cases} f(d_n, d_n, 0, \alpha d_n, d_n), & \text{if } n \text{ is even,} \\ f(d_n, d_n, \alpha d_n, 0, d_n), & \text{if } n \text{ is odd.} \end{cases}$$

In both cases, by (a) we get $d_n \leq \beta d_n < d_n$, for some $\beta < 1$, $\beta \in R^+$, a contradiction. Thus, $d_{n-1} \geq d_n$ for $n = 1,2,3,...$

Suppose $d > 0$. Without loss of generality, we can postulate that the origin of X belongs to K

$$\lim_{n \longrightarrow \infty} \sup \|y_n\| = \ell' > 0.$$

Let $\ell \in R^+$ be chosen in such a way that $\ell' < 1$ and $eta\left(1 - \frac{1}{2}\delta\left(\frac{d}{\ell}\right)\right) < \ell'$, then there exists a sequence $\{n(k)\}$, $k = 0, 1, 2, ...,$ $n(0) \geq 1$, of positive integers such that

$$\|y_{n(k)}\| \geq \left(1 - \frac{1}{2}\delta\left(\frac{d}{\ell}\right)\right),$$

where as it is $\|y_n\| \leq \ell$ for any $n \geq n(0)$.

Since $d_{n(k)-1} \geq d_{n(k)} \geq d > 0,\ k = 0, 1, 2, ...,$ from Lemma 1.1 it follows that

$$\|y_{n(k)-1} - y_{n(k)+1}\| \leq \eta\left(\frac{\ell'}{\ell}\right) d_{n(k)-1}, \tag{3.8}$$

where $\eta\left(\frac{\ell'}{\ell}\right) < 2$ being $\frac{\ell'}{\ell} < 1$.

Then, by (3.6), (3.7) and (3.8), we have

$$d_{n(k)} \leq \begin{cases} f\left(d_{n(k)-1}, d_{n(k)-1}, 0, \eta\left(\frac{\ell'}{\ell}\right) d_{n(k)-1}, d_{n(k)-1}\right), & \text{if } n \text{ is even,} \\ f\left(d_{n(k)-1}, d_{n(k)-1}, 0, \eta\left(\frac{\ell'}{\ell}\right) d_{n(k)-1}, d_{n(k)-1}\right), & \text{if } n \text{ is odd.} \end{cases}$$

In both cases, (a) implies

$$d_{n(k)} \leq \beta d_{n(k)-1} \text{ for some } \beta < 1.$$

Observing that β does not depend on k, the foregoing inequality gives, as $n \longrightarrow \infty$, that $d \leq \beta d < d$, a contradiction. This means that $d = 0$.

Now, we wish to prove that $\{y_n\}$ is a Cauchy sequence. Since $\lim_{n \longrightarrow \infty} d_n = 0$, it is sufficient to show that the sequence $\{y_{2n}\}$ is a Cauchy sequence. If not, then there is an $\varepsilon > 0$ such that for every even integer $2k$, $k = 0, 1, 2, ...,$ there exists two sequences $\{2_{n(k)}\}, \{2_{m(k)}\}$ with $2k \leq 2_{n(k)} \leq 2_{m(k)}$ for which

$$\|y_{n(k)} - y_{m(k)}\| > \varepsilon. \tag{3.9}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $n(k)$ and satisfying (3.9). Then

$$\|y_{2n(k)} - y_{2m(k)-2}\| \leq \varepsilon \text{ and } \|y_{2n(k)} - y_{2m(k)}\| > \varepsilon.$$

For each $k = 0, 1, 2, ...,$ we have

$$\begin{aligned} \varepsilon &< \|y_{2n(k)} - y_{2m(k)}\| \leq \|y_{2n(k)} - y_{2m(k)-2}\| + \|y_{2m(k)-2} - y_{2m(k)-1}\| \\ &+ \|y_{2m(k)-1} - y_{2m(k)}\| \\ &\leq \varepsilon + d_{2m(k)-2} + d_{2m(k)-1}, \end{aligned}$$

which implies

$$\lim_{k \to \infty} \|y_{2n(k)} - y_{2m(k)}\| = \varepsilon. \tag{3.10}$$

Further, from the triangular inequality, it follows that

$$\big|\|y_{2n(k)} - y_{2m(k)-1}\| - \|y_{2n(k)} - y_{2m(k)}\|\big| \le d_{2m(k)-1}$$

and

$$\big|\|y_{2n(k)+1} - y_{2m(k)-1}\| - \|y_{2n(k)} - y_{2m(k)}\|\big| \le d_{2m(k)-1} + d_{2n(k)}.$$

Hence, for $k \longrightarrow \infty$, we find by (3.10) that

$$\|y_{2n(k)} - y_{2m(k)-1}\| \longrightarrow \varepsilon \text{ and } \|y_{2n(k)+1} - y_{2m(k)-1}\| \longrightarrow \varepsilon. \tag{3.11}$$

On the other hand, using (3.2) we deduce that

$$\begin{aligned} \|y_{2n(k)} - y_{2m(k)}\| &\le d_{2n(k)} + \|y_{2n(k)+1} - y_{2m(k)}\| \\ &\le d_{2n(k)} + f(\|y_{2m(k)-1} - y_{2n(k)}\|, d_{2n(k)}, \\ &\quad \|y_{2m(k)-1} - y_{2n(k)+1}\|, \|y_{2n(k)} - y_{2m(k)}\|, d_{2n(k)}). \end{aligned} \tag{3.12}$$

By (3.10), (3.11), the upper-semicontinuity and non-decreasing properties of f and condition (b), we have from (3.12), for $k \longrightarrow \infty$, $\varepsilon \le f(\varepsilon, 0, \varepsilon, \varepsilon, 0) < \varepsilon$, which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in K and so is $\{y_n\}$. But K is a closed subset of a Banach space X, therefore $\{y_n\}$ converges to a point z in K. On the other hand, the subsequences $\{Px_{2n}\}$, $\{Qx_{2n-1}\}$, $\{STJUx_{2n+1}\}$ and $\{ABILx_{2n}\}$ of $\{y_n\}$ also converges to z.

Now, suppose that $STJU(K)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $STJU(K)$ and has a limit in $STJU(K)$ call it z.

Let $u \in (STJU)^{-1}z$. Then $STJUu = z$. By (3.2), we have

$$\begin{aligned} \|Pu - Qx_{2n+1}\| \le f(&\|ABILx_{2n+1} - STJUu\|, \|Pu - STJUu\|, \\ &\|Qx_{2n+1} - STJUu\|, \|Pu - ABILx_{2n+1}\|, \|Qx_{2n+1} - ABILx_{2n+1}\|). \end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|Pu - z\| &\le f(\|z - z\|, \|Pu - z\|, \|z - z\|, \|Pu - z\|, \|z - z\|) \\ \|Pu - z\| &\le f(0, \|Pu - z\|, 0, \|Pu - z\|, 0), \end{aligned}$$

which is a contradiction and so $Pu = z$. Therefore, $Pu = z = STJUu$, i.e., u is a coincidence point of P and $STJU$.

Let $v \in (ABIL)^{-1}z$, then $ABILv = z$. By (3.2), we have

$$\begin{aligned} \|Px_{2n} - Qv\| \le\ & f(\|ABILv - STJUx_{2n}\|, \|Px_{2n} - STJUx_{2n}\|, \\ & \|Qv - STJUx_{2n}\|, \|Px_{2n} - ABILv\|, \|Qv - ABILv\|). \end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|z - Qv\| &\le f(\|z - z\|, \|z - z\|, \|Qv - z\|, \|z - z\|, \|Qv - z\|) \\ \|z - Qv\| &\le f(0, 0, \|Qv - z\|, 0, \|Qv - z\|). \end{aligned} \tag{3.13}$$

Let be $\|z-Qv\| > 0$. Being f non-decreasing in each coordinate variable from (3.13), we obtain

$$\|z-Qv\| \leq f(\|z-Qv\|, \|z-Qv\|, \alpha\|z-Qv\|, 0, \|z-Qv\|),$$

where $1 \leq \alpha < 2$. Applying (a), then we deduce for some $\beta < 1$ that

$$\|z-Qv\| \leq \beta\|z-Qv\| < \|z-Qv\|,$$

which is a contradiction and so $Qv = z$. Since $ABILv = z$, thus $ABILv = Qv = z$, i.e., v is a coincidence point of $ABIL$ and Q.

If $P(K)$ is complete, then by (3.1), $z \in P(K) \subset STJU(K)$.

Similarly, if $Q(K)$ is complete, then $z \in Q(K) \subset ABIL(K)$.

Since the pair $\{P, STJU\}$ is weakly compatible, therefore P and $STJU$ commute at their coincidence point, i.e., if $Pu = STJUu$ for some $u \in X$, then

$$P(STJU)u = (STJU)Pu \ \text{ or } \ Pz = STJUz.$$

Similarly,

$$Q(ABIL)v = (ABIL)Qv \text{ or } Qz = ABILz.$$

Now, we prove $Pz = z$. By (3.2), we have

$$\begin{aligned}\|Pz - Qx_{2n+1}\| \leq f(&\|ABILx_{2n+1} - STJUz\|, \|Pz - STJUz\|,\\ &\|Qx_{2n+1} - STJUz\|, \|Pz - ABILx_{2n+1}\|, \|Qx_{2n+1} - ABILx_{2n+1}\|).\end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned}\|Pz - z\| &\leq f(\|z-Pz\|, \|Pz-Pz\|, \|z-Pz\|, \|Pz-z\|, \|z-z\|)\\ &= f(\|z-Pz\|, 0, \|z-Pz\|, \|Pz-z\|, 0)\\ \|Pz-z\| &< \|Pz-z\|,\end{aligned}$$

which is a contradiction and so $Pz = z$ and, therefore, $Pz = z = STJUz$.

Similarly, the pair $\{Q, ABIL\}$ is weakly compatible, therefore Q and $ABIL$ commute at their coincidence point, i.e., if $Qv = ABILv$, for some $v \in X$, then $Q(ABIL)v = (ABIL)Qv$ or $Qz = ABILz$.

Now, we prove $Qz = z$. By (3.2), we have

$$\begin{aligned}\|Px_{2n} - Qz\| \leq f(&\|ABILz - STJUx_{2n}\|, \|Px_{2n} - STJUx_{2n}\|,\\ &\|Qz - STJUx_{2n}\|, \|Px_{2n} - ABILz\|, \|Qz - ABILz\|).\end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned}\|z-Qz\| &\leq f(\|Qz-z\|, \|z-z\|, \|Qz-z\|, \|z-Qz\|, \|Qz-Qz\|)\\ \|z-Qz\| &\leq f(\|Qz-z\|, 0, \|Qz-z\|, \|z-Qz\|, 0)\\ \|z-Qz\| &< \|z-Qz\|,\end{aligned}$$

which is a contradiction and so $Qz = z$ and, therefore, $Qz = ABILz = z$.

By (3.2), we have

$$\|Pz - Q(Lz)\| \leq f(\|ABIL(Lz) - STJUz\|, \|Pz - STJUz\|, \|Q(Lz) - STJUz\|, \|Pz - ABIL(Lz)\|, \|Q(Lz) - ABIL(Lz)\|).$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|z - Lz\| &\leq f(\|Lz - z\|, \|z - z\|, \|Lz - z\|, \|Lz - z\|, \|Lz - Lz\|) \\ &\leq f(\|Lz - z\|, 0, \|Lz - z\|, \|Lz - z\|, 0) \\ \|Lz - z\| &< \|Lz - z\|, \end{aligned}$$

which is a contradiction and so $Lz = z$. Since $ABILz = z$, we have $ABIz = z$.

By using (3.2) and (3.4), we have

$$\|Pz - Q(Iz)\| \leq f(\|ABIL(Iz) - STJUz\|, \|Pz - STJUz\|, \|Q(Iz) - STJUz\|, \|Pz - ABIL(Iz)\|, \|Q(Iz) - ABIL(Iz)\|).$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|z - Iz\| &\leq f(\|Iz - z\|, \|z - z\|, \|Iz - z\|, \|z - Iz\|, \|Iz - Iz|) \\ &\leq f(\|Iz - z\|, 0, \|Iz - z\|, \|z - Iz\|, 0) \\ \|Iz - z\| &< \|Iz - z\|, \end{aligned}$$

which is a contradiction and so $Iz = z$. Since $ABIz = z$, we have $ABz = z$.

Now, we prove $Bz = z$. By putting $x = z$ and $y = Bz$ in (3.2) and (3.4), we have

$$\|Pz - Q(Bz)\| \leq f(\|ABIL(Bz) - STJUz\|, \|Pz - STJUz\|, \|Q(Bz) - STJUz\|, \|Pz - ABIL(Bz)\|, \|Q(Bz) - ABIL(Bz)\|).$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|z - Bz\| &\leq f(\|Bz - z\|, \|z - z\|, \|Bz - z\|, \|z - Bz\|, \|Bz - Bz\|) \\ &\leq f(\|Bz - z\|, 0, \|Bz - z\|, \|z - Bz\|, 0) \\ \|Bz - z\| &< \|Bz - z\|, \end{aligned}$$

which is a contradiction and so $Bz = z$. Since $ABz = z$, we have $Az = z$.

Now, we prove $Uz = z$. By using (3.2) and (3.4), we have

$$\|P(Uz) - Qz\| \leq f(\|ABILz - STJU(Uz)\|, \|P(Uz) - STJU(Uz)\|, \|Qz - STJU(Uz)\|, \|P(Uz) - ABILz\|, \|Qz - ABILz\|).$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|Uz - z\| &\leq f(\|z - Uz\|, \|Uz - Uz\|, \|z - Uz\|, \|Uz - z\|, \|z - z\|) \\ &\leq f(\|z - Uz\|, 0, \|z - Uz\|, \|Uz - z\|, 0) \\ \|Uz - z\| &< \|Uz - z\|, \end{aligned}$$

which is a contradiction and so $Uz = z$. Since $STJUz = z$, we have $STJz = z$.

Now, we prove $Jz = z$. By using (3.2) and (3.4), we have

$$\begin{aligned}\|P(Uz) - Qz\| \le f(&\|ABILz - STJU(Jz)\|, \|P(Jz) - STJU(Jz)\|,\\ &\|Qz - STJU(Jz)\|, \|P(Jz) - ABILz\|, \|Qz - ABILz\|).\end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned}\|Jz - z\| &\le f(\|z - Jz\|, \|Jz - Jz\|, \|z - Jz\|, \|Jz - z\|, \|z - z\|)\\ &\le f(\|z - Jz\|, 0, \|z - Jz\|, \|Jz - z\|, 0)\\ \|Jz - z\| &< \|Jz - z\|,\end{aligned}$$

which is a contradiction and so $Jz = z$. Since $STJz = z$, we have $STz = z$.

Now, we prove $Tz = z$. By using (3.2) and (3.4), we have

$$\begin{aligned}\|P(Tz) - Qz\| \le f(&\|ABILz - STJU(Tz)\|, \|P(Tz) - STJU(Tz)\|,\\ &\|Qz - STJU(Tz)\|, \|P(Tz) - ABILz\|, \|Qz - ABILz\|).\end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned}\|Tz - z\| &\le f(\|z - Tz\|, \|Tz - Tz\|, \|z - Tz\|, \|Tz - z\|, \|z - z\|)\\ &\le f(\|z - Tz\|, 0, \|z - Tz\|, \|Tz - z\|, 0)\\ \|Tz - z\| &< \|Tz - z\|,\end{aligned}$$

which is a contradiction and so $Tz = z$. Since $STz = z$, we have $Sz = z$.

By combining the above results, we have

$$Az = Bz = Sz = Tz = Iz = Jz = Lz = Uz = Pz = Qz = z.$$

That is z is a common fixed point of $A, B, S, T, I, J, L, U, P$ and Q.

For the uniqueness of the common fixed point, let w ($w \neq z$) be another common fixed point of $A, B, S, T, I, J, L, U, P$ and Q. Then, by (3.2), we have

$$\begin{aligned}\|Pz - Qw\| \le f(&\|ABILw - STJUz\|, \|Pz - STJUz\|,\\ &\|Qw - STJUz\|, \ \|Pz - ABILw\|, \|Qw - ABILw\|).\end{aligned}$$

This gives

$$\begin{aligned}\|z - w\| &\le f(\|w - z\|, \|z - z\|, \|w - z\|, \|z - w\|, \|w - w\|)\\ &\le f(\|w - z\|, 0, \|w - z\|, \|z - w\|, 0)\\ \|w - z\| &< \|w - z\|,\end{aligned}$$

which is a contradiction and so $w = z$.

This completes the proof of the Theorem. ∎

If we put $P = Q$ in Theorem 3.1, we have

Corollary 1. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S, T, I, J, L, U and P be mappings on K satisfying the following conditions:*

(1) $P(K) \subset ABIL(K)$ *and* $P(K) \subset STJU(K)$,

(2) *there exists a function* $f \in F$ *such that for every* $x, y \in K$:

$$\|Px - Py\| \leq f(\|ABILy - STJUx\|, \|Px - STJUx\|,$$
$$\|Py - STJUx\|, \|Px - ABILy\|, \|Py - ABILy\|)$$

(3) *if one of* $P(K), ABIL(K)$ *or* $STJU(K)$ *is a complete subspace of* X, *then*

- (i) *P and STJU have a coincidence point,*
- (ii) *P and ABIL have a coincidence point,*

(4) $AB = BA,\ AI = IA,\ AL = LA,\ BI = IB,\ BL = LB,\ IL = LI,$
$PL = LP,\ PI = IP,\ PB = BP,\ ST = TS,\ SJ = JS,\ SU = US,$
$TJ = JT,\ TU = UT,\ JU = UJ,\ PU = UP,\ PJ = JP,\ PT = TP.$

Further, if

(5) *the pairs* $\{P, STJU\}$ *and* $\{P, ABIL\}$ *are weakly compatible, then A, B, S, T, I, J, L, U and P have a common fixed point z in X.*

If we put $L = U = Ix$ (The identity map on X) in Theorem 3.1, we have
Corollary 2. *Let X be uniformly convex Banach space and K a non-empty closed*

subset of X. Let A, B, S, T, I, J, P and Q be mappings on K satisfying the following conditions:

(1) $P(K) \subset ABI(K)$ *and* $Q(K) \subset STJ(K)$,

(2) *there exists a function* $f \in F$ *such that for every* $x, y \in K$:

$$\|Px - Qy\| \leq f(\|ABIy - STJx\|, \|Px - STJx\|,$$
$$\|Qy - STJx\|, \|Px - ABIy\|, \|Qy - ABIy\|)$$

(3) *if one of P(K), ABI(K), STJ(K) or Q(K) is a complete subspace of X, then*

- (i) *P and STJ have a coincidence point,*
- (ii) *Q and ABI have a coincidence point,*

(4) $AB = BA,\ AI = IA,\ BI = IB,\ QI = IQ,\ QB = BQ,\ ST = TS,$
$SJ = JS,\ TJ = JT,\ PJ = JP,\ PT = TP.$

Further, if

(5) *the pairs* $\{P, STJ\}$ *and* $\{Q, ABI\}$ *are weakly compatible, then A, B, S, T, I, J, P and Q have a common fixed point z in X.*

If we put $P = Q$ in Corollary 2, we have the following.

Corollary 3. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S, T, I, J and P be mappings on K satisfying the following conditions:*

(1) $P(K) \subset ABI(K)$ *and* $P(K) \subset STJ(K)$,

(2) *there exists a function* $f \in F$ *such that for every* $x, y \in K$:

$$\|Px - Py\| \leq f(\|ABIy - STJx\|, \|Px - STJx\|, \|Py - STJx\|, \|Px - ABIy\|, \|Py - ABIy\|)$$

(3) *if one of* $P(K), ABI(K)$ *or* $STJ(K)$ *is a complete subspace of* X, *then*

  (i) *P and STJ have a coincidence point,*

  (ii) *P and ABI have a coincidence point,*

(4) $AB = BA,\ AI = IA,\ BI = IB,\ PI = IP,\ PB = BP,\ ST = TS,\ SJ = JS,\ TJ = JT,\ PJ = JP,\ PT = TP.$

Further, if

(5) *the pairs* $\{P, STJ\}$ *and* $\{P, ABI\}$ *are weakly compatible, then A, B, S, T, I, J and P have a common fixed point z in X.*

If we put $I = J = Ix$ (the identity map on X) in Corollary 3, we have the following.

Corollary 4. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S, T and P be mappings on K satisfying the following conditions:*

(1) $P(K) \subset AB(K)$ *and* $P(K) \subset ST(K)$,

(2) *there exists a function* $f \in F$ *such that for every* $x, y \in K$:

$$\|Px - Py\| \leq f(\|ABy - STx\|, \|Px - STx\|, \|Py - STx\|, \|Px - ABy\|,\ \|Py - ABy\|).$$

(3) *if one of P(K), AB(K) or ST(K) is complete subspace of X, then*

  (i) *P and ST have a coincidence point,*

  (ii) *P and AB have a coincidence point,*

(4) $AB = BA,\ PB = BP,\ ST = TS,\ PT = TP.$

Further, if

(5) *the pairs* $\{P, ST\}$ *and* $\{P, AB\}$ *are weakly compatible, then* A, B, S, T *and* P *have a common fixed point* z *in* X.

Remark 1. If we put $P = Ix$ (the identity map on X) in Corollary 4, we obtain the results due to Sharma and Bamboria [11], which improves the results of Rashwan [9].

If we put $B = P = Ix$ (the identity map on X) in Corollary 4, we improve results of Imdad, Khan and Sessa [3] in the following way.

Corollary 5. *Let* X *be uniformly convex and* K *a non-empty closed subset of* X. *Let* A, S *and* T *be three self-mappings of* K *satisfying the following conditions:*

(1) $AK \subset SK \cap TK$,

(2) $\{A, S\}$ *and* $\{A, T\}$ *are weakly compatible pairs,*

(3) *there exists a function* $f \in F$ *such that for every* $x, y \in K$:

$$\|Ax - Ay\| \leq f(\|Sx - Ty\|, \|Sx - Ax\|, \|Sx - Ay\|, \ \|Ty - Ax\|, \ \|Ty - Ay\|),$$

where f has the additional requirements:

(a) *for* $t > 0$, $f(t, t, 0, \alpha t, t) \leq \beta t$ *and* $f(t, t, \alpha t, 0, t) \leq \beta t$ *being* $\beta < 1$ *for* $\alpha < 2$ *and* $\beta = 1$ *for* $\alpha = 2$, $\alpha, \beta \in R^+$,

(b) $f(t, 0, t, t, 0) < t$ *for* $t > 0$.

Then, there exists a point z *in* K *such that* z *is the unique common fixed point of* A, S *and* T.

Now, we extend Theorem 3.1 for a finite number of mappings in the following way:

Theorem 3.2. *Let* X *be uniformly convex Banach space and* K *a non-empty closed subset of* X. *Let* $A_1, A_2, ..., A_n, S_1, S_2, ..., S_n$, P *and* Q *be mappings from* X *into itself such that*

(3.14) $P(K) \subset S_1S_2...S_n(K), Q(K) \subset A_1A_2...A_n(K)$,

(3.15) $$\|Px - Qy\| \leq f(\|A_1A_2...A_ny - S_1S_2...S_nx\|, \|Px - S_1S_2...S_nx\|, \|Qy - A_1A_2...A_ny\|, \|Qy - S_1S_2...S_nx\|, \|Px - A_1A_2...A_ny\|)$$ *for all* $x, y \in X$,

(3.16) *if one of* $P(K), A_1A_2...A_n(K), S_1S_2...S_n(K)$ *or* $Q(K)$ *is a complete subspace of* X, *then*

(i) P *and* $S_1S_2...S_n$ *have a coincidence point and*

(ii) Q *and* $A_1A_2...A_n$ *have a coincidence point.*

Further, if

(3.17) A_1 *commutes with* A_2, A_3, ..., A_n,
A_2 *commutes with* A_3, A_4, ..., A_n,
A_3 *commutes with* A_4, A_5,..., A_n,
....
A_{n-1} *commutes with* A_n.

Similarly,

S_1 *commutes with* S_2, S_3, ..., S_n,
S_2 *commutes with* S_3, S_4..., S_n,
S_3 *commutes with* S_4, S_5..., S_n,
....
S_{n-1} *commutes with* S_n,
P *commutes with* S_2, S_3, ..., S_n,
Q *commutes with* A_2, A_3, ..., A_n.

(3.18) *the pairs* $\{P, S_1S_2...S_n\}$ *and* $\{Q, A_1A_2...A_n\}$ *are weakly compatible, then*
(iii) $A_1, A_2, ..., A_n, S_1, S_2, ..., S_n$, P *and* Q *have a unique common fixed point in* X.

Proof. Since $P(K) \subset S_1S_2...S_n(K)$, for any point $x_0 \in X$ there exists a point $x_1 \in X$ such that $Px_0 = S_1S_2...S_nx_1$. Since $Q(K) \subset A_1A_2...A_n(K)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Qx_1 = A_1A_2...A_nx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that for $n = 0, 1, 2, ...,$

$$\begin{aligned} y_{2n} &= Qx_{2n-1} = A_1A_2...A_nx_{2n}, \\ y_{2n+1} &= Px_{2n} = S_1S_2...S_nx_{2n+1}. \end{aligned}$$

By using the method of the proof of Theorem 3.1, we can see that conclusions (i), (ii) and (iii) hold.

Observations. Now, we are giving a formula for commutative conditions:

(i) If the number of mappings are even and finite in above theorems and corollaries then there will be $\frac{n^2-2n-8}{4}$ commutativity conditions, where $n = 4, 6, 8, 10, 12, ...$ up to finite values. For example, if $n = 10$, then 18 commutativity conditions are required. (See (3.4)).

(ii) If the number of mappings are odd and finite in above theorems and corollaries, then there will be $\frac{n^2-9}{4}$ commutativity conditions, where $n = 5, 7, 9, 11...$ up to finite values. For example, if $n = 7$, then 10 commutativity conditions are required. (See (4) in Corollary 3).

(iii) If $n = 1, 2, 3, 4$, then any commutativity condition is not required. (See Theorem C and Corollary 5.)

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